NONLINEAR STABILITY IN L^P FOR A CONFINED SYSTEM OF CHARGED PARTICLES

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Abstract. We prove the nonlinear stability in L^p, with $1 \le p \le 2$, of particular steady solutions of the Vlasov-Poisson system for charged particles in the whole space $\mathbb{R}^6$. Our main tool is a functional associated to the relative entropy or Casimir-energy functional.

Key words. Kinetic equations, Vlasov-Poisson system, nonlinear stability, relative entropy, Csiszár-Kullback inequality, interpolation inequalities

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1. Introduction. We consider a gas of charged particles described by a distribution function $f(t, x, v) \ge 0$ which represents the probability density of particles at position x with velocity v at time t. The evolution of f is governed by the Liouville evolution equation

$$(1.1) \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = 0$$

in $\mathbb{R}_0^+ \times \mathbb{R}^3 \times \mathbb{R}^3$, where the electric field $F(t, x)$ is given by an external potential ϕ_e and by a mean field potential ϕ according to

$$(1.2) \qquad F(t, x) = -q\left(\nabla_x \phi(t, x) + \nabla_x \phi_e(x)\right).$$

The electrostatic potential $\phi \ge 0$ is self-consistently computed by

$$(1.3) \qquad \phi = K * \rho(f)$$

with $K = \frac{q}{4\pi\epsilon_0} |x|^{-1}$, where $\rho(f)$ is the spatial density of particles, which is defined by

$$\rho(f)(t, x) = \int_{\mathbb{R}^3} f(t, x, v) \, dv.$$

As usual, ϵ_0 and q are respectively the permittivity of the vacuum and the elementary charge of the particles that, in the sequel, we assume to be unity without loss of generality. We shall consider the initial value problem corresponding to

$$(1.4) \qquad f(0, x, v) = f_0(x, v) \ge 0.$$

This system is called *the Vlasov-Poisson system for charged particles.* The main feature we add to standard versions of the Vlasov-Poisson system is an external potential that confines particles and allows the existence of steady states. For this reason, we will refer to $\phi_e(x)$ as a *confinement potential*.

The aim of this paper is to establish the nonlinear stability of special stationary solutions in $L^p(\mathbb{R}^6)$ with $p \in [1, 2]$ and explicit constants, at least in some cases (see

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Section 3). For that purpose, we shall use an entropy, which is also called Casimir-energy, free energy, relative entropy or Lyapunov functional in the literature. The stationary solution is a minimizer, under constraints, of the entropy, or, reciprocally, the entropy functional is determined by the shape in energy of the stationary solution. Our first main result corresponds to a p which is fixed by the entropy.

THEOREM 1.1. *Let ϕ_e be a bounded from below function on $\mathbb{R}^3$ with $\phi_e(x) \to \infty$ as $|x| \to +\infty$, such that $(x,s) \mapsto s^{3/2-1}\gamma(s + \phi_e(x))$ belongs to $L^1 \cap L^\infty(\mathbb{R}^3, L^1(\mathbb{R}))$. Here γ is the inverse of $(-\sigma')$, eventually extended by 0, where σ is a bounded from below and strictly convex function of class C^2.*

Let f be a weak solution of the Vlasov-Poisson system corresponding to a nonnegative initial data f_0 in $L^1 \cap L^{p_0}$, $p_0 = (12+3\sqrt{5})/11$, such that $\sigma(f_0)$ and $(|\phi_e|+|v|^2)f_0$ belong to $L^1(\mathbb{R}^3)$. If $\inf_{s\in(0,+\infty)} \sigma''(s)/s^{p-2} > 0$ for some $p \in [1,2]$, then there exists an explicit constant $C > 0$, which depends only on f_0, such that for any $t > 0$, $f = f(t)$ satisfies

$$\|f-f_\infty\|_{L^p}^2 \le C \int_{\mathbb{R}^6} [\sigma(f_0)-\sigma(f_\infty)-\sigma'(f_\infty)(f_0-f_\infty)] \, d(x,v) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla(\phi_0-\phi_\infty)|^2 \, dx$$

where $\left(f_\infty(x,v) = \gamma(\frac{1}{2}|v|^2 + \phi_e(x) + \phi_\infty(x)), \phi_\infty\right)$ is a stationary solution of the Vlasov-Poisson system and ϕ_0 is given by (1.3) at $t = 0$.

The value of p_0 arises from the paper [34] by Hörst and Hunze in order to define weak solutions (see Section 2 for more details). Note that some of our results can be extended to weaker notions of solutions, like the renormalized solutions introduced by DiPerna and Lions in [27], as we shall see later.

Also, let us point out that assumptions over σ in Theorem 1.1 can be translated into assumptions over γ, if needed. We remark that our stationary states are obtained as minimizers of entropy functionals, thus hypotheses over σ are more natural.

Our second main result is a stability result in L^2, which can be written as follows in the case of maxwellian stationary solutions.

THEOREM 1.2. *Under the same assumptions as in Theorem 1.1, except that we assume now $p_0 = 2$ and $\sigma(s) = s\log s - s$, there exists a convex functional $\mathcal{F}$ reaching its minimum at $f = f_\infty$ such that any weak solution to (1.1)-(1.4) satisfies*

$$\|f(t,\cdot) - f_\infty\|_{L^2}^2 \le \mathcal{F}[f_0] \, .$$

With the notations of Theorem 1.1, $p = 1$, $\gamma(s) = e^{-s}$ and (f_∞, ϕ_∞) is given by $f_\infty(x,v) = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}} \rho_\infty(x)$ with $-\Delta\phi_\infty = \rho_\infty = \|f_0\|_{L^1} \frac{e^{-(\phi_\infty+\phi_e)}}{\int e^{-(\phi_\infty+\phi_e)} \, dx}$. More general statements will be given in the rest of the paper.

Theorem 1.1 is based on a somehow canonical method to relate entropies and special stationary solutions, at least for $p = 1$ or $p = 2$. Here we get an L^p-nonlinear stability result, $1 \le p \le 2$, for a whole family of stationary solutions. It is also possible to take advantage of the uniform boundedness of the stationary solution to introduce new possible choices of the entropy functional and get stability results in L^q with $q \ne p$: for instance $q = 2$ and $p = 1$ in Theorem 1.2. Note that Theorem 1.2 provides an L^2-stability result for the maxwellian stationary solutions, which is not included in Theorem 1.1 (see Section 4).

Similar ideas have been used previously in various contexts: for gravitational systems (without confinement) in [42, 44, 30, 31, 32] using the Casimir-energy method, and for systems in bounded domains in [6, 7], using entropy fluxes involving Darrozès & Guiraud type estimates. For confinement, we shall refer to [26], and also to [11, 24, 10] in case of models with a Fokker-Planck term. Entropy methods have recently been adapted to nonlinear diffusions: see for instance [2] in the linear case and [13, 14, 20, 39, 23, 22] in the nonlinear case, with applications to models where a Poisson coupling is involved [2, 8, 9] (also see references therein for earlier works). The estimates of Csiszár-Kullback type are indeed exactly the same in kinetic and parabolic frameworks.

In the electrostatic case of the Vlasov-Poisson system, the most relevant reference for our paper is [12] (also see [4, 5, 29] for earlier results in plasma physics). In [12], Braasch, Rein and Vukadinović consider compactly supported classical solutions to the Cauchy problem and stationary solutions which are compactly supported in the energy variable and depending on additional invariants of the particle motion. The scope of our paper is to extend their approach to general weak solutions and to emphasize the interplay of the regularity of the initial data and the various possible functionals and norms. We improve and complement results in [12] in several ways: we generalize stationary states, in two directions: we allow them to be not compactly supported in energy variable (maxwellian stationary states), and the dependence on energy and on other invariants of motion includes not factorized states (see Section 6 for details). Theorems 1.1 and 1.2 are valid for either weak or renormalized solutions (see below for details). And finally, we obtain stability bounds in L^q spaces $1 \le q \le 2$ (while in [12] only for $q = 2$).

