Mathematical Models and Methods in Applied Sciences
Vol. 1, No. 2 (1991) 183–208

STATIONARY STATES IN PLASMA PHYSICS: MAXWELLIAN SOLUTIONS OF THE VLASOV–POISSON SYSTEM

J. DOLBEAULT
CEREMADE, Université de Paris IX–Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
and
URA 1408, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex, France

Received 25 January 1991

We study the Maxwellian solutions of the stationary Vlasov–Poisson system, which describes stationary states for plasma. We prove existence, uniqueness and regularity results for these solutions.

Notations:

In this paper, we denote the derivative with respect to the time t by ∂_t and the gradient with respect to the position x by ∂_x. We do not specify the target space for the functional spaces when it is $\mathbb{R}$: $L^p(\mathbb{R}^N)=L^p(\mathbb{R}^N,\mathbb{R})$. χ_A is the characteristic function of the set A. The Marcinkiewicz space $L^{p,\infty}(\Omega)$ is defined for all $p\in]1,+\infty[$ by

$$L^{p,\infty}(\Omega) = \{f\in L^1_{\mathrm{loc}}(\Omega)\ |\sup_{\lambda>0}\lambda\cdot \mathrm{meas}\ \{x\in\Omega|\ |f(x)| > \lambda\}^{1/p} < \infty\}\ .$$

1. Introduction

In this paper, we study the stationary solutions for a simple plasma model. A plasma is a gas of charged particles interacting through electromagnetic forces. The study of stationary solutions of several kinetic models (see Refs. 15 and 17) leads to the Vlasov–Poisson model. The Vlasov equation is a first order partial differential equation which describes the evolution of the density f of the plasma in the phase space:

$$\partial_t f + \xi\cdot \partial_x f + (E(t,x) + E_0(x))\cdot \partial_\xi f = 0\,, \qquad (V)$$

Keywords: plasma physics, stationary states, Maxwellian solutions, Vlasov-Poisson system.

where E satisfies the Poisson equation:

$$\operatorname{div} E = \rho \tag{P}$$

and E_0 denotes an external electric field; the spatial density of the plasma ρ is given by

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,\xi)\, d\xi .$$

The Vlasov–Poisson system (see Ref. 11) describes the state of a rarefied gas of charged particles interacting with an external electric field E_0 and an electric field E created by the particles themselves. The field E_0 models another species of particles which are supposed to be stationary. This model gives a good approximation of the dynamics of a gas of charged light particles (say electrons) moving in a background of charged heavy particles (say ions).

The description is done at the kinetic level: f is the density of the particles which at time t and point x move with velocity ξ. We assume that the particles remain in an open set Ω and we are interested in the non-relativistic case: ξ belongs to $\mathbb{R}^N$. It is natural to assume that f is non-negative and that the solution has a finite total mass:

$$\int_{\Omega} \rho(t,x)\, dx .$$

In the following, we restrict ourselves to the study of stationary Maxwellian solutions of the Vlasov–Poisson system. We prove results of existence (using variational methods and representation of the solution with the Green function) and uniqueness (using considerations of convexity) when Ω is $\mathbb{R}^N$ or when it is a regular bounded open set of $\mathbb{R}^N$. These results are extensions of theorems obtained by Dressler in Ref. 15 (case $\Omega = \mathbb{R}^N$) and by Gogny and Lions in Ref. 17 (bounded case). In appendix A, we shall go back to the model and explain why Maxwellian solutions of the Vlasov–Poisson system are particularly interesting. We shall also generalize our results to the case of several species of light particles (see appendix B).

