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Optimal critical mass in the two dimensional Keller-Segel model in $\mathbb{R}^2$

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Abstract. The Keller-Segel system describes the collective motion of cells that are attracted by a chemical substance and are able to emit it. In its simplest form it is a conservative drift-diffusion equation for the cell density coupled to an elliptic equation for the chemo-attractant concentration. It is known that, in two space dimensions, for small initial mass there is global existence of classical solutions and for large initial mass blow-up occurs. In this note we complete this picture and give an explicit value for the critical mass when the system is set in the whole space.

Masse critique optimale pour le modèle de Keller-Segel dans $\mathbb{R}^2$

Résumé. *Le système de Keller-Segel décrit le mouvement collectif de cellules attirées par une substance chimique et qui sont capables de l'émettre. Dans sa forme la plus simple, il s'agit d'une équation de dérive-diffusion pour la densité de cellules, couplée à une équation elliptique pour la concentration de chémo-attracteur. Il est bien connu qu'en deux dimensions, il y a existence pour des masses petites et explosion pour des masses grandes. Dans cette note nous complétons ce résultat en donnant une expression de la masse critique dans le cas où le problème estposé dans tout l'espace.*

Version française abrégée

Le système de Keller et Segel décrit le mouvement collectif de cellules (bactéries ou amibes), de densité $n(x,t)$, attirées par une force induite par une substance chimique, le chémo-attracteur, de concentration $c(x,t)$, qu'elles émettent elles-même ([11, 14, 12, 16, 9]). Dans sa forme la plus simple, ce système s'écrit :

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \chi \nabla\cdot(n\nabla c) & x\in\mathbb{R}^2\,,\ t>0\,, \\ -\Delta c = n & x\in\mathbb{R}^2\,,\ t>0 \end{cases} \tag{1}$$

avec une donnée initial $n_0 \geqslant 0$. La *sensitivité* $\chi > 0$ est le paramètre fondamental, qui mesure la non-linéarité du système. En dimension deux, qui est la dimension pertinente pour la chimiotactie, la norme L^1 est critique ([10, 5]) et nous supposerons que

$$n_0 \in L^1(\mathbb{R}^2; (1+|x|^2)\,dx)\ , \qquad n_0\log n_0 \in L^1(\mathbb{R}^2; dx)\ . \tag{2}$$

Sous ces hypothèses, nous complétons les résultats de [10] en établissant que $8\pi/\chi$ est la masse critique, au sens où pour $M := \int_{\mathbb{R}^2} n_0(x)\,dx = \int_{\mathbb{R}^2} n(x,t)\,dx > 8\pi/\chi$, il y a explosion en temps fini (Cas 1), alors que la solution de (1) existe globalement pour tout temps si $M < 8\pi/\chi$ (Cas 2). Pour $M = 8\pi/\chi$, il existe une solution radiale stationnaire. Sur un ouvert borné la situation est bien plus complexe, voir [6, 8, 9].

Dans le premier cas, l'explosion en temps fini est une conséquence facile du lemme suivant démontré dans la section II.

LEMME 1. – *Considérons une solution positive ou nulle de (1) au sens des distributions sur un intervalle* $[0,T]$*, sous les hypothèses* $\int_{\mathbb{R}^2} |x|^2 n(x,t)\ dx < \infty$*,* $\int_{\mathbb{R}^2} n(x,t)\ dx = M$ *pour tout* $t > 0$ *et* $\int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\ dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$*. Alors cette solution vérifie*

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\ dx = 4M\left(1 - \frac{\chi}{8\pi} M\right)\ .$$

Si la solution existait globalement en t pour $M > 8\pi/\chi$, alors $\int_{\mathbb{R}^2} |x|^2\, n\, dx$ deviendrait négatif, ce qui est absurde. Quand la solution est à symétrie radiale en x, le profil d'explosion est connu explicitement :

$$n(x,t) \to \frac{8\pi}{\chi}\,\delta + \tilde{n}(|x|,t) \quad \text{lorsque } t \nearrow T^*\ ,$$

sans condition particulière sur le second moment en x. Ici $\tilde{n}$ est une fonction $L^1_{\rm loc}(\mathbb{R}^2\times\mathbb{R}^+)$, voir [7, 18]. Le cas général est moins connu (voir [6, 17] pour des concentrations, [13] pour des calculs numériques).