We are going to work in the framework of weak [34, 36] or renormalized solutions [27, 38], which of course contains the case of classical solutions. As we shall see below, there is a natural class of stationary solutions and L^p norms with respect to which the stability can be studied, but we will also consider other L^q norms. For instance, Maxwellian steady states are known to be asymptotically stable in $L^1(\mathbb{R}^6)$ for the Vlasov-Poisson-Fokker-Planck (VPFP) system [11, 10, 26, 24]. It turns out that they are stable for the Vlasov-Poisson system, in L^1 of course, but also in other norms. This question initially motivated our study and has been used to extend [12] (Theorem 1.2).

This paper is organized as follows. We start our discussion by doing an overview of the definitions and properties of the solutions to the Vlasov-Poisson system. We also introduce in Section 2 the family of stationary solutions we are dealing with and some of their properties. Section 3 contains the proof of a generalized version of Theorem 1.1. Theorem 1.2 is proved in Section 4. In Section 5, we establish some relations among various nonlinear stability results and generalize Theorem 1.2. Finally, in Section 6, we consider more general steady states depending on additional invariants, for which we prove an extension of Theorem 1.2.

2. Notions of solution and stationary solutions.

2.1. Weak and renormalized solutions to the Cauchy problem. A *classical solution* [41, 43, 33, 28] is a solution to the Cauchy problem (1.1)-(1.4) for which the derivatives hold in the classical sense and the force term F satisfies a Lipschitz condition. Our approach applies to weaker notions of solutions. By *weak solution* [3, 34, 36], we mean a solution in the distributional sense, for which the force field F is not smooth enough to apply the classical characteristics theory (see below for a pre-

cise definition). Essentially, we are going to use the framework of *weak solutions* $(\mathcal{W})$ of Hörst & Hunze [34], and as a special case, the one of Lions & Perthame [36] for which further interpolations identities are available. These last solutions are sometimes called *strong solution* [40] and we shall denote them by $(\mathcal{S})$. For solutions corresponding to initial data with very low regularity, we shall use the *renormalized solutions* $(\mathcal{R})$ of DiPerna & Lions [27, 38].

Before making precise these notions of solution, let us introduce some notations and a basic hypothesis on the initial data. We shall refer to the Cauchy problem for the Vlasov-Poisson system with initial data f_0 as the *initial value problem* (1.1)-(1.4). We assume

(H1): f_0 *is a nonnegative function in* $L^1(\mathbb{R}^6)$

and denote by $M := \|f_0\|_{L^1}$ its mass. Let ϕ_0 be the solution to the Poisson equation at $t = 0$, corresponding to $f = f_0$ in (1.3).

Throughout this paper, we consider global in time solutions: $\mathbb{R}_0^+ = [0, \infty)$ is the time interval. As a preliminary step, we can state the following result (see the Appendix for a proof).

PROPOSITION 2.1. *For any nonnegative function* f_0 *in* $L^1(\mathbb{R}^6)$, *there exists a nonnegative strictly convex function* σ *such that* $\lim_{s \to +\infty} \sigma(s)/s = +\infty$ *and* $\sigma(f_0) \in L^1(\mathbb{R}^6)$.

To obtain stability results, we are going to impose further constraints on σ, which will be strongly related to the choice of the entropy or to the choice of a special stationary solution. However, we first have to define a precise notion of solution.

DEFINITION 2.2. *Let* $p \in [1, \infty]$. *A function* $f \in L^\infty(\mathbb{R}_0^+, L^p(\mathbb{R}^6))$ *is a global weak solution of (1.1)-(1.4) with initial data* f_0 *if and only if:*
1. f is continuous on $\mathbb{R}_0^+$ with values in $L^s(\mathbb{R}^6)$, where $s \in [1, p)$ ($s = 1$ if $p = 1$), with respect to the $\sigma(L^p, L^{p'})$ topology (weak topology for $p < \infty$ and weak∗ topology for $p = \infty$). Here p and p' are the Hölder conjugates.
2. $f(0, \cdot) = f_0$.
3. The function $(x, v) \mapsto f(t, x, v)F(t, x)$ is locally integrable over $\mathbb{R}^6$ for all $t \geq 0$ (since $f(t) \in L^1(\mathbb{R}^6)$ for any fixed t, $F(t, \cdot)$ is defined almost everywhere on $\mathbb{R}^3$ and is locally integrable).
4. For all test functions $\chi \in C_c^{\,1}(\mathbb{R}^6)$, the function $\varrho(t) := \int \chi(x, v)f(t, x, v)\, d(x, v)$ is continuously differentiable on $\mathbb{R}_0^+$ and

$$\varrho'(t) = \int v \cdot \nabla_x \chi(x, v)\, f(t, x, v)\, d(x, v) + \int F(t, x) \cdot \nabla_v \chi(x, v)\, f(t, x, v)\, d(x, v)\ .$$

Note that a weak solution for $p > 1$ is a weak solution for all $q \in [1, p]$. According to Hörst & Hunze [34], such *weak solutions* exist in case $\phi_e \equiv 0$ globally in time if we assume that f_0 satisfies
$(\mathcal{W})$ $f_0 \geq 0$, $f_0 \in L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6)$, $p \geq p_0 = (12 + 3\sqrt{5})/11 = 1.70075...$ and

$$\int_{\mathbb{R}^6} \Big(|v|^2 + \phi_e(x) \Big) f_0(x, v)\, d(x, v) < \infty\ .$$

We shall also consider the subcase of the so-called *strong solutions* of Lions & Perthame [36]:

($\mathcal{S}$) $f_0 \geq 0$, $f_0 \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$, and for some $m > 3$,

$$\int_{\mathbb{R}^6} \Big(|v|^m + \phi_e(x) \Big) f_0(x,v) \, d(x,v) < \infty \, .$$

REMARK 2.3. *In case* ($\mathcal{W}$), $\nabla \phi_0 \in L^2(\mathbb{R}^3)^3$ *[34] as a consequence of the interpolation inequality:* $\|\rho\|_{L^q} \leq C \|f\|_{L^p}^\theta \| |v|^2 f \|_{L^1}^{1-\theta}$ *with* $q = \frac{5p-3}{3p-1}$, $\theta \in (0,1)$, *and of the Hardy-Littlewood-Sobolev inequality:* $\|\nabla \phi\|_{L^r} \leq C \|\rho\|_{L^q}$ *with* $\frac{1}{q} - \frac{1}{r} = \frac{1}{3}$. *The case* $p = p_0$ *is obtained by imposing* $r = p'$.

Without assumptions on the initial energy, it is still possible to give global existence results [15, 16]. Also note that if ($\mathcal{W}$) *is satisfied,* $f_0 \log f_0 \in L^1(\mathbb{R}^6)$, *as we shall see in Section 4, provided* $e^{-\beta \phi_e} \in L^1(\mathbb{R}^3)$ *for some* $\beta > 0$.

In this paper, we will also consider weaker notions of solutions.

DEFINITION 2.4. *Assume that*
($\mathcal{R}$) f_0 *is a nonnegative function in* $L^1(\mathbb{R}^6)$ *such that* $f_0 \log f_0 \in L^1(\mathbb{R}^6)$ *and*

$$\int_{\mathbb{R}^6} \Big(\frac{1}{2} |v|^2 + \phi_e(x) \Big) f_0(x,v) \, d(x,v) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi_0|^2 \, dx < \infty \, .$$

We shall say that $f \in C^0(\mathbb{R}_0^+, L^1(\mathbb{R}^6))$ *is a renormalized solution of (1.1)-(1.4) on* $\mathbb{R}_0^+$ *with initial data* f_0 *if and only if*
1. *The quantities*

$$\int_{\mathbb{R}^6} \Big(\frac{1}{2} |v|^2 + \phi_e(x) + \phi(x,t) \Big) f(x,v,t) \, d(x,v) \quad and \quad \int_{\mathbb{R}^6} f(x,v,t) \log f(x,v,t) \, d(x,v)$$

are bounded from above, uniformly in $t \geq 0$.
2. $\beta(f) = \log(1+f)$ *is a weak solution of*

$$\frac{\partial}{\partial t} \beta(f) + v \cdot \nabla_x \beta(f) + F(t,x) \cdot \nabla_v \beta(f) = 0$$

considered in the distributional sense, where F *is defined according to (1.2) and (1.3).*

In case $e^{-\beta \phi_e} \in L^1(\mathbb{R}^3)$ for some $\beta > 0$, weak solutions for $p > 1$ are also renormalized solutions (see Lemma 4.1).

