REVERSE HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES

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ABSTRACT. This paper is devoted to a new family of reverse Hardy-Littlewood-Sobolev inequalities which involve a power law kernel with positive exponent. We investigate the range of the admissible parameters and characterize the optimal functions. A striking open question is the possibility of concentration which is analyzed and related with non-linear diffusion equations involving mean field drifts.

We are concerned with the following minimization problem. For any $\lambda > 0$ and any (measurable) function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\,|x-y|^\lambda\,\rho(y)\,dx\,dy\,.$$

For $0 < q < 1$ we consider

$$\mathscr{C}_{N,\lambda,q} := \inf\left\{\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(2-\alpha)/q}} : 0 \leq \rho \in \mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)\,,\ \rho \not\equiv 0\right\},$$

where

$$\alpha := \frac{2N - q\,(2N+\lambda)}{N\,(1-q)}\,.$$

By convention, for any $p > 0$ we use the notation $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N}|\rho(x)|^p\,dx$ is finite. Note that α is determined by scaling and homogeneity: for given values of λ and q, the value of α is the only one for which there is a chance that the infimum is positive. We are asking whether $\mathscr{C}_{N,\lambda,q}$ is equal to zero or positive and, in the latter case, whether there is a minimizer. We note that there are three regimes $q < 2N/(2N+\lambda)$, $q = 2N/(2N+\lambda)$ and $q > 2N/(2N+\lambda)$, which respectively correspond to $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$. The case $q = 2N/(2N+\lambda)$ has already been dealt with in [7] by J. Dou and M. Zhu, and in [19] by Q.A. Ngô and V.H. Nguyen, who have explicitly computed $\mathscr{C}_{N,\lambda,q}$ and characterized all solutions of the corresponding Euler–Lagrange equation. In the following we will mostly concentrate on the other cases. Our main result is the following.

Theorem 1. *Let $\lambda > 0$, $q \in (0,1)$, $N \in \mathbb{N}^*$ and α as above. Then the inequality*

$$I_\lambda[\rho] \geq \mathscr{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(2-\alpha)/q} \tag{1}$$

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holds for any nonnegative function $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$*, for some positive constant* $\mathscr{C}_{N,\lambda,q}$*, if and only if* $q > N/(N+\lambda)$*. In this range, if either* $N = 1, 2$ *or* $N \geq 3$ *and* $q \geq \min\{1 - 2/N, 2N/(2N+\lambda)\}$*, then the equality case is achieved by a radial nonnegative function* $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$.

This theorem provides a necessary and sufficient condition for the validity of the inequality, namely $q > N/(N+\lambda)$ or equivalently $\alpha < 1$. Concerning the existence of an optimizer, the theorem completely answers this question in dimensions $N = 1$ and $N = 2$. In dimensions $N \geq 3$ we obtain a sufficient condition for the existence of an optimizer, namely, $q \geq \min\{1-2/N, 2N/(2N+\lambda)\}$. We emphasize that this is not a necessary condition and, in fact, in Proposition 17 we prove existence in a slightly larger, but less explicit region. However, in the *whole* region $q > N/(N+\lambda)$ we are able to prove the existence of an optimizer for a relaxed problem, with same optimal constant $\mathscr{C}_{N,\lambda,q}$, which allows for an additional Dirac mass at a single point. Therefore the question about existence of an optimizer in Theorem 1 is reduced to the much simpler, but still not obvious problem of whether the optimizer for this relaxed problem in fact has a Dirac mass. Fig. 1 summarizes these considerations.

The Hardy-Littlewood-Sobolev (HLS) inequality is named after G. Hardy and J.E. Littlewood, see [10, 11], and S.L. Sobolev, see [20, 21] (also see [12] for an early discussion of rearrangement methods applied to these inequalities). In 1983, E.H. Lieb in [17] proved the existence of optimal functions for negative values of λ and established optimal constants. His proof requires an analysis of the invariances which has been systematized under the name of *competing symmetries*, see [4]. A comprehensive introduction can be found in [18, 3]. Notice that rearrangement free proofs, which in some cases rely on the duality between Sobolev and HLS inequalities, have also been established more recently in various cases: see for instance [8, 9, 15]. Standard HLS inequalities, which correspond to negative values of λ in $I_\lambda[\rho]$, have many consequences in the theory of functional inequalities, particularly when the point comes to the identification of the optimal constants.

Relatively few results are known in the case $\lambda > 0$. The conformally invariant case, *i.e.*, $q = 2N/(2N+\lambda)$, appears in [7] and is motivated by some earlier results on the sphere (see references therein). Further results have been obtained in [19], which still correspond to the conformally invariant case. Another range of exponents, which has no intersection with the one considered in the present paper, was studied earlier in [22, Theorem G]. Here we focus on a non conformally invariant family of interpolation inequalities corresponding to a given $\mathrm{L}^1(\mathbb{R}^N)$ norm. In a sense, these inequalities play for HLS inequalities a role analogous to Gagliardo-Nirenberg inequalities compared to Sobolev's conformally invariant inequality.

Our study of (1) is motivated by the study of the nonnegative solutions of the evolution equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho^q + \nabla \cdot \left(\rho \, \nabla W_\lambda * \rho\right) \tag{2}$$

where the kernel is $W_\lambda(x) := \frac{1}{\lambda}\,|x|^\lambda$. Optimal functions for (1) provide energy minimizers for the *free energy* functional

$$\mathscr{F}[\rho] := \frac{1}{2}\int_{\mathbb{R}^N}\rho\,(W_\lambda*\rho)\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx = \frac{1}{2\lambda}\,I_\lambda[\rho] - \frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx$$

under a *mass* constraint $M = \int_{\mathbb{R}^N}\rho\,dx$. It is indeed an elementary computation to check that a smooth solution $\rho(t,\cdot)$ of (2) with sufficient decay properties as $|x|\to+\infty$ is such that $M = \int_{\mathbb{R}^N}\rho(t,x)\,dx$ does not depend on t while the free energy decays according to

$$\frac{d}{dt}\mathscr{F}[\rho(t,\cdot)] := -\int_{\mathbb{R}^N}\rho\left|\tfrac{q}{1-q}\nabla\rho^{q-1} - \nabla W_\lambda*\rho\right|^2 dx\,.$$

This identity allows us to identify the smooth stationary solutions as the solutions of

$$\rho_s(x) = \big(\mu + (W_\lambda*\rho_s)(x)\big)^{-\frac{1}{1-q}}\quad\forall\,x\in\mathbb{R}^N\,,$$

where μ is a constant which has to be determined by the mass constraint and observe that smooth minimizers of $\mathscr{F}$, whenever they exist, are stationary solutions.

Corollary 2. *With the notations of Theorem 1, $\mathscr{F}[\rho]$ is bounded from below on the set of nonnegative functions $\rho\in\mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)$ if and only if $q>N/(N+\lambda)$, and in the range of parameters for which equality is achieved in (1), if ρ_* is optimal for (1), then $\mathscr{F}[\rho]\ge\mathscr{F}[\rho_*]$ up to a scaling of ρ_*, if ρ and ρ_* have same mass.*

It is obvious that Eq. (2) is *translation* invariant and this is also a natural property of $\mathscr{F}[\rho]$. Concerning *scaling* properties, we introduce $\rho_\tau(x) := \tau^N\rho(\tau\,x)$ for an arbitrary $\tau>0$ and observe that

$$\tau\mapsto\mathscr{F}[\rho_\tau] = \frac{\tau^{-\lambda}}{2\lambda}\,I_\lambda[\rho] - \frac{\tau^{-N(1-q)}}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx$$

reaches a minimum at $\tau=\tau_0$ such that $\tau_0^{\lambda-N(1-q)} = \frac{I_\lambda[\rho]}{2N\int_{\mathbb{R}^N}\rho^q\,dx}$. As a consequence, we have that

$$\mathscr{F}[\rho]\ge\mathscr{F}[\rho_{\tau_0}] = -\kappa\,\frac{\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{\frac{\lambda}{\lambda-N(1-q)}}}{I_\lambda[\rho]^{\frac{N(1-q)}{\lambda-N(1-q)}}}\ge-\kappa\left(\mathscr{C}_{N,\lambda,q}\,M^\alpha\right)^{-\frac{N(1-q)}{\lambda-N(1-q)}}$$

where $\kappa := \left(\frac{1}{1-q}-\frac{N}{\lambda}\right)(2N)^{\frac{N(1-q)}{\lambda-N(1-q)}}>0$, which completes the proof of Corollary 2.