La preuve d'existence habituelle est donnée dans [10], dont nous suivons une variante ([5]). Elle consiste à montrer que l'entropie décroît en écrivant

$$\frac{d}{dt}\int_{\mathbb{R}^2} n\log n\ dx = -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n}\right|^2\ dx + \chi\int_{\mathbb{R}^2} n^2\ dx\ ,$$

et en utilisant l'inégalité de Gagliardo-Nirenberg-Sobolev avec $u = \sqrt{n}$,

$$\|u\|^2_{L^4(\mathbb{R}^2)} \leqslant C_{\rm GNS}\, \|\nabla u\|_{L^2(\mathbb{R}^2)}\, \|u\|_{L^2(\mathbb{R}^2)}\ .$$

La valeur numérique de la meilleure constante est connue numériquement (voir [19]), ce qui montre la décroissance si $\chi M \leqslant 4C_{\rm GNS}^{-2} \approx 1.862\times(4\pi) < 8\pi$, mais ne couvre pas l'ensemble du second cas.

Afin d'obtenir le résultat optimal, on peut utiliser l'énergie libre, ce qui permet des estimations plus fines.

LEMME 2. – *Considérons une solution positive ou nulle* n*, de classe* $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ *de (1) telle que* $n(1+|x|^2)$*,* $n\log n$ *sont bornées dans* $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ *et* $\nabla c \in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$*. Alors*

$$\frac{d}{dt}\left[\int_{\mathbb{R}^2} n\log n\ dx - \frac{\chi}{2}\int_{\mathbb{R}^2} n\, c\ dx\right] = -\int_{\mathbb{R}^2} n\left|\nabla(\log n) - \chi\nabla c\right|^2\ dx\ .$$

Rappelons l'inégalité de Hardy-Littlewood-Sobolev logarithmique.

LEMME 3. – [3, 1] *Soit* f *une fonction positive ou nulle de* $L^1(\mathbb{R}^2)$ *telle que* $f\log f$ *et* $f\log(1+|x|^2)$ *appartiennent à* $L^1(\mathbb{R}^2)$*. Si* $\int_{\mathbb{R}^2} f\, dx = M$*, alors*

$$\int_{\mathbb{R}^2} f\log f\ dx + \frac{2}{M}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y)\log|x-y|\ dx\, dy \geqslant M(1+\log\pi+\log M)\ .$$

En combinant les estimations des Lemmes 2 et 3, on démontre une borne *a priori* sur $\int_{\mathbb{R}^2} n\log n\, dx$, qui, combinée avec le moment en $|x|^2$ du Lemme 1 suffit à démontrer l'équiintegrabilité et résoud le problème dans le deuxième cas (voir aussi l'énoncé plus précis ci-dessous).

THÉORÈME 4. – *Sous l'hypothèse (2) et si* $M < 8\pi/\chi$*, le système de Keller-Segel (1) a une solution faible globale positive ou nulle telle que* $(1+|x|^2+|\log n|)\, n \in L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$.

I Introduction

In its simpler form, the Keller and Segel system describing chemotaxis reads

$$\begin{cases} \frac{\partial n}{\partial t} &= \Delta n - \chi \nabla \cdot (n \nabla c) \qquad x \in \mathbb{R}^2\,,\ t > 0\,, \\ -\Delta c &= n, \qquad c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, n(y,t)\, dy\,, \\ n(x,t=0) &= n_0(x) \geqslant 0 \qquad x \in \mathbb{R}^2\,. \end{cases} \tag{1}$$

It describes the collective motion of cells (usually bacteria or amoebae) that are attracted by a chemical substance and are able to emit it ([11, 14, 12, 16, 9]). Here $n(x,t)$ represents the cell density, and $c(x,t)$ is the concentration of chemo-attractant which induces a drift force. A fundamental parameter of the system is the *sensitivity* $\chi > 0$ of the bacteria to the chemo-attractant because it measures the nonlinearity in this system. For simplicity, we are going to consider the system in the full space $\mathbb{R}^2$, without boundary conditions (in bounded domains, no-flux conditions are used in general). Although there are related models in gravitation which are defined on $\mathbb{R}^3$, the relevant case for chemotaxis is the two-dimensional space. Then, the L^1-norm is critical as established by [10] (more generally the critical space is $L^{d/2}(\mathbb{R}^d)$ for $d \geqslant 2$, see [5] and the references therein). We assume that the initial data satisfy the following asssumptions:

$$n_0 \in L^1(\mathbb{R}^2, (1+|x|^2)\, dx)\,, \qquad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\,. \tag{2}$$

Then we define the conserved total mass

$$M := \int_{\mathbb{R}^2} n_0(x)\, dx = \int_{\mathbb{R}^2} n(x,t)\, dx\,. \tag{3}$$

Our purpose here is to complete the original result of [10] with an exact value for the critical mass. We prove that under assumption (2), there are two cases:

1. Classical solutions to (1) blow-up in finite time when $M > 8\pi/\chi$,

2. There exists a global in time solution of (1) when $M < 8\pi/\chi$.

For $M = 8\pi/\chi$, there is a radially symmetric stationary solution and in a bounded domain, the situation is much more involved [6, 8, 9]. For larger mass, an additional condition (large second x-moment) is needed for blow-up.