Assuming that the mean velocity of the particles (at point x) is zero (which is natural if Ω is bounded and not radially symmetric), stationary Maxwellian solutions are given (for all $t \in \mathbb{R}$) by

$$f(t,x,\xi) = m(x,\xi) ,$$

$$m(x,\xi) = \frac{1}{(2\pi T)^{N/2}} \cdot \rho(x) \cdot e^{-|\xi|^2/(2T)}, \qquad (x,\xi) \in \Omega \times \mathbb{R}^N ,$$

where Ω is an open set of $\mathbb{R}^N$. T is the temperature ($T>0$) and of course

$$\rho(x) = \int_{\mathbb{R}^N} m(x, \xi)\, d\xi .$$

The problem reduces to solving the following system:

$$\partial_x \rho = \frac{1}{T}(E + E_0)\rho ,$$

$$\operatorname{div} E = \rho .$$

Assuming that there exists a potential U_0 such that

$$E_0 = -\partial_x U_0 ,$$

it is enough to prove that there exists a potential U (with $E = -\partial_x U$) and a density ρ which satisfies

$$\partial_x \ln \rho = -\frac{1}{T}(\partial_x U + \partial_x U_0) ,$$

$$-\Delta_x U = \rho .$$

By normalizing ρ in the L^1-norm (see remark 2 if there is no renormalization)

$$\rho = \frac{e^{-(U+U_0)/T}}{\int_\Omega e^{-(U+U_0)/T} dx} ,$$

the problem reduces to solving the following equation:

$$-\Delta U = \frac{\rho_0 e^{-U/T}}{\int_\Omega \rho_0 e^{-U/T} dx} \qquad (x \in \Omega) , \tag{0a}$$

where ρ_0 is defined by setting

$$\rho_0 = e^{-U_0/T} ,$$

or, replacing U by TV:

$$-T\Delta V = \frac{\rho_0 e^{-V}}{\int_\Omega \rho_0 e^{-V} dx} \qquad (x \in \Omega) . \tag{0b}$$

The case $T \neq 1$ can be treated by the same method as the case $T=1$: in the following, we shall suppose that $T=1$. We therefore have to solve

$$-\Delta U = \frac{\rho_0 e^{-U}}{\int_\Omega \rho_0 e^{-U} dx} \qquad (x \in \Omega) . \tag{1}$$

Let us note that if $\Omega = \mathbb{R}^N$ with $N \geq 3$, Eq. (1) is equivalent to

$$\rho = \frac{\rho_0 e^{-g_N * \rho}}{\int_\Omega \rho_0 e^{-g_N * \rho} dx} \qquad (x \in \Omega) ,$$

where g_N is the Green function

$$g_N = \frac{|S^{N-1}|^{-1}}{|x|^{N-2}} .$$

This last situation is a particular case of the generalized Poisson–Boltzmann–Emden equation, which can be derived in several ways (see Ref. 2).

2. Existence and Uniqueness in $\mathbb{R}^N$ ($N \geq 3$)

We assume that $\Omega = \mathbb{R}^N$. In the following theorem, we give an extension (see Ref. 15) of Dressler's existence and uniqueness result (see also appendix A: the Vlasov–Fokker–Planck model for a precise statement of Dressler's theorem).

Theorem. Let ρ_0 be a non-negative function of $L^1(\mathbb{R}^N)$ with $N \geq 3$, such that ρ_0 is not identically equal to 0. Then there exists a solution of the equation:

$$-\Delta U = \frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N} \rho_0 e^{-U} dx} \tag{1a}$$

in $L^{N/(N-2),\infty}(\mathbb{R}^N)$, and ∇U belongs to $L^{N/(N-1),\infty}(\mathbb{R}^N)$.

The solution is unique in $L^{N/(N-2),\infty}(\mathbb{R}^N)$, up to an additive constant. Moreover, the following limit exists

$$U_\infty = \lim_{r \to +\infty} \Big(\sup_{x \in B(0, r+1)^c} \|U\|_{L^1(B(x,1))} \Big) ,$$

and we have:

$$U(x) = U_\infty + \frac{|S^{N-1}|^{-1}}{|x|^{N-2}} * \frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N} \rho_0 e^{-U} dx} .$$

Proof:

1st step: We assume that ρ_0 is a function of $L^1 \cap L^{2N/(N+2)}(\mathbb{R}^N)$. It is not restrictive to assume that

$$||\rho_0||_{L^1(\mathbb{R}^N)} = 1$$

(replacing of ρ_0 by $\dfrac{\rho_0}{||\rho_0||_{L^1(\mathbb{R}^N)}}$ does not change Eq. (1a)). Let us prove the existence of a positive solution of Eq. (1a) in $D^{1,2}(\mathbb{R}^N) = \{V \in L^{2N/(N-2)}(\mathbb{R}^N) \mid \nabla V \in L^2(\mathbb{R}^N)\}$. Let us consider a function V of $D^{1,2}(\mathbb{R}^N)$ and define the functional J by setting:

$$J(V) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla V|^2\, dx + \ln\left(\int_{\mathbb{R}^N} \rho_0 e^{-V}\, dx\right).$$

J is bounded below on $D^{1,2}(\mathbb{R}^N)$. Indeed, by Jensen's inequality, we have

$$\int_{\mathbb{R}^N} \rho_0\, e^{-V}\, dx \geq \exp - \int_{\mathbb{R}^N} \rho_0\, dx\,.$$

Hölder's inequality ensures that

$$\int_{\mathbb{R}^N} \rho_0\, V dx \leq ||\rho_0||_{L^{2N/(N+2)}(\mathbb{R}^N)} ||V||_{L^{2N/(N-2)}(\mathbb{R}^N)}\,,$$

and using Sobolev's embedding,

$$||V||_{L^{2N/(N-2)}(\mathbb{R}^N)} \leq C(N) \cdot ||\Delta V||_{L^2(\mathbb{R}^N)}\,, \tag{2}$$

where $C(N)$ is a strictly positive constant, we get

$$\int_{\mathbb{R}^N} \rho_0\, V dx \leq C(N) ||\rho_0||_{L^{2N/(N+2)}(\mathbb{R}^N)} \cdot ||\nabla V||_{L^2(\mathbb{R}^N)}\,.$$

We therefore have

$$J(V) \geq \frac{1}{2}||\nabla V||^2_{L^2(\mathbb{R}^N)} - C||\nabla V||_{L^2(\mathbb{R}^N)}\,, \tag{3}$$

with

$$C = C(N) \cdot ||\rho_0||_{L^{2N/(N+2)}(\mathbb{R}^N)}\,,$$

and consequently,

$$J(V) \geq -\frac{1}{2} C^2 .$$

Let us consider a minimizing sequence $(U_n)_{n \in \mathbb{N}}$:

$$\lim_{n \to +\infty} J(U_n) = \inf_{V \in D^{1,2}(\mathbb{R}^N)} J(V) .$$

It is not restrictive to assume that, for all $n \in \mathbb{N}$,

$$J(U_n) \leq J(0) = 0 .$$

Using Eq. (3), we obtain

$$\|\nabla U_n\|_{L^2(\mathbb{R}^N)} \leq 2C \qquad \forall\, n \in \mathbb{N} .$$

Moreover, it is clear that

$$\forall\, V \in D^{1,2}(\mathbb{R}^N), \qquad J(V^+) \leq J(V) ,$$

and we can then assume, without loss of generality, that

$$\forall\, n \in \mathbb{N}, \qquad U_n \geq 0 .$$

One can note that $D^{1,2}(\mathbb{R}^N) \subset H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Rellich–Kondrachov's theorem ensures that $(U_n)_{n \in \mathbb{N}}$ is strongly relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and therefore converges almost everywhere in $\mathbb{R}^N$, after extraction of a subsequence if necessary: there exists a non-negative function U of $D^{1,2}(\mathbb{R}^N)$ such that

$$\nabla U_n \to \nabla U \quad \text{weakly in} \quad L^2(\mathbb{R}^N) ,$$

$$U_n \to U \quad \text{a.e..}$$

The lower semi-continuity of the L^2-norm ensures that

$$\|\nabla U\|_{L^2(\mathbb{R}^N)} \leq \liminf_{n \to +\infty} \|\nabla U_n\|_{L^2(\mathbb{R}^N)}$$