Eq. (2) deserves some comments. It is a *mean field* equation in the overdamped regime, in which the *drift term* is an average of a spring force $\nabla W_\lambda(x)$ for any $\lambda>0$. The case $\lambda=2$ corresponds to linear springs obeying Hooke's law, while large λ reflect a force which is small at small distances, but becomes very large for large values of $|x|$. In this sense, it is a *strongly confining* force term. By expanding the diffusion term as $\Delta\rho^q = q\,\rho^{q-1}\left(\Delta\rho+(q-1)\,\rho^{-1}\,|\nabla\rho|^2\right)$ and considering ρ^{q-1} as a diffusion coefficient, it is obvious that this *fast diffusion* coefficient is large for small values of ρ and has to be balanced by a very large drift term to avoid a *runaway* phenomenon in which no stationary solutions may exist in $\mathrm{L}^1(\mathbb{R}^N)$. In the case of a drift term with linear growth as $|x|\to+\infty$, it is well known that the threshold is given by the exponent $q=1-2/N$ and it is also known according to, *e.g.*, [13] for the pure fast diffusion case (no drift) that $q = 1-2/N$ is the threshold for the global existence of nonnegative solutions in $\mathrm{L}^1(\mathbb{R}^N)$, with constant mass.

In the regime $q < 1-2/N$, a new phenomenon appears which is not present in linear diffusions. As emphasized in [23], the diffusion coefficient ρ^{q-1} becomes small for large values of ρ and does not prevent the appearance of singularities. This is one of the major issues investigated in this paper. Before giving details, let us observe that W_λ is a convolution kernel which averages the solution and can be expected to give rise to a smooth effective potential term $V_\lambda = W_\lambda * \rho$ at $x = 0$ if we consider a radial function ρ. This is why we expect that $V_\lambda(x) = V_\lambda(0) + O\big(|x|^2\big)$ for $|x|$ small. With these considerations at hand, let us illustrate these ideas with a simpler model involving only a given, external potential V. Assume that u solves the *fast diffusion with external drift* given by

$$\frac{\partial u}{\partial t} = \Delta u^q + \nabla\cdot\big(u\,\nabla V\big)\,.$$

To fix ideas, we shall take $V(x) = 1 + \frac{1}{2}|x|^2 + \frac{1}{\lambda}|x|^\lambda$, which is expected to capture the behavior of the potential $W_\lambda * \rho$ at $x = 0$ and as $|x| \to +\infty$ when $\lambda \ge 2$. Such an equation admits a free energy functional

$$u \mapsto \int_{\mathbb{R}^N} V\,u\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N} u^q\,dx\,,$$

whose minimizers are, under the mass constraint $M = \int_{\mathbb{R}^N} u\,dx$, given by

$$u_\mu(x) = \big(\mu + V(x)\big)^{-\frac{1}{1-q}} \quad \forall\, x \in \mathbb{R}^N\,.$$

A linear spring would simply correspond to a fast diffusion Fokker–Planck equation when $V(x) = |x|^2/2$. One can for instance refer to [16] for a general account on this topic. In that case, it is straightforward to observe that the so-called *Barenblatt profile* u_μ has finite mass if and only if $q > 1-2/N$. For a general parameter $\lambda \ge 2$, the corresponding integrability condition for u_μ is $q > 1-\lambda/N$. But $q = 1-2/N$ is also a threshold value for the regularity. Let us assume that $\lambda > 2$ and $1-\lambda/N < q < 1-2/N$, and consider the stationary solution u_μ, which depends on the parameter μ. The mass of u_μ can be computed for any $\mu \ge -1$ by inverting the monotone decreasing function of $\mu \mapsto M(\mu)$ defined by

$$M(\mu) := \int_{\mathbb{R}^N} \big(\mu + V(x)\big)^{-\frac{1}{1-q}}\,dx \le M(-1) = \int_{\mathbb{R}^N} \Big(\tfrac{1}{2}|x|^2 + \tfrac{1}{\lambda}|x|^\lambda\Big)^{-\frac{1}{1-q}}\,dx\,.$$

Now, if one tries to minimize the free energy under the mass contraint $\int_{\mathbb{R}^N} u\,dx = M$ for an arbitrary $M > M(-1)$, it is left to the reader to check that the limit of a minimizing sequence is simply the measure $\big(M - M(-1)\big)\,\delta + u_{-1}$ where δ denotes the Dirac distribution centered at $x = 0$. For the model described by Eq. (2), the situation is by far more complicated because the mean field potential $V_\lambda = W_\lambda * \rho$ depends on the regular part ρ and we have no simple estimate on a critical mass as in the case of an external potential V.

Eq. (2) is a special case of a larger family of Keller-Segel type equations, which covers the cases with $q = 1$ (linear diffusions) and $q \ge 1$ (diffusions of porous medium type), and also the range of exponents $\lambda < 0$. Of particular interest is the original parabolic–elliptic Keller–Segel system which corresponds in dimension $N = 2$ to a limit case as $\lambda \to 0$, in which the kernel is $W_0(x) = \frac{1}{2\pi} \log|x|$ and the diffusion exponent is $q = 1$. The reader is invited to refer to [14] for a global overview of this class of problems and for a detailed list of references and applications. A research project related with the present paper, [5], focuses on the role of the free energy of the gradient flow equation (2) using a dyadic decomposition, and nicely complements the results presented here.

1. Validity of the inequality

The following proposition gives a necessary and sufficient condition for inequality (1).

Proposition 3. *Let* $\lambda > 0$.

(1) *If* $0 < q \le N/(N+\lambda)$*, then* $\mathscr{C}_{N,\lambda,q} = 0$.
(2) *If* $N/(N+\lambda) < q < 1$*, then* $\mathscr{C}_{N,\lambda,q} > 0$.

The result in (1) for $q < N/(N+\lambda)$ is obtained in [6] using a different method. The result in (1) for $q = N/(N+\lambda)$, as well as the result in (2) is new.

Proof of Proposition 3. Part (1). Let $\rho \ge 0$ be bounded with compact support and let $\sigma \ge 0$ be a smooth function with $\int_{\mathbb{R}^N} \sigma(x)\,dx = 1$. With another parameter $M > 0$ we consider

$$\rho_\varepsilon(x) = \rho(x) + M\,\varepsilon^{-N}\,\sigma(x/\varepsilon)\,.$$

Then $\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,dx = \int_{\mathbb{R}^N} \rho(x)\,dx + M$ and, by simple estimates,

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)^q\,dx \to \int_{\mathbb{R}^N} \rho(x)^q\,dx \quad \text{as} \quad \varepsilon \to 0_+ \tag{3}$$

and

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx \quad \text{as} \quad \varepsilon \to 0_+\,.$$

Thus, taking ρ_ε as a trial function,

$$\mathscr{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}} =: \mathscr{Q}[\rho, M]\,. \tag{4}$$

This inequality is valid for any M and therefore we can let $M \to +\infty$. If $\alpha > 1$, which is the same as $q < N/(N+\lambda)$, we immediately obtain $\mathscr{C}_{N,\lambda,q} = 0$ by letting $M \to +\infty$. If $\alpha = 1$, *i.e.*, $q = N/(N+\lambda)$, by taking the limit as $M \to +\infty$, we obtain

$$\mathscr{C}_{N,\lambda,q} \le \frac{2\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}}\,.$$

Let us show that by a suitable choice of ρ the right side can be made arbitrarily small. For any $R > 1$, we take

$$\rho_R(x) := |x|^{-(N+\lambda)}\,\mathbb{1}_{1\le|x|\le R}(x)\,.$$

Then

$$\int_{\mathbb{R}^N} |x|^\lambda \rho_R\,dx = \int_{\mathbb{R}^N} \rho_R^q\,dx = \left|\mathbb{S}^{N-1}\right| \log R$$

and, as a consequence,

$$\frac{\int_{\mathbb{R}^N} |x|^\lambda \rho_R(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho_R^{N/(N+\lambda)}\,dx\right)^{(N+\lambda)/N}} = \left(\left|\mathbb{S}^{N-1}\right| \log R\right)^{-\lambda/N} \to 0 \quad \text{as} \quad R\to\infty\,.$$

This proves that $\mathscr{C}_{N,\lambda,q} = 0$ for $q = N/(N+\lambda)$. □

In order to prove that $\mathscr{C}_{N,\lambda,q} > 0$ in the remaining cases we need the following simple bound.