II Critical mass for blow-up (Case 1)

LEMMA 2.1. – *Consider a nonnegative distributional solutions to (1) on an interval* $[0,T]$ *that satisfies (3),* $\int_{\mathbb{R}^2} |x|^2 n(x,t)\, dx < \infty$ *and* $\int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$. *Then it also satisfies*

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx = 4M\left(1 - \frac{\chi}{8\pi}M\right)\,.$$

As a consequence, we obtain the above statement in Case 1, namely there is a blow-up time T^*. Similar statements in bounded domains are available, see [15, 2, 8, 9, 6] and the references therein.

Proof. – Consider a smooth function $\varphi_\varepsilon(|x|)$ with compact support that grows nicely to $|x|^2$ as $\varepsilon \to 0$. Then, we compute

$$\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_\varepsilon\, n\, dx = \int_{\mathbb{R}^2} \Delta\varphi_\varepsilon\, n\, dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2} \frac{(\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y))\cdot(x-y)}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy\,.$$

As ε vanishes we may pass to the limit and obtain Lemma 2.1. □

Let us point out that, when the solution is radially symmetric in x, the second x-moment is not needed and the blow-up profile has been explicited, namely

$$n(x,t) \to \frac{8\pi}{\chi}\,\delta + \tilde{n}(|x|,t) \quad \text{as } t \nearrow T^* \;,$$

where $\tilde{n}$ is a $L^1_{\rm loc}(\mathbb{R}^2 \times \mathbb{R}^+)$ function, see [7, 18]. Except that solutions blow-up for large mass, in the general case very little is known on the blow-up profile (see [6, 17] for concentrations estimates, [13] for numerical computations).

III Existence: shortcoming of the usual proof

The usual proof of existence, due to [10], is based on the following computation (use the equation for n, an integration by parts and the equation for c):

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_{\mathbb{R}^2} n \log n \, dx &= -4 \textstyle\int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \, dx + \chi \textstyle\int_{\mathbb{R}^2} \nabla n \cdot \nabla c \, dx \\ &= -4 \textstyle\int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \, dx + \chi \textstyle\int_{\mathbb{R}^2} n^2 \, dx \;. \end{aligned} \tag{4}$$

This shows that two terms compete, namely the diffusion based entropy dissipation term $\int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \, dx$ and the hyperbolic production of entropy.

Thus the entropy is nonincreasing if $\chi M \leqslant 4 C_{\rm GNS}^{-2}$, where $C_{\rm GNS} = C_{\rm GNS}^{(2)}$ is the best constant in the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|^2_{L^p(\mathbb{R}^2)} \leqslant C_{\rm GNS}^{(p)} \, \|\nabla u\|^{4/p}_{L^2(\mathbb{R}^2)} \, \|u\|^{2-4/p}_{L^2(\mathbb{R}^2)} \quad \forall\, u \in H^1(\mathbb{R}^2)\;, \quad \forall\, p \in [2,\infty)\;. \tag{5}$$

The explicit value of $C_{\rm GNS}$ is not known but can be computed numerically (see [19]) and one finds that the entropy is nonincreasing if $\chi M \leqslant 4 C_{\rm GNS}^{-2} \approx 1.862... \times (4\pi) < 8\pi$. Such an estimate is therefore not sufficient to cover the whole range of M for global existence in the second case.

In [10] it is also shown that equiintegrability (deduced from the $n \log n$ estimate for instance) is enough to propagate any L^p initial norm.

IV Existence: the entropy method (Case 2)

To obtain sharper estimates, we use the well known free energy (see [6, 5])

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} n \log n \, dx + \frac{\chi}{4\pi} \int\!\!\int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x,t)\, n(y,t)\, \log|x-y| \, dx\, dy. \tag{6}$$

LEMMA 4.1. – *Consider a nonnegative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (1) such that $n(1+|x|^2)$, $n \log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\nabla\sqrt{n} \in L^1{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c \in L^\infty_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^2)$. Then*

$$\frac{d}{dt}\mathcal{E}(t) = -\int_{\mathbb{R}^2} n \, |\nabla(\log n) - \chi \nabla c|^2 \, dx \leqslant 0 \;. \tag{7}$$

On the other hand, we recall the logarithmic Hardy-Littlewood-Sobolev inequality.