and, using Lebesgue's theorem of dominated convergence, we obtain

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} \rho_0 e^{-U_n}\, dx = \int_{\mathbb{R}^N} \rho_0 e^{-U}\, dx$$

which proves that

$$J(U) = \inf_{V \in D^{1,2}(\mathbb{R}^N)} J(V) .$$

To avoid technical considerations, let us consider the more regular functional J^+ defined on $D^{1,2}(\mathbb{R}^N)$ by setting

$$J^+(V) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla V|^2 \, dx + \ln\left(\int_{\mathbb{R}^N} \rho_0 e^{-V^+} \, dx \right).$$

We have

$$\forall\, V \in D^{1,2}(\mathbb{R}^N), \qquad J(V^+) \leq J^+(V) \leq J(V)$$

and therefore

$$J^+(U) = \inf_{V \in D^{1,2}(\mathbb{R}^N)} J^+(V) .$$

J^+ is a functional of class C^1 on $D^{1,2}(\mathbb{R}^N)$:

$$dJ^+(V) = -\Delta V - \frac{\rho_0 e^{-V^+}}{\int_{\mathbb{R}^N} \rho_0 e^{-V^+} dx} .$$

Consequently we have

$$0 = dJ^+(U) = -\Delta U - \frac{\rho_0 e^{-U^+}}{\int_{\mathbb{R}^N} \rho_0 e^{-U^+} dx} ,$$

and

$$-\Delta U = \frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N} \rho_0 e^{-U} dx}$$

because

$$U \geq 0 \quad \text{a.e.} .$$

<u>2nd step:</u> The solution of the first step is unique in $D^{1,2}(\mathbb{R}^N)$. Indeed, let us define the convex cone D^+ by setting

$$D^+ = \{ V \in D^{1,2}(\mathbb{R}^N) \,|\, V \geq 0 \text{ a.e. on } \mathbb{R}^N \} .$$

By the maximum principle, every solution of (la) is non-negative and therefore belongs to D^+. As a consequence, every solution U of (la) satisfies

$$dJ^+(U) = 0.$$

Now, we have

$$J^+(V) = J(V) \quad \forall V \in D^+ ,$$

and J is strictly convex. Indeed, the L^2-norm is strictly convex and if we consider two functions U_1 and U_2 belonging to $D^{1,2}(\mathbb{R}^N)$ and $t \in [0, 1]$, we get

$$\ln\left(\int_{\mathbb{R}^N} \rho_0 e^{-(tU_1+(1-t)U_2)}\,dx\right) = \ln\left(\int_{\mathbb{R}^N} (\rho_0 e^{-U_1})^t \cdot (\rho_0 e^{-U_2})^{(1-t)}\,dx\right)$$

$$\leq t\ln\left(\int_{\mathbb{R}^N} \rho_0 e^{-U_1}\,dx\right) + (1-t)\ln\left(\int_{\mathbb{R}^N} \rho_0 e^{-U_2}\,dx\right),$$

by Hölder's inequality. Finally

$$J(tU_1 + (1-t)U_2) \leq tJ(U_1) + (1-t)J(U_2),$$

and this inequality is strict if $\|\nabla U_2 - \nabla U_1\|_{L^2(\mathbb{R}^N)} \neq 0$ and $t \in]0, 1[$.

The solution of

$$dJ^+(U) = 0$$

is therefore unique in D^+: the solution of (1a) is unique in $D^{1,2}(\mathbb{R}^N)$ if ρ_0 belongs to $L^1 \cap L^{2N/(N+2)}(\mathbb{R}^N)$.

3rd step: In this part, we use classical results of interpolation theory in the Marcinkiewicz spaces (see Refs. 21, 22 and 26) to prove the existence of a solution of Eq. (1a) in the general case ($\rho_0 \in L^1(\mathbb{R}^N)$).