PROPOSITION 2.5. *Let* f_0 *verify* ($\mathcal{R}$) *and assume that* ϕ_e *is a nonnegative potential such that:* $\lim_{|x| \to +\infty} \phi_e(x) = +\infty$. *If* ϕ_e *is in* $W^{1,1}_{loc}(\mathbb{R}^3)$, *(1.1)-(1.4) admits a global in time renormalized solution. If moreover* ϕ_e *belongs to* $W^{1,q}_{loc}$ *for* $q \geq \frac{5p-3}{2(p-1)}$ *and if* ($\mathcal{W}$) *holds, then (1.1)-(1.4) admits a weak solution.*

Proof. This result can be obtained by adapting the proofs of [34, 36, 27, 38]. For renormalized solutions, characteristics can be defined according to [25, 35] as soon as ϕ_e is in $W^{1,1}_{loc}(\mathbb{R}^3)$. Details are left to the reader. □

Weak or renormalized solutions have the following properties:
1. The distribution function is nonnegative for all $t \geq 0$.

2. Conservation of mass: for any $t \geq 0$,

$$\int_{\mathbb{R}^6} f(t,x,v) \, d(x,v) \; = \; \int_{\mathbb{R}^6} f_0(x,v) \, d(x,v) \; = \; M \; .$$

3. Finite kinetic energy, potential energy and entropy: for any $t \geq 0$,

$$\int_{\mathbb{R}^6} \left(\tfrac{1}{2}|v|^2 + \phi_e(x) + \phi(x) \right) f \, d(x,v) \leq \int_{\mathbb{R}^6} \left(\tfrac{1}{2}|v|^2 + \phi_e(x) + \phi_0(x) \right) f_0 \, d(x,v)$$

and $\int_{\mathbb{R}^6} f \, \log f \, d(x,v) \leq \int_{\mathbb{R}^6} f_0 \, \log f_0 \, d(x,v) \; ,$

with equality in the case of classical solutions (see Corollary 2.8 for an application).
4. In case $(\mathcal{S})$, for any $t \geq 0$,

$$\|f(t,\cdot)\|_{L^\infty(\mathbb{R}^6)} \leq \|f_0\|_{L^\infty(\mathbb{R}^6)} \; .$$

5. Moreover, if we assume that

$$\text{(H2):} \qquad \int_{\mathbb{R}^6} \sigma(f_0) \, d(x,v) < \infty$$

for some strictly convex continuous function $\sigma : \mathbb{R}_0^+ \to \mathbb{R}$, then for any $t \geq 0$,

$$\int_{\mathbb{R}^6} \sigma(f) \, d(x,v) \; \leq \; \int_{\mathbb{R}^6} \sigma(f_0) \, d(x,v) \; ,$$

with equality in the case of classical solutions (see Corollary 2.8 for an application).

2.2. Stationary solutions and entropy functionals. Let us introduce further notations. For any function $f \in L^1(\mathbb{R}^6)$, let $\phi = \phi[f]$ be the solution of $-\Delta\phi = \int_{\mathbb{R}^3} f \, dv$ in $L^{3,\infty}(\mathbb{R}^3)$ given by the convolution with the Green function of the Laplacian. The operator ϕ is linear and satisfies:

$$\int_{\mathbb{R}^6} f \, \phi[g] \, d(x,v) = \int_{\mathbb{R}^6} g \, \phi[f] \, d(x,v) \; .$$

Any function $f_{\infty,\sigma}$ such that

$$(2.1) \qquad f_{\infty,\sigma}(x,v) = \gamma \left(\frac{1}{2}|v|^2 + \phi[f_{\infty,\sigma}](x) + \phi_e(x) - \alpha \right)$$

is a stationary solution of the Vlasov-Poisson system. Such a solution exists if and only if

$$-\Delta\phi_{\infty,\sigma} = G_\sigma(\phi_{\infty,\sigma} + \phi_e - \alpha) \quad \text{with} \quad G_\sigma(\phi) = 4\pi\sqrt{2} \int_0^{+\infty} \sqrt{s} \, \gamma(s+\phi) \, ds$$

has a solution $\phi_{\infty,\sigma} = \phi[f_{\infty,\sigma}]$ such that $\int_{\mathbb{R}^6} f_{\infty,\sigma} \, d(x,v) = M$. The constant α is therefore determined by the total mass M. Under assumptions that we are going to specify now, we will prove that such a stationary exists and is unique (see Lemma 2.7).

Let us consider σ such that γ is the generalized inverse of $-\sigma'$ (eventually extended by 0): σ is convex (resp. strictly convex) if and only if γ is monotone nonincreasing (resp. decreasing in its support). With these notations, we assume that σ and ϕ_e verify:

(H3): $\quad\sigma \in C^2(\mathbb{R}^+) \cap C^0(\mathbb{R}_0^+)$ *is a bounded from below strictly convex function such that*

$$\lim_{s \to +\infty} \frac{\sigma(s)}{s} = +\infty .$$

(H4): $\quad\quad \phi_e : \mathbb{R}^3 \to \mathbb{R}$ *is a measurable bounded from below function such that*

$$\lim_{|x| \to +\infty} \phi_e(x) = +\infty$$

and $x \mapsto G_\sigma(\phi_e(x)) = 4\pi\sqrt{2} \int_0^{+\infty} \sqrt{s}\, \gamma(s + \phi_e(x))\, ds$ *belongs to* $L^1 \cap L^\infty(\mathbb{R}^3)$.

The conditions on the growth of ϕ_e and on the decay of γ will be refered as *confinement conditions*. We are going to adapt the proofs given in [26] for the case $\gamma(s) = e^{-s}$ and in [6, 7] for the bounded domain case to prove the existence of a stationary solution $f_{\infty,\sigma}$. The existence of $\alpha = \alpha(M)$ will be a consequence of the proof.

Let $M > 0$ and consider on $L_M^1(\mathbb{R}^6) = \{f \in L^1(\mathbb{R}^6) \ : \ f \geq 0 \text{ a.e.}, \ \|f\|_{L^1} = M\}$ the functional

$$K_\sigma[f] = \int_{\mathbb{R}^6} \left[\sigma(f) + \left(\frac{1}{2}|v|^2 + \phi_e(x) \right) f \right] d(x, v) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\phi[f]|^2 \, dx .$$

DEFINITION 2.6. *Given f and g in $L_M^1(\mathbb{R}^6)$, the relative entropy of f with respect to g is*

$$(2.2) \qquad\qquad \Sigma_\sigma[f|g] := K_\sigma[f] - K_\sigma[g] .$$

LEMMA 2.7. *Under Assumptions (H3)-(H4), K_σ is a strictly convex bounded from below functional on $L_M^1(\mathbb{R}^6)$. It has a unique global minimum, $f_{\infty,\sigma}$, which takes the form (2.1) and is therefore a stationary solution of the Vlasov-Poisson system. Moreover $\Sigma_\sigma[f|f_{\infty,\sigma}]$ can be written as*

$$(2.3) \qquad \Sigma_\sigma[f|f_{\infty,\sigma}] = \int_{\mathbb{R}^6} [\sigma(f) - \sigma(f_{\infty,\sigma}) - \sigma'(f_{\infty,\sigma})(f - f_{\infty,\sigma})] \, d(x, v)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x(\phi - \phi_{\infty,\sigma})|^2 \, dx$$

and $\sigma(f_{\infty,\sigma})$ and $\sigma'(f_{\infty,\sigma})f_{\infty,\sigma}$ belong to $L^1(\mathbb{R}^6)$.