Lemma 4. *Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. Then there is a constant $c_{N,\lambda,q} > 0$ such that for all $\rho \ge 0$,*

$$\left(\int_{\mathbb{R}^N} \rho\,dx\right)^{1-\frac{N(1-q)}{\lambda q}} \left(\int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx\right)^{\frac{N(1-q)}{\lambda q}} \ge c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{1/q}.$$

Proof. Let $R > 0$. Using Hölder's inequality, we obtain

$$\int_{\{|x|<R\}} \rho^q\,dx \le \left(\int_{\mathbb{R}^N} \rho\,dx\right)^q |B_R|^{1-q} = C_1 \left(\int_{\mathbb{R}^N} \rho\,dx\right)^q R^{N(1-q)}$$

and

$$\begin{aligned}\int_{\{|x|\ge R\}} \rho^q\,dx = \int_{\{|x|\ge R\}} \left(|x|^\lambda \rho\right)^q |x|^{-\lambda q}\,dx &\le \left(\int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx\right)^q \left(\int_{\{|x|\ge R\}} |x|^{-\frac{\lambda q}{1-q}}\,dx\right)^{1-q} \\ &= C_2 \left(\int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx\right)^q R^{-\lambda q+N(1-q)}\,.\end{aligned}$$

The fact that $C_2 < \infty$ comes from the assumption $q > N/(N+\lambda)$, which is the same as $\lambda q/(1-q) > N$. To conclude, we add these two inequalities and optimize over R. □

Proof of Proposition 3. Part (2). By rearrangement inequalities it suffices to prove the inequality for symmetric non-increasing ρ's. For such functions, by the simplest rearrangement inequality,

$$\int_{\mathbb{R}^N} |x-y|^\lambda \rho(y)\,dx \ge \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Thus,

$$I_\lambda[\rho] \ge \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx \int_{\mathbb{R}^N} \rho\,dx\,.$$

In the range $\frac{N}{N+\lambda} < q < \frac{2N}{2N+\lambda}$, we recall that by Lemma 4, we have for any symmetric non-increasing function ρ,

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx\right)^\alpha} \ge \left(\int_{\mathbb{R}^N} \rho\,dx\,dx\right)^{1-\alpha} \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx \ge c_{N,\lambda,q}^{2-\alpha} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{\frac{2-\alpha}{q}}$$

because $2-\alpha=\frac{\lambda\,q}{N\,(1-q)}$. As a consequence, we obtain that

$$\mathscr{C}_{N,\lambda,q}\ge c_{N,\lambda,q}^{2-\alpha}\,.$$

□

Remark 5. The above computation explains a surprising feature of (1): $I_\lambda[\rho]$ controls a product of two terms. However, in the range $N/(N+\lambda)<q<2N/(2N+\lambda)$ which corresponds to $\alpha\in(0,1)$, the problem is actually reduced to the interpolation of $\int_{\mathbb{R}^N}\rho^q\,dx$ between $\int_{\mathbb{R}^N}\rho\,dx$ and $\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx$, which has a more classical structure.

Remark 6. There is an alternative way to prove the inequality in the range $2N/(2N+\lambda)<q<1$ using the results from [7, 19]. We can indeed rely on Hölder's inequality to get that

$$\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{1/q}\le\left(\int_{\mathbb{R}^N}\rho(x)^{\frac{2N}{2N+\lambda}}\,dx\right)^{\eta\,\frac{2N+\lambda}{2N}}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^{1-\eta}$$

with $\eta=\frac{2N\,(1-q)}{\lambda\,q}$. By applying the inequality

$$I_\lambda[\rho]\ge\mathscr{C}_{N,\lambda,\frac{2N}{2N+\lambda}}\left(\int_{\mathbb{R}^N}\rho(x)^{\frac{2N}{2N+\lambda}}\,dx\right)^{\frac{2N+\lambda}{N}}$$

shown in [7, 19] with an explicit constant, we obtain that

$$\mathscr{C}_{N,\lambda,q}\ge\mathscr{C}_{N,\lambda,\frac{2N}{2N+\lambda}}=\pi^{\frac{\lambda}{2}}\,\frac{\Gamma\left(\frac{N}{2}-\frac{\lambda}{2}\right)}{\Gamma\left(N-\frac{\lambda}{2}\right)}\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1-\frac{\lambda}{N}}\,.$$

We notice that $\alpha=-2\,(1-\eta)/\eta$ is negative.

Remark 7. In the proof of Lemma 4, the computation of the lower bound can be made more explicit and shows that

$$c_{N,\lambda,q}\ge O\left(\left((N+\lambda)\,q-N\right)^{(1-q)/q}\right)\quad\text{as}\quad q\to N/(N+\lambda)_+\,.$$

The optimal constant $c_{N,\lambda,q}$ can be explicitly computed after observing that a minimizer exists and that $x\mapsto(1+|x|^\lambda)^{-1/(1-q)}$ solves the Euler-Lagrange equation (after taking into account translations, scalings and homogeneity), thus realizing the equality case. An elementary computation shows that $\lim_{q\to N/(N+\lambda)_+}c_{N,\lambda,q}=0$. This limit is compatible with the fact that

$$\lim_{q\to N/(N+\lambda)_+}\mathscr{C}_{N,\lambda,q}=0$$

because the map $(\lambda,q)\mapsto\mathscr{C}_{N,\lambda,q}$ is *upper semi-continuous*. The proof of this last property goes as follows. Let us denote by

$$\mathsf{Q}_{q,\lambda}[\rho]:=\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(2-\alpha)/q}}$$

the energy quotient in which we emphasize the dependence in q and λ. The infimum of $\mathsf{Q}_{q,\lambda}[\rho]$ over ρ is $\mathscr{C}_{N,\lambda,q}$. Let (q,λ) be a given point in $(0,1)\times(0,\infty)$ and let (q_n,λ_n) be a sequence converging to (q,λ). Let $\varepsilon>0$ and choose a ρ which is bounded, has compact support and is such that $\mathrm{Q}_{q,\lambda}[\rho] \le \mathscr{C}_{N,\lambda,q} + \varepsilon$. Then, by the definition as an infimum, $\mathscr{C}_{N,q_n,\lambda_n} \le \mathrm{Q}_{q_n,\lambda_n}[\rho]$. On the other hand, the assumptions on ρ easily imply that $\lim_{n\to\infty} \mathrm{Q}_{q_n,\lambda_n}[\rho] = \mathrm{Q}_{q,\lambda}[\rho]$. We conclude that $\limsup_{n\to\infty} \mathscr{C}_{N,q_n,\lambda_n} \le \mathscr{C}_{N,\lambda,q} + \varepsilon$. Since ε is arbitrary, we obtain the claimed upper semi-continuity.

2. Existence of minimizers

We now investigate whether there are minimizers for $\mathscr{C}_{N,\lambda,q}$ if $N/(N+\lambda) < q < 1$. As mentioned before, the conformally invariant case $q = 2N/(2N+\lambda)$ has been dealt with before and will be excluded from our considerations. We start with the simpler case $2N/(2N+\lambda) < q < 1$, which corresponds to $\alpha < 0$.