Let us note that the solution found in the first step (case $\rho_0 \in L^1 \cap L^{2N/(N+2)}(\mathbb{R}^N)$) can be expressed with the Green function g_N:

$$U = g_N * \frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N} \rho_0 e^{-U} dx} \text{ a.e. in } \mathbb{R}^N,$$

where

$$g_N(x) = -\frac{|S^{N-1}|^{-1}}{|x|^{N-2}}.$$

Indeed, $V = g_N * \dfrac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N}\rho_0 e^{-U}dx}$ belongs to $D^{1,2}(\mathbb{R}^N)$ because of

the Hardy–Littlewood–Sobolev inequalities in L^p

Let $0<\mu<N$, $1<p<\dfrac{N}{N-\mu}$ and let q satisfy: $\dfrac{1}{p}+\dfrac{\mu}{N}=1+\dfrac{1}{q}$. Then

$$\left\| \frac{1}{|x|^{\mu}} * v \right\|_{L^q(\mathbb{R}^N)} \le C_0 \|v\|_{L^p(\mathbb{R}^N)} \qquad \forall\, v \in L^p(\mathbb{R}^N)\,.$$

Therefore:

1) $g_N(x)=k\dfrac{1}{|x|^{\mu}}$ with $k=|S^{N-1}|^{-1}$ and $\mu=N-2$. $\rho_0\in L^1\cap L^{2N/(N+2)}(\mathbb{R}^N)$ and U is non-negative: $v=\rho_0 e^{-U}\in L^p(\mathbb{R}^N)$, with $p=\dfrac{2N}{N+2}$, which ensures that V belongs to $L^q(\mathbb{R}^N)$ with $q=\dfrac{2N}{N-2}$.
2) $|\nabla g_N(x)|=\dfrac{k}{N-1}\dfrac{1}{|x|^{\mu}}$ with $\mu=N-1$. $v\in L^p(\mathbb{R}^N)$, with $p=\dfrac{2N}{N+2}$, which ensures that ∇V belongs to $L^q(\mathbb{R}^N)$, with $q=2$.

But U and V are both solutions of Eq. (1a) in $D^{1,2}(\mathbb{R}^N)$, and in $D^{1,2}(\mathbb{R}^N)$ the solution is unique:

$$U = V\,.$$

Since $\dfrac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N}\rho_0 e^{-U}dx}$ belongs to $L^1(\mathbb{R}^N)$, it is obvious that U belongs to $L^{N/(N-2),\infty}$ $(\mathbb{R}^N)$, and that ∇U belongs to $L^{N/(N-1),\infty}(\mathbb{R}^N)$ because of

the Hardy–Littlewood–Sobolev inequalities in $L^{p,\infty}$:

Let $p\in]1,+\infty[$. Then

$$\|f * v\|_{L^{p,\infty}(\mathbb{R}^N)} \le C_0 \|f\|_{L^{p,\infty}(\mathbb{R}^N)} \|v\|_{L^1(\mathbb{R}^N)} \qquad \forall\, f\in L^{p,\infty}(\mathbb{R}^N) \qquad \forall\, v\in L^1(\mathbb{R}^N)\,.$$

Therefore

$$\|\nabla U\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)} \le C_0 \|\nabla g_N\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)} \cdot \left\|\frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N}\rho_0 e^{-U}dx}\right\|_{L^1(\mathbb{R}^N)}$$

$$= C_0 \|\nabla g_N\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)}\,,$$

which does not depend on U.