Proof. Assumption (H4) gives that $K_\sigma[f]$ is bounded from below by Jensen's inequality. By Hypothesis (H3) K_σ is convex, so we may pass to the limit in a minimizing sequence involving the semi-continuity property. The limit $f_{\infty,\sigma}$ belongs to $L_M^1(\mathbb{R}^6)$ because of Dunford-Pettis' criterion. Equation (2.1) is the corresponding Euler-Lagrange (where α enters as the Lagrange multiplier associated to the constraint on the L^1 norm). Identity (2.3) easily follows by a direct computation, using (2.1). $\square$

Note that $\Sigma_\sigma[f|f_{\infty,\sigma}]$ is obviously nonnegative, since $K_\sigma[f]$ attains its unique minimum at $f = f_{\infty,\sigma}$.

COROLLARY 2.8. *Consider a renormalized or weak solution f of (1.1)-(1.4) under Assumptions* (H1), (H2), (H3) *and* (H4). *Then* $\Sigma_\sigma[f(t)|f_{\infty,\sigma}] \leq \Sigma_\sigma[f_0|f_{\infty,\sigma}]$.

The proof relies on standard semi-continuity arguments and is left to the reader.

EXAMPLE 2.9. *1) Let $\sigma_q(s) = s^q$, with $\gamma_q(s) = (-s/q)_+^{1/(q-1)}$, for some given $q > 1$. With the notations $f_{\infty,q} = f_{\infty,\sigma_q}$ and $\phi_{\infty,q} = \phi[f_{\infty,\sigma_q}]$, this stationary solution satisfies the nonlinear Poisson equation*

$$-\Delta\phi_{\infty,q} = C_q \left(\alpha(M) - \phi_e - \phi_{\infty,q}\right)_+^{\frac{3}{2}+\frac{1}{q-1}}$$

where $C_q = (2\pi)^{3/2}\, q^{-\frac{1}{q-1}}\, \Gamma(\frac{q}{q-1})/\Gamma(\frac{5q-3}{2(q-1)})$.
2) The limit case as $q \to 1$ corresponds to $\sigma_1(s) = s\log s - s$ and $\gamma_1(s) = e^{-s}$. In this case we obtain the maxwellian stationary solution

$$(2.4) \qquad f_{\infty,1}(x,v) \;=\; m(x,v) \;=\; M\, \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{3/2}}\, \frac{e^{-(\phi_{\infty,1}(x)+\phi_e(x))}}{\int_{\mathbb{R}^3} e^{-(\phi_{\infty,1}(x)+\phi_e(x))}\, dx}\,,$$

where $\phi_{\infty,1}$ is given by the Poisson-Boltzmann equation

$$(2.5) \qquad -\Delta_x\phi_{\infty,1} \;=\; \int_{\mathbb{R}^3} m(x,v)\, dv \;=\; M\, \frac{e^{-(\phi_{\infty,1}+\phi_e)}}{\int_{\mathbb{R}^3} e^{-(\phi_{\infty,1}+\phi_e)}\, dx}\,.$$

3) A less standard case is given by

$$\sigma(t) = \begin{cases} 2\int_1^{\sqrt{-\log t}} s^2 e^{-s^2}\, ds & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$$

which corresponds to: $\gamma(t) = e^{-t^2}$.

In next sections, the various cases of this example will be analyzed. They will motivate a more general treatment. For simplicity, we shall write $\Sigma_q[f|f_{\infty,q}]$ instead of $\Sigma_{\sigma_q}[f|f_{\infty,\sigma_q}]$, for $q \geq 1$.

3. L^p-nonlinear stability. In this section, we give a L^p-nonlinear stability result for $f_{\infty,\sigma}$, $1 \leq p \leq 2$, with minimal convexity assumptions on the initial data and an explicit stability constant. It is based on the following result.

PROPOSITION 3.1. *Let f and g be two nonnegative functions in $L^1(\mathbb{R}^6)\cap L^p(\mathbb{R}^6)$, $p \in [1,2]$ and consider a strictly convex function $\sigma : \mathbb{R}_0^+ \to \mathbb{R}$ in $C^2(\mathbb{R}^+)\cap C^0(\mathbb{R}_0^+)$. Let $A = \inf\{\sigma''(s)/s^{p-2}:\ s \in (0,\infty)\}$. If $A > 0$, then the following inequality holds:*

$$(3.1) \qquad \Sigma_\sigma[f|g] \geq 2^{-2/p}A\left[\max\left(\|f\|_{L^p}^{2-p}, \|g\|_{L^p}^{2-p}\right)\right]^{-1}\|f-g\|_{L^p}^2$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x(\phi[f]-\phi[g])|^2\, dx\,.$$

Proof. The case $p = 1$ is the well known Csiszár-Kullback inequality (see for instance [1]) that we are going to adapt to the case $p \geq 1$.

Assume first that $f > 0$. By a Taylor development at order two of σ we deduce that we can write the relative entropy for f and g as

$$\Sigma_\sigma[f|g] = \frac{1}{2} \int_{\mathbb{R}^6} \sigma''(\xi)|f-g|^2 \, d(x,v) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x(\phi[f] - \phi[g])|^2 \, dx$$

$$(3.2) \qquad \geq \frac{A}{2} \int_{\mathbb{R}^6} \xi^{p-2}|f-g|^2 \, d(x,v) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x(\phi[f] - \phi[g])|^2 \, dx$$

where ξ lies between f and g. If $p = 2$, the result is obvious. Let $1 \leq p < 2$. By Hölder's inequality, for any $h > 0$ and for any measurable set $\mathcal{A} \subset \mathbb{R}^6$, we get

$$\int_{\mathcal{A}} |f-g|^p \, h^{-\alpha} \, h^\alpha \, d(x,v) \leq \left(\int_{\mathcal{A}} |f-g|^2 \, h^{p-2} \, d(x,v) \right)^{p/2} \left(\int_{\mathcal{A}} h^{\alpha s} \, d(x,v) \right)^{1/s}$$

with $\alpha = p\,(2-p)/2$, $s = 2/(2-p)$. Thus

$$\left(\int_{\mathcal{A}} |f-g|^2 \, h^{p-2} \, d(x,v) \right)^{p/2} \geq \left(\int_{\mathcal{A}} |f-g|^p \, d(x,v) \right) \left(\int_{\mathcal{A}} h^p \, d(x,v) \right)^{(p-2)/2}.$$

We apply this formula to two different sets.

i) On $\mathcal{A} = \mathcal{A}_1 = \{(x,v) \in \mathbb{R}^6 : f(x,v) > g(x,v)\}$, use $\xi^{p-2} > f^{p-2}$ and take $h = f$:

$$\left(\int_{\mathcal{A}_1} |f-g|^2 \xi^{p-2} \, d(x,v) \right)^{p/2} \geq \left(\int_{\mathcal{A}_1} |f-g|^p \, d(x,v) \right) \|f\|_{L^p}^{-(2-p)\,p/2}.$$

ii) On $\mathcal{A} = \mathcal{A}_2 = \{(x,v) \in \mathbb{R}^6 : f(x,v) \leq g(x,v)\}$, use $\xi^{p-2} \geq g^{p-2}$ and take $h = g$:

$$\left(\int_{\mathcal{A}_2} |f-g|^2 \xi^{p-2} \, d(x,v) \right)^{p/2} \geq \left(\int_{\mathcal{A}_2} |f-g|^p \, d(x,v) \right) \|g\|_{L^p}^{-(2-p)\,p/2}.$$

To prove (3.2) in the case $f > 0$, we just add the two previous inequalities in (3.2) and use the inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for any $a, b \geq 0$ and $r \geq 1$. To handle the case $f \geq 0$, we proceed by a density argument: apply (3.2) to $f_\epsilon(x,v) = f(x,v) + \epsilon\, e^{-|x|^2 - |v|^2}$ and let $\epsilon \to 0$ using Lebesgue's convergence theorem. $\qquad \square$

This proposition can be applied to weak or renormalized solutions, thus proving the first main result of this paper, which is a more detailed version of Theorem 1.1.