Proposition 8. *Let $\lambda > 0$ and $2N/(2N+\lambda) < q < 1$. Then there is a minimizer for $\mathscr{C}_{N,\lambda,q}$.*

Proof. Let $(\rho_j)_{j\in\mathbb{N}}$ be a minimizing sequence. By rearrangement inequalities we may assume that the ρ_j are symmetric non-increasing. By scaling and homogeneity, we may also assume that

$$\int_{\mathbb{R}^N} \rho_j(x)\,dx = \int_{\mathbb{R}^N} \rho_j(x)^q\,dx = 1 \quad \text{for all } j \in \mathbb{N}.$$

This together with the symmetric non-increasing character implies that

$$\rho_j(x) \le C \min\left\{|x|^{-N}, |x|^{-N/q}\right\}$$

with C independent of j. By Helly's selection theorem we may assume, after passing to a subsequence if necessary, that $\rho_j \to \rho$ almost everywhere. The function ρ is symmetric non-increasing and satisfies the same upper bound as ρ_j.

By Fatou's lemma we have

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] \quad \text{and} \quad 1 \ge \int_{\mathbb{R}^N} \rho(x)\,dx.$$

To complete the proof we need to show that $\int_{\mathbb{R}^N} \rho(x)^q\,dx = 1$ (which implies, in particular, that $\rho \not\equiv 0$) and then ρ will be an optimizer.

Modifying an idea from [1] we pick $p \in \big(N/(N+\lambda), q\big)$ and apply (1) with the same λ and $\alpha(p) = \big(2N - p(2N+\lambda)\big)/\big(N(1-p)\big)$ to get

$$I_\lambda[\rho_j] \ge \mathscr{C}_{N,\lambda,p} \left(\int_{\mathbb{R}^N} \rho_j^p\,dx\right)^{(2-\alpha(p))/p}.$$

Since the left side converges to a finite limit, namely $\mathscr{C}_{N,\lambda,q}$, we find that the ρ_j are uniformly bounded in $\mathrm{L}^p(\mathbb{R}^N)$ and therefore we have as before

$$\rho_j(x) \le C'\,|x|^{-N/p}.$$

Since $\min\left\{|x|^{-N}, |x|^{-N/p}\right\} \in \mathrm{L}^q(\mathbb{R}^N)$, we obtain by dominated convergence

$$\int_{\mathbb{R}^N} \rho_j^q\,dx \to \int_{\mathbb{R}^N} \rho^q\,dx,$$

which, in view of the normalization, implies that $\int_{\mathbb{R}^N} \rho(x)^q\,dx = 1$, as claimed. □

Next we prove the existence of minimizers in the range $N/(N+\lambda) < q < 2N/(2N+\lambda)$ by considering the *minimization of a relaxed problem*. The idea behind the relaxed problem is to allow ρ to contain a Dirac function at the origin. The motivation for this comes from the proof of the first part of Proposition 3. The expression on the right side of (4) comes precisely from ρ together with a delta function of strength M at the origin. We have seen that in the regime $q \le N/(N+\lambda)$ (that is, $\alpha \ge 1$) it is advantageous to increase M to infinity. This is no longer so if $N/(N+\lambda) < q < 2N/(2N+\lambda)$. While it is certainly disadvantageous to move M to infinity, it depends on ρ whether the optimum M is 0 or a positive finite value.

Let

$$\mathscr{C}^{\rm rel}_{N,\lambda,q} := \inf\left\{\mathscr{Q}[\rho,M] : 0 \le \rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)\,,\ \rho \not\equiv 0\,,\ M \ge 0\right\}$$

where $\mathscr{Q}[\rho,M]$ is defined by (4). We know that $\mathscr{C}^{\rm rel}_{N,\lambda,q} \le \mathscr{C}_{N,\lambda,q}$ by restricting the minimization to $M = 0$. On the other hand, (4) gives $\mathscr{C}^{\rm rel}_{N,\lambda,q} \ge \mathscr{C}_{N,\lambda,q}$. Therefore,

$$\mathscr{C}^{\rm rel}_{N,\lambda,q} = \mathscr{C}_{N,\lambda,q}\,,$$

which justifies our interpretation of $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ as a *relaxed minimization* problem. Let us start with a preliminary observation.

Lemma 9. *Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. If $\rho \ge 0$ is an optimal function for either $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ (for an $M > 0$) or $\mathscr{C}_{N,\lambda,q}$ (with $M = 0$), then ρ is radial (up to a translation), monotone non-increasing and positive almost everywhere on $\mathbb{R}^d$.*

Proof. Since $\mathscr{C}_{N,\lambda,q}$ is positive, we observe that ρ is not identically 0. By rearrangement inequalities and up to a translation, we know that ρ is radial and monotone non-increasing. Assume by contradiction that ρ vanishes on a set $E \subset \mathbb{R}^N$ of finite, positive measure. Then

$$\mathscr{Q}\big[\rho, M + \varepsilon\,\mathbb{1}_E\big] = \mathscr{Q}[\rho,M]\left(1 - \frac{2-\alpha}{q}\,\frac{|E|}{\int_{\mathbb{R}^N}\rho(x)^q\,dx}\,\varepsilon^q + o(\varepsilon^q)\right)$$

as $\varepsilon \to 0_+$, a contradiction to the minimality for sufficiently small $\varepsilon > 0$. □

Proposition 10. *Let $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$. Then there is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$.*

We will later show that for $N = 1$ and $N = 2$ there is a minimizer for the original problem $\mathscr{C}_{N,\lambda,q}$ in the full range of λ's and q's covered by Proposition 10. If $N \ge 3$, the same is true under an additional restriction.

Proof of Proposition 10. The beginning of the proof is similar to that of Proposition 8. Let (ρ_j, M_j) be a minimizing sequence. By rearrangement inequalities we may assume that ρ_j is symmetric non-increasing. Moreover, by scaling and homogeneity, we may assume that

$$\int_{\mathbb{R}^N}\rho_j\,dx + M_j = \int_{\mathbb{R}^N}\rho_j^q = 1\,.$$

In a standard way this implies that

$$\rho_j(x) \le C \min\left\{|x|^{-N}, |x|^{-N/q}\right\}$$

with C independent of j. By Helly's selection theorem we may assume, after passing to a subsequence if necessary, that $\rho_j \to \rho$ almost everywhere. The function ρ is symmetric non-increasing and satisfies the same upper bound as ρ_j. Passing to a further subsequence, we can also assume that (M_j) and $\left(\int_{\mathbb{R}^N} \rho_j\, dx\right)$ converge and define $M := L + \lim_{j\to\infty} M_j$ where $L = \lim_{j\to\infty} \int_{\mathbb{R}^N} \rho_j\, dx - \int_{\mathbb{R}^N} \rho\, dx$, so that $\int_{\mathbb{R}^N} \rho\, dx + M = 1$. In the same way as before, we show that

$$\int_{\mathbb{R}^N} \rho(x)^q\, dx = 1\,.$$

We now turn our attention to the L^1-term. We cannot invoke Fatou's lemma because $\alpha \in (0,1)$. The problem with this term is that $|x|^{-N}$ is not integrable at the origin and we cannot get a better bound there. We have to argue via measures, so let $d\mu_j(x) := \rho_j(x)\, dx$. Because of the upper bound on ρ_j we have

$$\mu_j\left(\mathbb{R}^N \setminus B_R(0)\right) = \int_{\{|x|\ge R\}} \rho_j(x)\, dx \le C \int_{\{|x|\ge R\}} \frac{dx}{|x|^{N/q}} = C'\, R^{-N(1-q)/q}\,.$$

This means that the measures are tight. After passing to a subsequence if necessary, we may assume that $\mu_j \to \mu$ weak * in the space of measures on $\mathbb{R}^N$. Tightness implies that $\mu(\mathbb{R}^N) = L + \int_{\mathbb{R}^N} \rho\, dx$. Moreover, since the bound $C|x|^{-N/q}$ is integrable away from any neighborhood of the origin, we see that μ is absolutely continuous on $\mathbb{R}^N \setminus \{0\}$ and $d\mu/dx = \rho$. In other words,

$$d\mu = \rho\, dx + L\delta\,.$$

Using weak convergence in the space of measures one can show that

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\, \rho(x)\, dx\,.$$

Finally, by Fatou's lemma,

$$\liminf_{j\to\infty} \int_{\mathbb{R}^N} |x|^\lambda\, \rho_j(x)\, dx \ge \int_{\mathbb{R}^N} |x|^\lambda \left(\rho(x)\, dx + L\delta\right) = \int_{\mathbb{R}^N} |x|^\lambda\, \rho(x)\, dx\,.$$

Thus,

$$\liminf_{j\to\infty} \mathcal{Q}[\rho_j, M_j] \ge \mathcal{Q}[\rho, M]\,.$$

By definition of $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$ the right side is bounded from below by $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$. On the other hand, by choice of ρ_j and M_j the left side is equal to $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$. This proves that (ρ, M) is a minimizer for $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$. □

Next, we show that under certain assumptions a minimizer (ρ_*, M_*) for the relaxed problem must, in fact, have $M_* = 0$ and is therefore a minimizer of the original problem.