LEMMA 4.2. – [3, 1] *Let f be a nonnegative function in $L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f \, dx = M$, then*

$$\frac{M}{2} \int_{\mathbb{R}^2} f \log f \, dx + \int\!\!\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| \, dx\, dy \geqslant C(M) := \frac{M^2}{2}(1 + \log\pi + \log M)\;. \tag{8}$$

Combining (6) and (8), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx &\leqslant \mathcal{E}_0 - \tfrac{\chi}{4\pi}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)\,n(y,t)\,\log|x-y|\,dx\,dy \\ &\leqslant \mathcal{E}_0 - \tfrac{\chi}{4\pi}C(M) + \tfrac{M\chi}{8\pi}\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx\ .\end{aligned}$$

This inequality gives an *a priori* bound on $\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx$, which combined with the $|x|^2$ moment bound in Lemma 2.1 is enough for equiintegrability (classically we deduce an *a priori* bound on $\int_{\mathbb{R}^2} n(x,t)|\log n(x,t)|\,dx$). Using this information in the method of [10], we obtain in Case 2 the following existence result of weak solutions, in the spirit of [4].

THEOREM 4.3. – *Under assumption (2) and $M < 8\pi/\chi$, the Keller-Segel system (1) has a global weak nonnegative solution such that*

$$(1+|x|^2+|\log n|)\,n \in L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2)), \qquad \int_0^\infty\int_{\mathbb{R}^2} n|\nabla\log n - \chi\nabla c|^2\,dx\,dt < \infty,$$

$$n,\ \nabla\sqrt{n} \in L^2([0,T]\times\mathbb{R}^2), \qquad \forall T>0.$$

Moreover, the energy relation (7) holds as an inequality and if n_0 is bounded in $L^p(\mathbb{R}^2)$ for some $p>1$, then $n \in L^\infty_{\rm loc}(\mathbb{R}^+, L^p(\mathbb{R}^2))$.

In particular, the equation holds in the distributions sense. Indeed, writing,

$$\Delta n - \chi\nabla\cdot(n\nabla c) = \nabla\cdot[n(\nabla\log n - \chi\nabla c)],$$

we can see that the flux is well defined in L^1 since

$$\int_0^T\int_{\mathbb{R}^2} n\,|\nabla\log n - \chi\nabla c|\,dx\,dt \leqslant \left(\int\int_{[0,T]\times\mathbb{R}^2} n\,dx\,dt\right)^{1/2}\left(\int\int_{[0,T]\times\mathbb{R}^2} n\,|\nabla\log n - \chi\nabla c|^2\,dx\,dt\right)^{1/2} < \infty\ .$$

V Proof of Theorem 4.3

As usual the existence proof consists in three steps (i) regularize the problem (in order to solve it we use Banach fixed point theorem), (ii) prove estimates similar to the above, (iii) prove space time compactness and pass to the limit.
We only indicate here the step (i) using the approximate convolution kernel κ^ε by $\kappa^\varepsilon(z) = -\frac{1}{2\pi}\log|z|$ if $|z|>\varepsilon$, $\kappa^\varepsilon(z) = -\frac{1}{2\pi}\log|\varepsilon|$ if $|z|\leqslant\varepsilon$, and the solution of

$$(n^\varepsilon)_t = \Delta n^\varepsilon - \chi\nabla\cdot(n^\varepsilon\nabla(\kappa^\varepsilon * n^\varepsilon)), \tag{9}$$

with initial data n_0. Firstly, note that $\int_{\mathbb{R}^2}|x|^2\,n^\varepsilon(x,t)\,dx$ can be estimated as in Lemma 2.1 thanks to

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,n^\varepsilon(x,t)\,dx = 4M - \frac{\chi}{2\pi}\int_{|x-y|>\varepsilon} n^\varepsilon(x,t)\,n^\varepsilon(y,t)\,dx\,dy \leqslant 4M\ .$$

Secondly, this regularized system comes with an energy as well, namely

$$\frac{d}{dt}\left[\int_{\mathbb{R}^2} n^\varepsilon\log n^\varepsilon\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} n^\varepsilon\,(\kappa^\varepsilon * n^\varepsilon)\,dx\right] = -\int_{\mathbb{R}^2} n^\varepsilon\,|\nabla(\log n^\varepsilon) - \chi\nabla(\kappa^\varepsilon * n^\varepsilon)|^2\,dx \tag{10}$$

To extend the estimate on $n\log n$, we need to prove a priori bounds on each term of (10) which follow from the estimate

$$\left(1-\frac{\chi M}{8\pi}\right)\int_{\mathbb{R}^2} n^\varepsilon\log n^\varepsilon\,dx \leqslant \mathcal{E}_0^\varepsilon - \frac{\chi M^2}{8\pi}(1+\log\pi+\log M).$$

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