THEOREM 3.2. *Let f_0 verify* (H1), (H2) *and either* ($\mathcal{R}$) *or* ($\mathcal{W}$). *Assume* (H3) *and* (H4). *If f is a weak or renormalized solution of (1.1)-(1.4) with initial value f_0, then*

$$\|\nabla\phi - \nabla\phi_{\infty,\sigma}\|_{L^2}^2 \quad \leq \quad 2\,\Sigma_\sigma[f_0|f_{\infty,\sigma}]$$

Assume that $A = \inf\left\{ \sigma''(s)/s^{p-2} : s \in (0,\infty) \right\}$ is positive for some $p \in [1,2]$. If $p = 1$, assume moreover that $e^{-\phi_e} \in L^1$. Then $f_0 \in L^p(\mathbb{R}^6)$ and

$$\|f(t) - f_{\infty,\sigma}\|_{L^p}^2 \leq C(f_0, \sigma)\, \Sigma_\sigma[f_0|f_{\infty,\sigma}]$$

for any $t \geq 0$, where $C(f_0, \sigma)$ is a constant, which takes the explicit form

$$C(f_0, \sigma) = \begin{cases} \dfrac{2^{2/p}}{A} \max\left(\|f_0\|_{L^p}^{2-p}, \|f_{\infty,\sigma}\|_{L^p}^{2-p} \right) & \text{if } p > 1 \\[2mm] \dfrac{4}{A} M & \text{if } p = 1 \end{cases}$$

In case $(\mathcal{S})$, $C(f_0, \sigma)$ *is also bounded by* $\frac{2^{2/p}}{A} M^{(2-p)/p} \mathcal{M}^{(2-p)(p-1)/p}$ *with* $\mathcal{M} = \max(\|f_0\|_{L^\infty}, \|f_{\infty,\sigma}\|_{L^\infty})$.

Proof. The proof is a straightforward consequence of Lemma 2.7, Corollary 2.8 and Proposition 3.1 once it is known that $C(f_0, \sigma)$ is finite. Although we directly prove an estimate of $\|f(t) - f_{\infty,\sigma}\|_{L^p}^2$ in terms of $\Sigma_\sigma[f_0|f_{\infty,\sigma}]$, we may notice that, for $p > 1$, two integrations give the inequality

$$\sigma(s) - \sigma(s_0) - \sigma'(s_0)(s - s_0) \geq \frac{A}{p(p-1)}\left[s^p - s_0^p - p\,s_0^{p-1}(s - s_0)\right]$$

for any $(s, s_0) \in (0 + \infty)^2$. Applied to f and $f_{\infty,\sigma}$, this means that on $\mathbb{R}^6$

$$(3.3)\ \ \sigma(f) - \sigma(f_{\infty,\sigma}) - \sigma'(f_{\infty,\sigma})(f - f_{\infty,\sigma}) \geq \frac{A}{p(p-1)}\left[f^p - f_{\infty,\sigma}^p - p\,f_{\infty,\sigma}^{p-1}(f - f_{\infty,\sigma})\right],$$

which proves that f belongs to $L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^6))$ (by $\|f_0\|_{L^p}$, according to Corollary 2.8 applied with $\sigma(s) = \sigma_p(s) = s^p$). The constant $C(f_0, \sigma)$ involves $\|f_0\|_{L^p}$, which is therefore itself bounded in terms of $\sigma(f_0)$ and $f_0\,\sigma'(f_0)$.

If $p = 1$, the condition that $e^{-\phi_e} \in L^1$ shows that $f_{\infty,\sigma}$ also belongs to L^1. In that case, inequality (3.3) is replaced by

$$\sigma(f) - \sigma(f_{\infty,\sigma}) - \sigma'(f_{\infty,\sigma})(f - f_{\infty,\sigma}) \geq A\left[f \log\left(\frac{f}{f_{\infty,\sigma}}\right) - (f - f_{\infty,\sigma})\right].$$

The details of the proof are left to the reader. □

REMARK 3.3. *Note that* $A = p(p-1)$ *if* $\sigma = \sigma_p$, $p > 1$, *and* $A = 1$ *if* $p = 1$ *and* $C(f_0, \sigma_2) = 1$. *The expression of* $C(f_0, \sigma)$ *is optimal at least for* $\sigma = \sigma_p$ *in the limit* $\|f_0 - f_{\infty,\sigma}\|_{L^p} \to 0$ *(see [1] for a discussion in the case* $p = 1$*).*

For $p > 2$, *Hölder's inequality holds in the reverse sense:* $\|f(t) - f_{\infty,\sigma}\|_{L^p}^2 + \|\nabla\phi - \nabla\phi_{\infty,\sigma}\|_{L^2}^2$ *controls* $\Sigma_\sigma[f_0|\sigma]$.

For $p = 1$, *we recover the classical Csiszár-Kullback inequality in Proposition 3.1 and a stability result in* L^1 *(see [1, 2]) which is natural in the framework of renormalized solutions (if* $f \log f$ *belongs to* L^1*: see Lemma 4.1 below).*

4. L^2-Nonlinear stability of maxwellian steady states. In [12], Braasch, Rein and Vukadinović introduce modified Lyapunov functionals for proving L^2-stability for certain steady states (see Section 5 for more details). In this section, we shall extend this approach to the maxwellian case. The main idea is the following. Although $\sigma''(s) = 1/s$ is not bounded from below uniformly away from 0 (which would be the condition to apply directly Proposition 3.1 in L^2), since $f_{\infty,1}$ is bounded in L^∞ by a constant $\bar{s}$, it is sufficient to consider the infimum of σ'' in $(0, \bar{s})$.

In the maxwellian case, we first notice that (H2) follows from the other assumptions.

LEMMA 4.1. *Assume that* $e^{-\beta\phi_e}$ *belongs to* $L^1(\mathbb{R}^3)$ *for some* $\beta > 0$. *Let* f *be a nonnegative function in* $L^1 \cap L^q(\mathbb{R}^6)$, $q > 1$, *such that* $(x, v) \mapsto (|v|^2 + \phi_e(x))f(x, v) \in L^1(\mathbb{R}^6)$. *Then* $f \log f$ *belongs to* $L^1(\mathbb{R}^6)$.

Proof. Depending on the sign of $\log f$, we are going to consider two cases.

1) Define $g(x, v) = e^{-\frac{\beta}{2}|v|^2 - \beta \phi_e(x)}$. On $\mathcal{A} = \{(x, v) \in \mathbb{R}^6 \ : \ f(x, v) < 1\}$, using Jensen's inequality, we get

$$0 \geq \int_{\mathcal{A}} \left[f \log f + \beta \left(\tfrac{1}{2}|v|^2 + \phi_e \right) f \right] d(x, v) = \int_{\mathcal{A}} f \log \left(\tfrac{f}{g} \right) d(x, v)$$
$$\geq \|f\|_{L^1(\mathcal{A})} \log \left(\frac{\|f\|_{L^1(\mathcal{A})}}{\|g\|_{L^1(\mathcal{A})}} \right) .$$

2) On $\mathbb{R}^6 \setminus \mathcal{A}$, we conclude using the next lemma. $\qquad \square$

LEMMA 4.2. *Let f be a nonnegative function in $L^1 \cap L^q(\Omega)$, $q > 1$, for some arbitrary domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Then*

$$\int_\Omega f(z) \log f(z) \, dz \leq \frac{1}{q-1} \|f\|_{L^1(\Omega)} \log \left(\frac{\|f\|_{L^q(\Omega)}^q}{\|f\|_{L^1(\Omega)}} \right) .$$

Proof. According to Hölder's inequality,

$$\|f\|_{L^r}^r \leq \|f\|_{L^1}^{\frac{q-r}{q-1}} \|f\|_{L^q}^{\frac{q(r-1)}{q-1}} ,$$

for $1 \leq r \leq q$. At $r = 1$, this is an equality, and thus, we may derive the inequality with respect to r at $r = 1$. $\qquad \square$