Proposition 11. *Let $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$. If either $N = 1, 2$ or $N \geq 3$ and $\lambda > 2N/(N-2)$, assume, in addition, that $q \geq 1-2/N$. If (ρ_*, M_*) is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$, then $M_* = 0$.*

Note that for $N \geq 3$, we are implicitly assuming $\lambda < 4N/(N-2)$ since otherwise the two assumptions $q < 2N/(2N+\lambda)$ and $q \geq 1-2/N$ cannot be simultaneously satisfied. For the proof of Proposition 11 we need the following lemma which identifies the sub-leading term in (3).

Lemma 12. *Let $0 < q < p$, let $f \in \mathrm{L}^p \cap \mathrm{L}^q(\mathbb{R}^N)$ be a symmetric non-increasing function and let $g \in \mathrm{L}^q(\mathbb{R}^N)$. Then, for any $\mu > 0$, as $\varepsilon \to 0_+$,*

$$\int_{\mathbb{R}^N} \left| f(x) + \varepsilon^{-N/p} \mu\, g(x/\varepsilon) \right|^q dx = \int_{\mathbb{R}^N} f^q\, dx + \varepsilon^{N(1-q/p)} \mu^q \int_{\mathbb{R}^N} |g|^q\, dx + o\left(\varepsilon^{N(1-q/p)} \mu^q\right).$$

Proof of Lemma 12. We first note that

$$f(x) = o\left(|x|^{-N/p}\right) \quad \text{as} \quad x \to 0 \tag{5}$$

in the sense that for any $c > 0$ there is a $r > 0$ such that for all $x \in \mathbb{R}^N$ with $|x| \leq r$ one has $f(x) \leq c\,|x|^{-N/p}$. To see this, we note that, since f is symmetric non-increasing,

$$f(x)^p \leq \frac{1}{\left|\left\{y \in \mathbb{R}^N : |y| \leq |x|\right\}\right|} \int_{|y| \leq |x|} f(y)^p\, dy.$$

The bound (5) now follows by dominated convergence.

It follows from (5) that, as $\varepsilon \to 0_+$,

$$\varepsilon^{N/p} f(\varepsilon x) \to 0 \quad \text{for any} \quad x \in \mathbb{R}^N,$$

and therefore, in particular, $\mu\, g(x) + \varepsilon^{N/p} f(\varepsilon x) \to \mu\, g(x)$ for any $x \in \mathbb{R}^N$. From the Brézis–Lieb lemma (see [2]) we know that

$$\int_{\mathbb{R}^N} \left|\mu\, g(x) + \varepsilon^{N/p} f(\varepsilon x)\right|^q dx = \mu^q \int_{\mathbb{R}^N} |g(x)|^q\, dx + \int_{\mathbb{R}^N} \left(\varepsilon^{N/p} f(\varepsilon x)\right)^q dx + o(1).$$

By scaling this is equivalent to the assertion of the lemma. □

Proof of Proposition 11. We argue by contradiction and assume that $M_* > 0$. Let $0 \leq \sigma \in \left(\mathrm{L}^1 \cap \mathrm{L}^q\left(\mathbb{R}^N\right)\right) \cap \mathrm{L}^1\left(\mathbb{R}^N, |x|^\lambda\, dx\right)$ with $\int_{\mathbb{R}^N} \sigma\, dx = 1$. We compute the value of

$$\mathscr{Q}[\rho, M] = \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\, dx}{\left(\int_{\mathbb{R}^N} \rho(x)\, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\, dx\right)^{(2-\alpha)/q}}$$

for the family $(\rho, M) = \left(\rho_* + \varepsilon^{-N} \mu\, \sigma(\cdot/\varepsilon), M^* - \mu\right)$ with a parameter $\mu < M_*$.

1) We have

$$I_\lambda\left[\rho_* + \varepsilon^{-N} \mu\, \sigma(\cdot/\varepsilon)\right] + 2\,(M_* - \mu) \int_{\mathbb{R}^N} |x|^\lambda \left(\rho_*(x) + \varepsilon^{-N} \mu\, \sigma(x/\varepsilon)\right) dx$$
$$= I_\lambda[\rho_*] + 2\, M_* \int_{\mathbb{R}^N} |x|^\lambda \rho_*(x)\, dx + R_1$$

with

$$R_1 = 2\,\mu \iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_*(x)\left(|x-y|^\lambda - |x|^\lambda\right)\varepsilon^{-N}\sigma(y/\varepsilon)\,dx\,dy \\ + \varepsilon^\lambda\,\mu^2\,I_\lambda[\sigma] + 2\,(M_*-\mu)\,\mu\,\varepsilon^\lambda \int_{\mathbb{R}^N}|x|^\lambda\,\sigma(x)\,dx\,.$$

Let us show that $R_1 = O\left(\varepsilon^\beta\,\mu\right)$ with $\beta := \min\{2,\lambda\}$. This is clear for the last two terms in the definition of R_1, so it remains to consider the double integral. If $\lambda \le 1$ we use the simple inequality $|x-y|^\lambda - |x|^\lambda \le |y|^\lambda$ to conclude that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_*(x)\left(|x-y|^\lambda - |x|^\lambda\right)\varepsilon^{-N}\sigma(y/\varepsilon)\,dx\,dy \le \varepsilon^\lambda \int_{\mathbb{R}^N}|x|^\lambda\,\sigma(x)\,dx \int_{\mathbb{R}^N}\rho_*\,dx\,.$$

If $\lambda > 1$ we use the fact that, with a constant C depending only on λ,

$$|x-y|^\lambda - |x|^\lambda \le -\lambda|x|^{\lambda-2}x\cdot y + C\left(|x|^{(2-\lambda)_+}|y|^\beta + |y|^\lambda\right).$$

Since ρ_* is radial, we obtain

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_*(x)\left(|x-y|^\lambda - |x|^\lambda\right)\varepsilon^{-N}\sigma(y/\varepsilon)\,dx\,dy \\ \le C\left(\varepsilon^\beta \int_{\mathbb{R}^N}|x|^{(2-\lambda)_+}\rho_*(x)\,dx \int_{\mathbb{R}^N}|y|^\beta\sigma(y)\,dy + \varepsilon^\lambda \int_{\mathbb{R}^N}|x|^\lambda\,\sigma(x)\,dx \int_{\mathbb{R}^N}\rho_*(x)\,dx\right).$$

Using Hölder's inequality and the fact that $\rho_*,\,\sigma \in \mathrm{L}^1\left(\mathbb{R}^N\right)\cap\mathrm{L}^1\left(\mathbb{R}^N,|x|^\lambda\,dx\right)$ it is easy to see that the integrals on the right side are finite, so indeed $R_1 = O\left(\varepsilon^\beta\,\mu\right)$.
2) For the terms in the denominator of $\mathcal{Q}[\rho,M]$ we note that

$$\int_{\mathbb{R}^N}\left(\rho_*(x) + \varepsilon^{-N}\mu\,\sigma(x/\varepsilon)\right)dx + (M_*-\mu) = \int_{\mathbb{R}^N}\rho_*\,dx + M_*$$

and, by Lemma 12 applied with $p=1$,

$$\int_{\mathbb{R}^N}\left(\rho_*(x) + \varepsilon^{-N}\mu\,\sigma(x/\varepsilon)\right)^q\,dx = \int_{\mathbb{R}^N}\rho_*^q\,dx + \varepsilon^{N(1-q)}\mu^q\int_{\mathbb{R}^N}\sigma^q\,dx + o\left(\varepsilon^{N(1-q)}\mu^q\right).$$

Thus,

$$\left(\int_{\mathbb{R}^N}\left(\rho_*(x) + \varepsilon^{-N}\mu\,\sigma(x/\varepsilon)\right)^q\,dx\right)^{-\frac{2-\alpha}{q}} \\ = \left(\int_{\mathbb{R}^N}\rho_*^q\,dx\right)^{-\frac{2-\alpha}{q}}\left(1 - \frac{2-\alpha}{q}\,\varepsilon^{N(1-q)}\mu^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx} + R_2\right)$$

with $R_2 = o\left(\varepsilon^{N(1-q)}\mu^q\right)$.