Now, we only assume that ρ_0 belongs to $L^1(\mathbb{R}^N)$. Let us define ρ^n by setting

$$\rho^n = \min(\rho, n) \qquad (\forall\, n \in \mathbb{N}) .$$

Of course ρ^n belongs to $L^1 \cap L^{2N/(N+2)}(\mathbb{R}^N)$ and the equation

$$-\Delta U = \frac{\rho^n e^{-U}}{\int_{\mathbb{R}^N} \rho^n e^{-U} dx}$$

has a unique solution U^n in $D^{1,2}(\mathbb{R}^N)$, which satisfies

$$\|\nabla U^n\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)} \leq \frac{3N}{N-1} \|\nabla g_N\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)} .$$

But there is a continuous embedding of $L^{N/(N-1),\infty}(\mathbb{R}^N)$ into $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for all q in $[1, \frac{N}{N-1}[$: according to the Rellich–Kondrachov theorem, there exists a function U of $L^{N/(N-1),\infty}(\mathbb{R}^N)$ such that, after extraction of a subsequence if necessary,

$$U^n \to U \quad \text{a.e. in } \mathbb{R}^N .$$

Lebesgue's theorem ensures that

$$\rho^n e^{-U^n} \to \rho_0 e^{-U} \qquad \text{in } L^1(\mathbb{R}^N) ,$$

and therefore U is a solution of Eq. (1a), which satisfies

$$U = g_N \frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N} \rho_0 e^{-U} dx} .$$

<u>4th step:</u> Let us note that $L^{N/(N-2),\infty}(\mathbb{R}^N)$ is continuously embedded in $L^q_{\mathrm{unif}}(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-2}[$: there exists a positive constant C such that for all function V of $L^{N/(N-2),\infty}(\mathbb{R}^N)$

$$\sup_{x \in \mathbb{R}^N} \|V\|_{L^1(B(x,1))} \leq C \|V\|_{L^{N/(N-2),\infty}(\mathbb{R}^N)} < \infty$$

which ensures that the following limit exists

$$V_\infty = \lim_{r \to +\infty} \Big(\sup_{x \in B(0,r+1)^c} \|V\|_{L^1(B(x,1))} \Big) .$$

Moreover, if U is the solution of the 3rd step, then $U_\infty = 0$.

Now, if U is a solution of Eq. (1a), $(U+k)$ is also a solution for all constant $k \in \mathbb{R}$. Indeed

$$\Delta(U + k) = \Delta U ,$$

and

$$\frac{\rho_0 e^{-(U+k)}}{\int_{\mathbb{R}^N} \rho_0 e^{-(U+k)}\, dx} = \frac{\rho_0 e^{-U} \cdot e^{-k}}{\int_{\mathbb{R}^N} \rho_0 e^{-U} \cdot e^{-k}\, dx} = \frac{\rho_0 e^{-U}}{\int_{\mathbb{R}^N} \rho_0 e^{-U}\, dx} .$$

Let us prove the uniqueness of the solution (up to an additive constant). Let U_1 and U_2 be two solutions of Eq. (1a) such that

$$\lim_{r \to +\infty} \Big(\sup_{x \in B(0,r+1)^c} \|U_1\|_{L^1(B(x,1))} \Big) = \lim_{r \to +\infty} \Big(\sup_{x \in B(0,r+1)^c} \|U_2\|_{L^1(B(x,1))} \Big) .$$

If $U_1 = U_2$ a.e. on $[\rho_0 > 0]$, then

$$\Delta U_1 = \Delta U_2 \quad \text{on } \mathbb{R}^N .$$

If meas$[U_1 \neq U_2$ and $\rho_0 > 0] > 0$, it is not restrictive to assume that meas$[U_1 > U_2$ and $\rho_0 > 0] > 0$.

Let us compute

$$- \int_{\mathbb{R}^N} (\Delta U_1 - \Delta U_2)\, \chi_{[U_1 \geq U_2]}\, dx .$$

The maximum principle ensures that

$$- \int_{\mathbb{R}^N} (\Delta U_1 - \Delta U_2)\, \chi_{[U_1 \geq U_2]}\, dx \geq 0 .$$

Using Eq. (1a), we get

$$\begin{aligned} &- \int_{\mathbb{R}^N} (\Delta U_1 - \Delta U_2)\, \chi_{[U_1 \geq U_2]}\, dx \\ &= \int_{\mathbb{R}^N} \left(\frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_1}\, dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_2}\, dy} \right) \chi_{[U_1 \geq U_2]}\, dx . \end{aligned}$$