Let ϕ_e and f_0 verify respectively (H4) for $\sigma_1(s) = s \log s - s$, and (H1), $(\mathcal{W})$. Consider a weak or renormalized solution f of (1.1)-(1.4) with initial value f_0 and the corresponding stationary solution $f_{\infty,1} = m$ given by (2.4)-(2.5). According to Theorem 3.2, m is L^1-stable:

$$\Sigma_1[f|m] \geq \frac{1}{4M} \|f - m\|_{L^1}^2 .$$

We shall now prove a L^2-stability result for m using an appropriate cut-off functional as in [12]. Let $E_1(x, v) := \frac{1}{2}|v|^2 + \phi_{\infty,1}(x) + \phi_e(x)$. According to (H4),

$$E_{min} := \inf\{E_1(x, v) \ : \ (x, v) \in \mathbb{R}^6\} \geq \inf\{\phi_e(x) \ : \ x \in \mathbb{R}^3\} > -\infty .$$

Denote $m = \varphi \circ E_1$ with $\varphi(s) = \kappa \, e^{-s}$ where

$$(4.1) \qquad \kappa = \frac{M}{(2\pi)^{3/2}} \left[\int e^{-\phi_{1,\infty} - \phi_e} \, dx \right]^{-1} .$$

Consider $\bar{s} = \varphi(E_{min})$ and define

$$\tau_1(s) := \begin{cases} s \log s - s & \text{if } s \in [0, \bar{s}] \\ \frac{1}{2\kappa} e^{E_{min}} (s - \bar{s})^2 - (E_{min} - \log \kappa)(s - \bar{s}) + \bar{s} \log \bar{s} - \bar{s} & \text{if } s \in (\bar{s}, +\infty) \end{cases}$$

The function τ_1 is of class $C([0, \infty)) \cap C^2((0, \infty))$, with $\min(\tau_1'') = e^{E_{min}}/\kappa > 0$. Since $0 \leq m(x, v) \leq \varphi(E_{min}) = \bar{s}$ for any $(x, v) \in \mathbb{R}^6$ and φ is decreasing, m is a minimizer of the modified free energy (or Casimir) functional $\Sigma_{\tau_1}[f|m] = K_{\tau_1}[f] - K_{\tau_1}[m]$, where

$$K_{\tau_1}[f] = \int_{\mathbb{R}^6} \left(\frac{1}{2}|v|^2 + \frac{1}{2}\phi + \phi_e \right) f \, d(x, v) + \int_{\mathbb{R}^6} \tau_1(f) \, d(x, v)$$

and we can apply Theorem 3.2 with $p = 2$. This proves a refined version of Theorem 1.2. Since f belongs to L^2, $\tau_1(f)$ makes sense in L^1 according to Lemma 4.1. Let us remark that the construction of τ_1 is done in such a way that $K_{\tau_1}[m] = K_{\sigma_1}[m]$, and then Corollary 2.8 can be applied. In this framework, it is natural to work with weak rather than renormalized solutions.

THEOREM 4.3. *Assume* (H1), (H3), (H4) *for* $\sigma = \sigma_1$ *and* $(\mathcal{W})$ *for* $p = 2$. *Consider the stationary solution given by* (2.4)-(2.5). *With the above notations, every weak solution* f *of* (1.1)-(1.4) *with initial data* $f_0 \in L^1 \cap L^2(\mathbb{R}^6)$ *verifies*

$$\Sigma_{\tau_1}[f_0|m] \geq \Sigma_{\tau_1}[f(t)|m] \geq \frac{1}{2\,\bar{s}}\,\|f(t) - m\|_{L^2}^2 \quad \forall\, t \geq 0\,.$$

REMARK 4.4. 1) *A simpler version of Theorem 4.3 holds for solutions satisfying* $(\mathcal{S})$. *In this case, it is not necessary to modify* σ, *since* $\sigma_1''(s) = \frac{1}{s}$ *is bounded from below in* $(0, \max(\|f_0\|_{L^\infty}, \|m\|_{L^\infty})]$ *by* $\max(\|f_0\|_{L^\infty}, \|m\|_{L^\infty})^{-1}$.

2) *Theorem 4.3 can be generalized to any stationary solution* $f_{\infty,\sigma}$ *and any* L^q *norm with* $p \neq q \in (1,2)$: *see next section.*

3) *Note that in the maxwellian case the value of* κ *defined by* (4.1) *is* $e^{-\alpha(M)}$ *where* $\alpha = \alpha(M)$ *is the constant in* (2.1) *which is fixed by the mass constraint.*

5. General nonlinear stability results. In this section, we generalize to L^q, $1 \leq q \leq 2$, and to arbitrary steady states $f_{\infty,\sigma}$ the stability results of Sections 3-4. We are also going to generalize the techniques used in the L^2-stability result of Braasch, Rein and Vukadinović in [12], which can be summarized as follows. Let γ be a C^1 function on $\mathbb{R}$ such that $\gamma' < 0$ on $(-\infty, E_{\max})$ and $\gamma \equiv 0$ on $[E_{max}, +\infty)$ and define σ as a primitive of $-(\gamma^{-1})$, which is well defined at least on some subinterval in $\mathbb{R}^+$ (see for instance [14] for more details). Then $f_{\infty,\sigma}$ is a compactly supported steady state which is L^2-stable among weak or renormalized solutions of (1.1)-(1.4).

For $q > p$, the main idea is again to bound $\sigma''(s)/s^{q-2}$ from below only on the interval $(0, \bar{s} = \|f_{\infty,\sigma}\|_{L^\infty})$ and to modify σ on $(\bar{s}, +\infty)$. In this case, let us establish a useful consequence of Proposition 3.1. Let $E_\sigma(x,v) := \frac{1}{2}|v|^2 + \phi_{\infty,\sigma}(x) + \phi_e(x)$ and $E_{min} := \inf\{E_\sigma(x,v)\ :\ (x,v) \in \mathbb{R}^6\}$, which is finite by assumption (H4). With the notations of Sections 2–3, $f_{\infty,\sigma} = \gamma \circ (E_\sigma - \alpha)$, where α is such that $\|f_{\infty,\sigma}\|_{L^1} = M$. Take $\bar{s} = \gamma(E_{min} - \alpha)$ and define

$$\tau_\sigma(s) := \begin{cases} \sigma(s) & \text{if } s \in [0, \bar{s}] \\ \psi(s) & \text{if } s \in (\bar{s}, +\infty) \end{cases}$$

with $\psi(s) = \frac{\sigma''(\bar{s})}{\sigma_q''(\bar{s})}\sigma_q(s) + \left(\sigma'(\bar{s}) - \frac{\sigma''(\bar{s})}{\sigma_q''(\bar{s})}\sigma_q'(\bar{s})\right)(s - \bar{s}) + \sigma(\bar{s}) - \frac{\sigma''(\bar{s})}{\sigma_q''(\bar{s})}\sigma_q(\bar{s})$ and $\sigma_q(t) = t^q$. With the truncated Lyapunov functional $\Sigma_{\tau_\sigma}[f|f_{\infty,\sigma}] = K_{\tau_\sigma}[f] - K_{\tau_\sigma}[f_{\infty,\sigma}]$, we immediately get the following variant of Proposition 3.1.

COROLLARY 5.1. *Let* f *and* g *be two nonnegative functions in* $L^1(\mathbb{R}^6) \cap L^q(\mathbb{R}^6)$, $q \in [1,2]$ *and consider a strictly convex function* $\sigma : \mathbb{R}_0^+ \to \mathbb{R}$ *in* $C^2(\mathbb{R}^+) \cap C^0(\mathbb{R}_0^+)$. *With the above notations, let* $B = \inf\left\{\sigma''(s)/s^{q-2}\ :\ s \in (0, \bar{s})\right\}$. *If* $B > 0$, *then there exists a constant* $C > 0$ *such that*

$$\Sigma_{\tau_\sigma}[f|g] \geq C\,\|f - g\|_{L^q}^2 + \frac{1}{2}\|\nabla\phi - \nabla\phi_{\infty,\sigma}\|_{L^2}^2\,.$$

As in the case of Section 4, this estimate can be applied to get nonlinear stability results.