Now we collect the estimates. Since (ρ_*, M_*) is a minimizer, we obtain that

$$\mathcal{Q}\left[\rho_* + \varepsilon^{-N}\mu\,\sigma(\cdot/\varepsilon), M_*-\mu\right] = \mathcal{C}_{N,\lambda,q}\left(1 - \frac{2-\alpha}{q}\,\varepsilon^{N(1-q)}\mu^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx} + R_2\right) \\ + R_1\left(\int_{\mathbb{R}^N}\rho_*\,dx + M_*\right)^{-\alpha}\left(\int_{\mathbb{R}^N}\left(\rho_*(x) + \varepsilon^{-N}\mu\,\sigma(x/\varepsilon)\right)^q\,dx\right)^{-\frac{2-\alpha}{q}}.$$

If $\beta = \min\{2,\lambda\} > N(1-q)$, we can choose μ to be a fixed number in $(0, M_*)$, so that $R_1 = o\big(\varepsilon^{N(1-q)}\big)$ and therefore

$$\mathcal{Q}\left[\rho_* + \varepsilon^{-N}\mu\,\sigma(\cdot/\varepsilon), M_* - \mu\right] \le \mathcal{C}_{N,\lambda,q}\left(1 - \frac{2-\alpha}{q}\,\varepsilon^{N(1-q)}\mu^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx} + o\big(\varepsilon^{N(1-q)}\big)\right).$$

Since $\alpha < 2$, this is strictly less than $\mathcal{C}_{N,\lambda,q}$ for $\varepsilon > 0$ small enough, contradicting the definition of $\mathcal{C}_{N,\lambda,q}$ as an infimum. Thus, $M_* = 0$.

Note that if either $N = 1$, 2 or $N \ge 3$ and $\lambda \le 2N/(N-2)$, then the assumption $q > N/(N+\lambda)$ implies that $\beta > N(1-q)$. If $N \ge 3$ and $\lambda > 2N/(N-2)$, then $\beta = 2 \ge N(1-q)$ by assumption. Thus, it remains to deal with the case where $N \ge 3$, $\lambda > 2N/(N-2)$ and $2 = N(1-q)$. In this case we have $R_1 = O\big(\varepsilon^2\mu\big)$ and therefore

$$\mathcal{Q}\left[\rho_* + \varepsilon^{-N}\mu\,\sigma(\cdot/\varepsilon), M_* - \mu\right] \le \mathcal{C}_{N,\lambda,q}\left(1 - \frac{2-\alpha}{q}\,\varepsilon^2\,\mu^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx} + O\big(\varepsilon^2\mu\big)\right).$$

By choosing μ small (but independently of ε) we obtain a contradiction as before. This completes the proof of the proposition. □

Remark 13. In the proof of Proposition 11, we used the bound $R_1 = O\big(\varepsilon^2\mu\big)$. For any $\lambda \ge 2$, this bound is optimal. Namely, one has

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho_*(x)\left(|x-y|^\lambda - |x|^\lambda\right)\varepsilon^{-N}\sigma(y/\varepsilon)\,dx\,dy$$
$$= \varepsilon^2\,\frac{\lambda}{2}\left(1 + \frac{\lambda-2}{N}\right)\int_{\mathbb{R}^N}|x|^{\lambda-2}\rho_*(x)\,dx\int_{\mathbb{R}^N}|y|^2\sigma(y)\,dy + o\big(\varepsilon^2\big)$$

for $\lambda \ge 2$. This follows from the fact that, for any given $x \ne 0$,

$$|x-y|^\lambda - |x|^\lambda = -\lambda\,|x|^{\lambda-2}x\cdot y + \frac{\lambda}{2}\,|x|^{\lambda-2}\left(|y|^2 + (\lambda-2)\frac{(x\cdot y)^2}{|x|^2}\right) + O\left(|y|^{\min\{3,\lambda\}} + |y|^\lambda\right).$$

The assumption $\beta \ge N(1-q)$ is dictated by the ε^2 behavior of R_1, for $\lambda \ge 2$, which cannot be improved.

3. Additional results

In this section we discuss the existence of a minimizer in the regime that is not covered by Proposition 11. In particular, we will find a connection between the regularity of a minimizer of the relaxed problem and the presence or absence of a delta mass, and we will also establish the existence of a minimizer in a certain region which is not covered by Proposition 11.

Proposition 14. *Let $N \ge 3$, $\lambda > 2N/(N-2)$ and $N/(N+\lambda) < q < \min\big\{1 - 2/N, 2N/(2N+\lambda)\big\}$. If (ρ_*, M_*) is a minimizer for $\mathcal{C}^{\mathrm{rel}}_{N,\lambda,q}$ such that $(\rho_*, M_*) \in \mathrm{L}^{N(1-q)/2}(\mathbb{R}^N)\times[0,+\infty)$, then $M_* = 0$.*

The condition that the. minimizer (ρ_*, M_*) of $\mathcal{C}^{\mathrm{rel}}_{N,\lambda,q}$ belongs to $\mathrm{L}^{N(1-q)/2}(\mathbb{R}^N)\times[0,+\infty)$ has to be understood as a regularity condition on ρ_*.

Proof. We argue by contradiction assuming that $M_*>0$ and consider a test function $\big(\rho_*+\varepsilon^{-N}\mu_\varepsilon\,\sigma(\cdot/\varepsilon),M^*-\mu_\varepsilon\big)$ such that $\int_{\mathbb{R}^N}\sigma\,dx=1$. We choose $\mu_\varepsilon=\mu_1\,\varepsilon^{N-2/(1-q)}$ with a constant μ_1 to be determined below. As in the proof of Proposition 11, one has

$$I_\lambda\left[\rho_*+\varepsilon^{-N}\mu_\varepsilon\,\sigma(\cdot/\varepsilon)\right]+2(M_*-\mu_\varepsilon)\int_{\mathbb{R}^N}|x|^\lambda\left(\rho_*(x)+\varepsilon^{-N}\sigma(x/\varepsilon)\right)dx\\ =I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx+R_1$$

with $R_1=O\big(\varepsilon^2\mu_\varepsilon\big)$. Note that here we have $\lambda\ge 2$. For the terms in the denominator we note that

$$\int_{\mathbb{R}^N}\left(\rho_*(x)+\varepsilon^{-N}\mu_\varepsilon\,\sigma(x/\varepsilon)\right)dx+(M_*-\mu_\varepsilon)=\int_{\mathbb{R}^N}\rho_*\,dx+M_*$$

and, by Lemma 12 applied with $p=N(1-q)/2$ and $\mu=\mu_\varepsilon$, *i.e.*, $\varepsilon^{-N}\mu_\varepsilon=\varepsilon^{-N/p}\mu_1$, we have

$$\int_{\mathbb{R}^N}\left(\rho_*(x)+\varepsilon^{-N}\mu_\varepsilon\,\sigma(x/\varepsilon)\right)^q\,dx=\int_{\mathbb{R}^N}\rho_*^q\,dx+\varepsilon^{N(1-q)}\mu_\varepsilon^q\int_{\mathbb{R}^N}\sigma^q\,dx+o\left(\varepsilon^{N(1-q)}\mu_\varepsilon^q\right).$$

Because of the choice of μ_ε we have

$$\varepsilon^{N(1-q)}\mu_\varepsilon^q=\varepsilon^\gamma\mu_1^q\quad\text{and}\quad\varepsilon^2\mu_\varepsilon=\varepsilon^\gamma\mu_1\quad\text{with}\quad\gamma:=\frac{N-q\,(N+2)}{1-q}>0$$

and thus

$$\mathcal{Q}\left[\rho_*+\varepsilon^{-N}\mu_\varepsilon\,\sigma(\cdot/\varepsilon),M_*-\mu_\varepsilon\right]\le\mathcal{C}_{N,\lambda,q}\left(1-\frac{2-\alpha}{q}\,\varepsilon^\gamma\mu_1^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx}+O\big(\varepsilon^\gamma\mu_1\big)\right).$$

By choosing μ_1 small (but independent of ε) we obtain a contradiction as before. □

Proposition 14 motivates the investigation of the regularity of the minimizer (ρ_*,M_*) of $\mathcal{C}^{\rm rel}_{N,\lambda,q}$. We are not able to prove the regularity required in Proposition 14, but we can state a dichotomy result which is interesting by itself.

Proposition 15. *Let $N\ge 3$, $\lambda>2N/(N-2)$ and $N/(N+\lambda)<q<\min\{1-2/N,2N/(2N+\lambda)\}$. Let (ρ_*,M_*) be a minimizer for $\mathcal{C}^{\rm rel}_{N,\lambda,q}$. Then the following holds:*

(1) *If $\int_{\mathbb{R}^N}\rho_*\,dx>\frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$, then $M_*=0$ and ρ_* is bounded with*

$$\rho_*(0)=\left(\frac{(2-\alpha)\,I_\lambda[\rho_*]\int_{\mathbb{R}^N}\rho_*\,dx}{\left(\int_{\mathbb{R}^N}\rho_*^q\,dx\right)\left(2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx\int_{\mathbb{R}^N}\rho_*\,dx-\alpha\,I_\lambda[\rho_*]\right)}\right)^{1/(1-q)}$$

(2) *If $\int_{\mathbb{R}^N}\rho_*\,dx=\frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$, then $M_*=0$ and ρ_* is unbounded with*

$$\rho_*(x)=C\,|x|^{-2/(1-q)}\left(1+o(1)\right)\quad as\quad x\to 0$$

for some $C>0$.

(3) *If $\int_{\mathbb{R}^N}\rho_*\,dx<\frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$, then*

$$M_*=\frac{\alpha\,I_\lambda[\rho_*]-2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx\,\int_{\mathbb{R}^N}\rho_*\,dx}{2\,(1-\alpha)\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}>0$$

and ρ_* is unbounded with

$$\rho_*(x) = C\,|x|^{-2/(1-q)}\big(1+o(1)\big) \quad as \quad x\to 0$$

for some $C>0$.

Notice that the restrictions on q and λ in Proposition 15 are intended to cover the cases which are not already dealt with in Proposition 11. The only assumptions that we shall use are $0<\alpha<1$ and $\lambda>2$. To prove Proposition 15, let us begin with an elementary lemma.

Lemma 16. *For constants A, $B>0$ and $0<\alpha<1$, define*

$$f(M) = \frac{A+M}{(B+M)^\alpha} \quad for \quad M\ge 0\,.$$

Then f attains its minimum on $[0,\infty)$ at $M=0$ if $\alpha A\le B$ and at $M=(\alpha A-B)/(1-\alpha)>0$ if $\alpha A>B$.

Proof. We consider the function on the larger interval $(-B,\infty)$. Let us compute

$$f'(M) = \frac{(B+M)-\alpha(A+M)}{(B+M)^{\alpha+1}} = \frac{B-\alpha A+(1-\alpha)M}{(B+M)^{\alpha+1}}\,.$$

Note that the denominator of the right side vanishes exactly at $M=(\alpha A-B)/(1-\alpha)$, except possibly if this number coincides with $-B$.

We distinguish two cases. If $A\le B$, which is the same as $(\alpha A-B)/(1-\alpha)\le -B$, then f is increasing on $(-B,\infty)$ and then f indeed attains its minimum on $[0,\infty)$ at 0. Thus it remains to deal with the other case, $A>B$. Then f is decreasing on $\big(-B,(\alpha A-B)/(1-\alpha)\big]$ and increasing on $\big[(\alpha A-B)/(1-\alpha),\infty\big)$. Therefore, if $\alpha A-B\le 0$, then f is increasing on $[0,\infty)$ and again the minimum is attained at 0. On the other hand, if $\alpha A-B>0$, then f has a minimum at the positive number $M=(\alpha A-B)/(1-\alpha)$. □

Proof of Proposition 15. Step 1. We vary $\mathcal{Q}[\rho_*,M]$ with respect to M. By the minimizing property the function

$$M\mapsto \mathcal{Q}[\rho_*,M] = \frac{2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}{\left(\int_{\mathbb{R}^N}\rho_*^q\,dx\right)^{(2-\alpha)/q}}\,\frac{A+M}{(B+M)^\alpha}$$

with

$$A := \frac{I_\lambda[\rho_*]}{2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx} \quad\text{and}\quad B := \int_{\mathbb{R}^N}\rho_*(x)\,dx$$

attains its minimum on $[0,\infty)$ at M_*. From Lemma 16 we infer that

$$M_*=0 \quad\text{if and only if}\quad \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx} \le \int_{\mathbb{R}^N}\rho_*(x)\,dx\,,$$

and that $M_* = \frac{\alpha I_\lambda[\rho_*]-2\left(\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx\right)\left(\int_{\mathbb{R}^N}\rho_*(y)\,dy\right)}{2\,(1-\alpha)\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$ if $\alpha\,\frac{I_\lambda[\rho_*]}{2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx} > \int_{\mathbb{R}^N}\rho_*(x)\,dx.$

Step 2. We vary $\mathcal{Q}[\rho,M_*]$ with respect to ρ. We begin by observing that ρ_* is positive almost everywhere according to Lemma 9. Because of the positivity of ρ_* we obtain the

Euler–Lagrange equation on $\mathbb{R}^N$,

$$2\,\frac{\int_{\mathbb{R}^N}|x-y|^\lambda\rho_*(y)\,dy+M_*|x|^\lambda}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy}-\alpha\,\frac{1}{\int_{\mathbb{R}^N}\rho_*\,dy+M_*}-(2-\alpha)\,\frac{\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}=0\,.$$

Letting $x\to 0$, we find that

$$2\,\frac{\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy}-\alpha\,\frac{1}{\int_{\mathbb{R}^N}\rho_*(y)\,dy+M_*}=(2-\alpha)\,\frac{\rho_*(0)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}\ge 0\,,$$

with equality if and only if ρ_* is unbounded. We can rewrite this as

$$M_*\ge\frac{\alpha I_\lambda[\rho_*]-2\left(\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy\right)\left(\int_{\mathbb{R}^N}\rho_*\,dy\right)}{2\,(1-\alpha)\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy}$$

with equality if and only if ρ_* is unbounded.