THEOREM 5.2. *Let f_0 verify* (H1), (H2) *and either* ($\mathcal{R}$) *or* ($\mathcal{W}$). *Assume that σ and ϕ_e satisfy* (H3) *and* (H4). *Assume that* $\inf \left\{ \sigma''(s)/s^{p-2} : s \in (0, \bar{s}) \right\}$ *is positive for some $p \in [1, 2]$, where $\bar{s}$ is defined as above. Then $f_{\infty,\sigma}$ is L^q-nonlinearly stable among weak or renormalized solutions of (1.1)-(1.4) for any $q \in (1, 2]$, provided $f_0 \in L^q(\mathbb{R}^6)$ if $q > p$.*

Proof. The case $q = p$ is covered by Theorem 3.2. In case $q > p$, the proof is an easy application of Corollary 5.1: $f_{\infty,\sigma}$ is L^q-stable in the sense that there exists a constant $C > 0$ such that for any $t \geq 0$,

$$\|f(t) - f_{\infty,\sigma}\|_{L^q} \leq C \, \Sigma_{\tau_\sigma}[f_0|f_{\infty,\sigma}] \ .$$

The case $1 < q < p$ relies on Hölder's inequality and Theorem 3.2:

$$\|f(t) - f_{\infty,\sigma}\|_{L^q} \leq (2M)^{\frac{p-q}{q\,(p-1)}} \left(C(f_0, \sigma) \, \Sigma_\sigma[f_0|f_{\infty,\sigma}] \right)^{\frac{p\,(q-1)}{2q\,(p-1)}} \ .$$

$\square$

The case $p = q = 1$ is covered by Theorem 3.2. Only the case $1 = q < p$ is left open. In the case $q > p$, notice that the L^q norm is bounded in terms of $\Sigma_{\tau_\sigma}[f_0|f_{\infty,\sigma}]$ and not in terms of $\Sigma_\sigma[f_0|f_{\infty,\sigma}]$ (as it is also the case in Theorem 4.3, with $p = 1$, $q = 2$).

6. Steady states depending on additional invariants. In the previous sections, we dealt with stationary solutions depending only on the energy. Our stability analysis can be extended to steady states which depend on additional invariants of the particle motion. To avoid lengthy statements, we shall only state the generalization of Theorem 4.3. In order to emphasize the connection with the previous results, we shall abusively use the same notations.

Consider the ODE system

$$\dot{X} = V \ , \quad \dot{V} = -\nabla_x \phi(t, X) - \nabla_x \phi_e(X)$$

which describes the characteristics of the Vlasov equation (1.1). We shall assume that either both ϕ and ϕ_e are locally Lipschitz (classical solutions), or at least in $W_{loc}^{1,1}$ (using the generalized characteritics of DiPerna & Lions, see [25, 35]). A function $I : \mathbb{R}^6 \to \mathbb{R}^m$, is an *invariant of the motion* if and only if

$$\frac{d}{dt} I(X(t), V(t)) = 0$$

in an appropriate sense. Classical examples of invariants are, for instance, the angular momentum $I(x, v) = x \times v$ in case of a central force motion (*i.e.* if $\phi + \phi_e$ is radially symmetric), its modulus, or one of its components: $I(x, v) \cdot \nu$, in the axisymmetric case with axis of direction $\nu \in S^2$, corresponding to a system invariant under rotations of axis ν. References on existence results of classical solutions with symmetries can be found in [28] (for stationary solutions, see [18]).

Consider stationary solutions in the form

(6.1) $$f_{\infty,\sigma}(x, v) = \mu \Big(E(x, v) - \alpha_M[\phi_{\infty,\sigma}, \phi_e, I], I(x, v) \Big)$$

where α_M is a constant to be determined by $\|f_{\infty,\sigma}\|_{L^1} = M$, E is the energy and I is an invariant of the motion. Note that E depends on $\phi_{\infty,\sigma} = \phi[f_{\infty,\sigma}]$. For simplicity, we suppose that I is a scalar quantity.

In [12], Braasch, Rein & Vukadinović consider the case where μ can be factorized as

$$\mu(E,I) = \gamma(E-\alpha)\,\nu(I) \quad \forall\,(E,I) \in \mathbb{R}^2 \ ,$$

where γ is compactly supported and $\alpha \in \mathbb{R}$. If γ satisfies (H3) and (H4) and if ν is a C^1 uniformly positive function, our previous results can easily be extended. In this section, we are going to consider general steady states corresponding to functions μ which cannot be factorized in terms of two functions γ and ν (such an extension has already been considered by Guo and Rein in [32] for gravitational systems) or which do not necessarily have a compact support in E.

In order to obtain the existence of these stationary solutions, we have to assume the following hypotheses on μ and ϕ_e, which are generalizations of (H3) and (H4) of Section 2.

(H3'): *Let* $\sigma : \ \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ *be such that* $\frac{\partial \sigma}{\partial s}(s,I) = -\mu^{-1}(s,I)$ *and assume that for any fixed* $I \in \mathbb{R}$, $\sigma(.,I)$ *has a* $C^0(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ *regularity, is bounded from below, strictly convex and such that* $\lim_{s\to+\infty} \sigma(s,I)/s = +\infty$. *Here* μ^{-1} *is the generalized inverse of* $s \mapsto \mu(s,I)$, *for fixed* I.

(H4'): *The external potential* $\phi_e : \mathbb{R}^3 \to \mathbb{R}$ *is a measurable bounded from below function such that* $\lim_{|x|\to+\infty} \phi_e(x) = +\infty$ *and*

$$x \mapsto \int_{\mathbb{R}^3} \mu\left(\frac{1}{2}|v|^2 + \phi_e(x), I(x,v)\right)\ dv$$

belongs to $L^1 \cap L^\infty(\mathbb{R}^3)$.

The stationary solution $f_{\infty,\sigma}$ is characterized as the unique nonnegative critical point of a strictly convex coercive functional K_σ, with

$$K_\sigma[f] = \int_{\mathbb{R}^6}\left[\sigma(f,I) + \left(\frac{1}{2}|v|^2 + \phi_e(x)\right)f\right]\ d(x,v) + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi[f]|^2\ dx \ ,$$

under the constraint $\int_{\mathbb{R}^6} f_{\infty,\sigma}\ d(x,v) = M$ for some given $M > 0$. As in Section 2, α_M in (6.1) is the Lagrange multiplier associated to the constraint on the masss and is uniquely determined by the condition $\int_{\mathbb{R}^6} f_{\infty,\sigma}\ d(x,v) = M$. To σ, we associate a relative entropy functional defined by

$$\begin{aligned}
\Sigma_\sigma[f|f_{\infty,\sigma}] &:= K_\sigma[f] - K_\sigma[f_{\infty,\sigma}] \\
&= \int_{\mathbb{R}^6}[\sigma(f,I) - \sigma_\infty - \frac{\partial\sigma_\infty}{\partial s}\,(f-f_{\infty,\sigma})]d(x,v) + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla_x(\phi[f]-\phi_{\infty,\sigma})|^2 dx
\end{aligned}$$

with $\sigma_\infty = \sigma(f_{\infty,\sigma},I)$ and $\phi_{\infty,\sigma} = \phi[f_{\infty,\sigma}]$.

If there exists a function $A_\sigma(I) > 0$ such that $\frac{\partial^2\sigma}{\partial s^2}(s,I) \geq A_\sigma(I)$ for any $(s,I) \in \mathbb{R}_0^+ \times \mathbb{R}$, by Taylor expansion it follows that

$$\Sigma_\sigma[f|f_{\infty,\sigma}] \geq \int_{\mathbb{R}^6} A_\sigma(I)|f-f_{\infty,\sigma}|^2\ d(x,v) \ ,$$

which proves a weighted L^2-stability result. Exactly as before, we can use a cut-off argument and get a generalization of Theorem 4.3.