Step 3. Combining Steps 1 and 2 we obtain all the assertions of Proposition 15 except for the behavior of ρ_* in the unbounded case. To compute the behavior near the origin we obtain, similarly as in Remark 13, using $\lambda>2$,

$$\int_{\mathbb{R}^N}|x-y|^\lambda\rho_*(y)\,dy+M_*|x|^\lambda=\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy+C\,|x|^2\left(1+o(1)\right)\quad\text{as}\quad x\to 0\,,$$

with

$$C:=\frac{1}{2}\,\lambda\,(\lambda-1)\int_{\mathbb{R}^N}|y|^{\lambda-2}\rho_*(y)\,dy\,.$$

Thus, the Euler–Lagrange equation from Step 2 becomes

$$\frac{2C\,|x|^2\left(1+o(1)\right)}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\rho_*(y)\,dy}=(2-\alpha)\,\frac{\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}\quad\text{as}\quad x\to 0\,.$$

This completes the proof of Proposition 15. □

For any $\lambda>1$ we deduce from

$$|x-y|^\lambda\le\left(|x|+|y|\right)^\lambda\le 2^{\lambda-1}\left(|x|^\lambda+|y|^\lambda\right)$$

that

$$I_\lambda[\rho]<2^\lambda\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx\int_{\mathbb{R}^N}\rho(x)\,dx\,.$$

For all $\alpha\le 2^{-\lambda+1}$, which can be translated into

$$q>\frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda}\,,$$

Case (1) of Proposition 15 applies and we know that $M_*=0$. Note that this bound for q is in the range $\left(N/(N+\lambda)\,,2N/(2N+\lambda)\right)$ for all $\lambda>1$.

A better range for which $M_*=0$ can be obtained as follows when $N\ge 3$. The superlevel sets of a symmetric non-increasing function are balls. From the layer cake representation we deduce that

$$I_\lambda[\rho]\le 2\,A_{N,\lambda}\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx\int_{\mathbb{R}^N}\rho(x)\,dx$$

where

$$A_{N,\lambda}=\sup_{0\le R,S<\infty}F(R,S)\quad\text{where}\quad F(R,S):=\frac{\iint_{B_R\times B_S}|x-y|^\lambda\,dx\,dy}{|B_R|\int_{B_S}|x|^\lambda\,dx+|B_S|\int_{B_R}|y|^\lambda\,dy}\,.$$

For any $\lambda>1$, we have $2\,A_{N,\lambda}\le 2^\lambda$, and also $A_{N,\lambda}\ge 1/2$ because $I_\lambda[\mathbb{1}_{B_1}]\ge|B_1|\int_{B_1}|y|^\lambda\,dy$. The bound $A_{N,\lambda}\ge 1/2$ can be improved to $A_{N,\lambda}>1$ for any $\lambda>2$ as follows. We know that

$$A_{N,\lambda}\ge F(1,1)=\frac{N\,(N+\lambda)}{2}\iint_{0\le r,\,s\le 1}r^{N-1}\,s^{N-1}\left(\int_0^\pi\left(r^2+s^2-2\,r\,s\cos\varphi\right)^{\lambda/2}\frac{\sin\varphi^{N-2}\,d\varphi}{W_N}\right)dr\,ds\,.$$

where W_N is the Wallis integral $W_N:=\int_0^\pi\sin\varphi^{N-2}\,d\varphi$. For any $\lambda>2$, we can apply Jensen's inequality twice and obtain

$$\int_0^\pi\left(r^2+s^2-2\,r\,s\cos\varphi\right)^{\lambda/2}\frac{\sin\varphi^{N-2}\,d\varphi}{W_N}\ge\left(\int_0^\pi\left(r^2+s^2-2\,r\,s\cos\varphi\right)\frac{\sin\varphi^{N-2}\,d\varphi}{W_N}\right)^{\lambda/2}=\left(r^2+s^2\right)^{\lambda/2}$$

and

$$\iint_{0\le r,\,s\le 1}r^{N-1}\,s^{N-1}\left(r^2+s^2\right)^{\lambda/2}dr\,ds\ge\frac{1}{N^2}\left(\iint_{0\le r,\,s\le 1}r^{N-1}\,s^{N-1}\left(r^2+s^2\right)N^2\,dr\,ds\right)^{\lambda/2}=\frac{1}{N^2}\left(\frac{2\,N}{N+2}\right)^{\lambda/2}.$$

Hence

$$A_{N,\lambda}\ge\frac{N+\lambda}{2\,N}\left(\frac{2\,N}{N+2}\right)^{\lambda/2}:=B_{N,\lambda}$$

where $\lambda\mapsto B_{N,\lambda}$ is monotone increasing, so that $A_{N,\lambda}\ge B_{N,\lambda}>B_{N,2}=1$ for any $\lambda>2$. In this range we can therefore define

$$\bar q(\lambda,N):=\frac{2N\left(1-A_{N,\lambda}^{-1}\right)}{2N\left(1-A_{N,\lambda}^{-1}\right)+\lambda}\,.$$

Based on a numerical computation, the curve $\lambda\mapsto\bar q(\lambda,N)$ is shown on Fig. 1. The next result summarizes the above considerations.

Proposition 17. *Assume that $N\ge3$ and $\lambda>2N/(N-2)$. Then, with the above notations,*

$$\bar q(\lambda,N)\le\frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda}<\frac{2\,N}{2\,N+\lambda}$$

and, for $\lambda>2$ large enough,

$$\bar q(\lambda,N)>\frac{N}{N+\lambda}\,.$$

If q is such that $\max\left\{\bar q(\lambda,N),N/(N+\lambda)\right\}<q<\min\left\{1-2/N\,,2N/(2N+\lambda)\right\}$ and if (ρ_*,M_*) is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$, then $M_*=0$ and $\rho_*\in\mathrm{L}^\infty(\mathbb{R}^N)$.

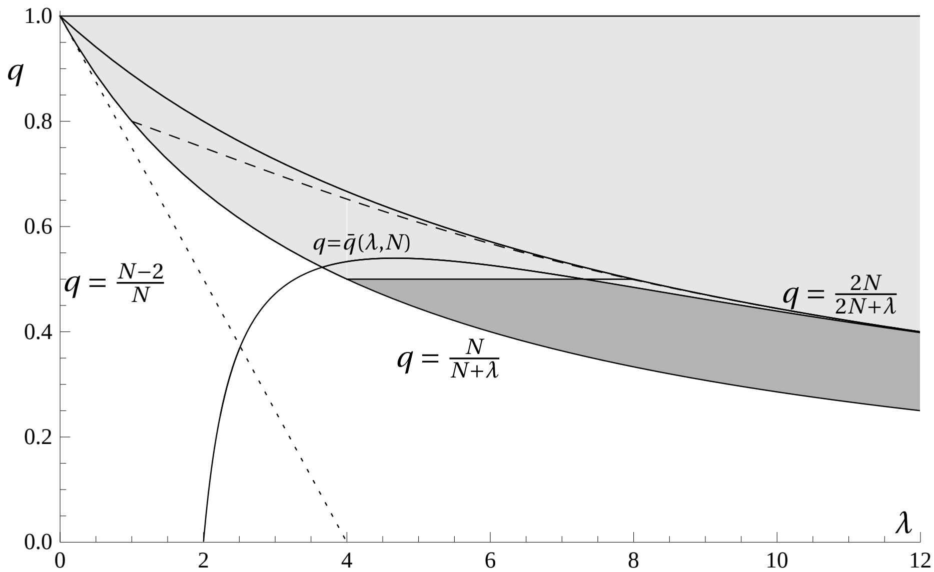

FIGURE 1. Main regions of the parameters (here $N = 4$). The case $q = 2N/(2N+\lambda)$ has already been treated in [7, 19]. Inequality (1) holds with a positive constant $\mathscr{C}_{N,\lambda,q}$ if $q > N/(N+\lambda)$, which determines the admissible range corresponding to the grey area, and it is achieved by a function ρ (without any Dirac mass) in the light grey area. The dotted line is $q = 1-\lambda/N$: it is tangent to the admissible range of parameters at $(\lambda, q) = (0,1)$. In the dark grey region, Dirac masses with $M_* > 0$ are not excluded. The dashed curve corresponds to the curve $q = 2N\left(1-2^{-\lambda}\right)/\left(2N\left(1-2^{-\lambda}\right)+\lambda\right)$ and can hardly be distinguished from $q = 2N/(2N+\lambda)$ when q is below $1-2/N$. The curve $q = \bar{q}(\lambda, N)$ of Corollary 17 is also represented . Above this curve, no Dirac mass appears when minimizing the relaxed problem corresponding to (1). Whether Dirac masses appear in the region which is not covered by Corollary 17 is an open question.

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