Let $E_\sigma(x,v) := \frac{1}{2}|v|^2 + \phi_{\infty,\sigma}(x) + \phi_e(x)$ and $E_{min} := \inf\{E_\sigma(x,v) : (x,v) \in \mathbb{R}^6\}$, which is finite by assumption (H4'). With evident notations, $f_{\infty,\sigma} = \mu(E_\sigma(\cdot) - \alpha_M, I(\cdot,\cdot))$. Take $\bar{s}(I) = \mu(E_{min} - \alpha_M, I)$ and define for any $I \in \mathbb{R}$

$$(6.2) \qquad \tau_\sigma(s,I) := \begin{cases} \sigma(s,I) & \text{if } s \in [0, \bar{s}(I)] \\ \psi(s,I) & \text{if } s \in (\bar{s}(I), +\infty) \end{cases}$$

with $\psi(s,I) = \frac{\sigma''(\bar{s},I)}{\sigma_2''(\bar{s})}\sigma_2(s) + \left(\sigma'(\bar{s},I) - \frac{\sigma''(\bar{s},I)}{\sigma_2''(\bar{s})}\sigma_2'(\bar{s})\right)(s - \bar{s}) + \sigma(\bar{s},I) - \frac{\sigma''(\bar{s},I)}{\sigma_2''(\bar{s})}\sigma_2(\bar{s})$, $\bar{s} = \bar{s}(I)$ and $\sigma_2(s) = s^2$. With the truncated Lyapunov functional $\Sigma_{\tau_\sigma}[f|f_{\infty,\sigma}] = K_{\tau_\sigma}[f] - K_{\tau_\sigma}[f_{\infty,\sigma}]$, we immediately get the following variant of Theorem 4.3.

THEOREM 6.1. *Let I be a function in $C^1(\mathbb{R}^6)$ and assume that ϕ_e, μ verify (H3')-(H4'). Assume moreover that*

$$B_\sigma(I) = \inf\{s \in [E_{min} - \alpha_M, \mu^{-1}(0,I)] : \frac{\partial^2\sigma}{\partial s^2}(s,I)\} > 0 \quad \text{for any } I \in \mathbb{R} .$$

Let f_0 be a nonnegative function in $L^1(\mathbb{R}^6) \cap L^2(\mathbb{R}^6, B_\sigma(I(x,v))\,d(x,v))$, such that $(x,v) \mapsto \sigma(f_0(x,v), I(x,v))$ belongs to $L^1(\mathbb{R}^6)$ and consider a weak (resp. renormalized) solution of the Vlasov-Poisson system with initial data f_0 satisfying $(\mathcal{W})$ (resp. $(\mathcal{R})$). Then for any $t \geq 0$

$$\Sigma_{\tau_\sigma}[f_0|f_{\infty,\sigma}] \geq \Sigma_{\tau_\sigma}[f(t)|f_{\infty,\sigma}] \geq \int_{\mathbb{R}^6} B_\sigma(I(x,v))\,|f(t,x,v) - f_{\infty,\sigma}(x,v)|^2\,d(x,v) .$$

Weighted L^q estimates can also be established, if one replaces σ_2 by σ_q in (6.2), under the condition that $\inf\{s \in [E_{min} - \alpha, \mu^{-1}(0,I)] : s^{2-q}\frac{\partial^2\sigma}{\partial s^2}(s,I)\} > 0$ for any $I \in \mathbb{R}$.

REMARK 6.2. *Equation (1.1) is a special case (parabolic-band approximation) of the Vlasov-Poisson system for semiconductors*

$$\frac{\partial f}{\partial t} + v(p) \cdot \nabla_x f + F(t,x) \cdot \nabla_p f = 0,$$

on $\mathbb{R}_0^+ \times \mathbb{R}^3 \times \mathbb{R}^3$, with $v(p) = \nabla_p \epsilon(p)$. If we assume that ϵ is a nonnegative C^1 function such that $e^{-\epsilon(p)} \in L^1(\mathbb{R}^3)$, then using abusively the same notations as for (1.1) (which corresponds to the special case $\epsilon(p) = \frac{1}{2}p^2$), one can for instance prove that there exists a maxwellian type stationary solution given by

$$m(x,p) = M\frac{e^{-\epsilon(p) - q\,(\phi(x) + \phi_e(x))}}{\int_{\mathbb{R}^6} e^{-\epsilon - q\,(\phi + \phi_e)}\,d(x,p)} ,$$

where ϕ is given by (1.3) with $\rho(f)(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,dp$. Nonlinear stability results for m and more general stationary states can be easily obtained using the previous ideas. Realistic models include collisions, which usually determine a special class of stationary solutions (and the appropriate Lyapunov functional is then decreasing even for classical solutions). We refer to [37, 37, 6, 7, 17, 19] for more details on this subject.

7. Appendix: a convexity property of L^1 functions. Let f_0 be a nonnegative function in $L^1(\Omega)$ for some (not necessarily bounded) domain Ω in $\mathbb{R}^d$, $d \geq 1$. It is straightforward to check that $\sigma(f_0) \in L^1(\Omega)$ if σ is a C^2 convex function on $\mathbb{R}^+$ such that $s \mapsto \sigma(s)/s$ is bounded (consider for example $\sigma(s) = 2s + e^{-s} - 1$). The result of Proposition 2.1, which is a special case of the following Proposition, is much stronger.

PROPOSITION 7.1. *Let $(E, d\mu)$ be a measurable space. For any nonnegative function f_0 in $L^1(E, d\mu)$, there exists a nonnegative strictly convex function σ of class C^2 such that $\lim_{s \to +\infty} \sigma(s)/s = +\infty$ and $\sigma(f_0) \in L^1(E, d\mu)$.*

This result is more or less standard. We are going to give a proof for the completeness of the paper, which is based on the following elementary lemma.

LEMMA 7.2. *Consider a sequence $\{\alpha_n\}$ with $\alpha_n > 0$ for any n and $\sum \alpha_n < \infty$. Then there exists an increasing sequence $\{\beta_n\}$ with $\beta_n > 0$ for any $n \in \mathbb{N}$, and $\lim_{n \to \infty} \beta_n = +\infty$ such that $\sum \alpha_n \beta_n < \infty$.*

Proof of Lemma 7.2. We prove this result by an explicit construction of β_n. Let $\epsilon_n = \sum_{m \geq n} \alpha_m$ and take $\beta_n = \frac{1}{2\sqrt{\epsilon_n}}$:

$$\alpha_n \beta_n = (\epsilon_n - \epsilon_{n+1}) \frac{1}{2\sqrt{\epsilon_n}} \leq \sqrt{\epsilon_n} - \sqrt{\epsilon_{n+1}} \;,$$

which immediately gives $\sum_{m \geq n} \alpha_m \beta_m \leq \sqrt{\epsilon_n}$. $\square$

Proof of Proposition 7.1. Let $\alpha_n = \int_{n \leq f_0 < n+1} f_0 \, d\mu$ and take β_n given by Lemma 7.2. One can find a convex function σ with $s \mapsto \sigma(s)/s$ nondecreasing, such that $\sigma(n+1) = (n+1)\beta_n$. Thus

$$\int_{n \leq f_0 < n+1} \sigma(f_0) \, d\mu \leq \int_{n \leq f_0 < n+1} f_0 \, d\mu \cdot \frac{\sigma(n+1)}{n+1} = \alpha_n \beta_n \;,$$

which ends the proof. $\square$

REMARK 7.3. *From Proposition 7.1, it is clear that there is no optimal convex function σ corresponding to a given initial data f_0 (reapply the Proposition to $\sigma(f_0)$). To any σ, one can however associate a function γ. Is there an optimal condition on the growth of ϕ_e so that both the stationary solution and the relative entropy are well-defined? This would indeed define a notion of* confinement *which would depend only on f_0. On the other hand, if the growth condition is not satisfied, is it possible to give some dispersion estimate (like in the case $\phi_e \equiv 0$, or $(x - x_0) \cdot \nabla \phi_e \geq 0$ for some given $x_0 \in \mathbb{R}^3$) ?*

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