Let us define U^t by setting, for all $t \in [0,1]$

$$U^t = tU_1 + (1-t)U_2 .$$

Integrating by parts, we have

$$\left(\frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N}\rho_0 e^{-U_1}dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N}\rho_0 e^{-U_2}dy}\right)\chi_{[U_1\geq U_2]} = \int_0^1 \frac{d}{dt}\left\{\frac{\rho_0 e^{-U^t}}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}\right\}\chi_{[U_1\geq U_2]}\,dt ,$$

$$\left(\frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N}\rho_0 e^{-U_1}dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N}\rho_0 e^{-U_2}dy}\right)\chi_{[U_1\geq U_2]} = -\int_0^1 dt\,\frac{\rho_0 e^{-U^t}}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}(U_1-U_2)$$

$$+\int_0^1 dt\,\frac{\rho_0 e^{-U^t}}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}\chi_{[U_1\geq U_2]}\int_{\mathbb{R}^N}\rho_0 e^{-U^t}(U_1-U_2)\,dy$$

because, according to Lebesgue's theorem

$$\frac{d}{dt}\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy = -\int_{\mathbb{R}^N}\rho_0 e^{-U^t}(U_1-U_2)\,dy .$$

Now, using Fubini's theorem, we get

$$\int_{\mathbb{R}^N}dx\int_0^1 dt\left(\frac{\rho_0 e^{-U^t}}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}\right)(U_1-U_2)^+ = \int_0^1 dt\,\frac{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}(U_1-U_2)^+dx}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy},$$

which proves that

$$\int_{\mathbb{R}^N}dx\int_0^1 dt\left(\frac{\rho_0 e^{-U^t}}{(\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy)^2}\chi_{[U_1\geq U_2]}\cdot\int_{\mathbb{R}^N}\rho_0 e^{-U^t}(U_1-U_2)^+\,dy\right)$$

$$=\int_0^1 dt\left(\frac{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}\chi_{[U_1\geq U_2]}\,dx}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}\cdot\frac{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}(U_1-U_2)dx}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}\right)$$

$$\leq\int_0^1 dt\left(1\cdot\frac{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}(U_1-U_2)^+\,dx}{\int_{\mathbb{R}^N}\rho_0 e^{-U^t}dy}\right).$$

Finally

$$\int_{\mathbb{R}^N}\left(\frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N}\rho_0 e^{-U_1}dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N}\rho_0 e^{-U_2}dy}\right)\chi_{[U_1\geq U_2]}\,dx \leq 0 ,$$

and therefore

$$\int_{\mathbb{R}^N} (\Delta U_1 - \Delta U_2)\, \chi_{[U_1 \geq U_2]}\, dx = 0\,,$$

$$\int_{\mathbb{R}^N} \left(\frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_1} dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_2} dy} \right) \chi_{[U_1 \geq U_2]}\, dx = 0\,.$$

More precisely, this last equality is equivalent to

$$\frac{\int_{\mathbb{R}^N} \rho_0 e^{-U_1}\, \chi_{[U_1 \geq U_2]}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_1} \chi_{[U_1 \geq U_2]}\, dy + \int_{\mathbb{R}^N} \rho_0 e^{-U_1} \chi_{[U_1 < U_2]}\, dy}$$

$$= \frac{\int_{\mathbb{R}^N} \rho_0 e^{-U_2}\, \chi_{[U_1 \geq U_2]}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_2} \chi_{[U_1 \geq U_2]}\, dy + \int_{\mathbb{R}^N} \rho_0 e^{-U_2} \chi_{[U_1 < U_2]}\, dy}\, dx\,,$$

or

$$\int_{\mathbb{R}^N} \rho_0 e^{-U_2} \chi_{[U_1 < U_2]}\, dx \cdot \int_{\mathbb{R}^N} \rho_0 e^{-U_1} \chi_{[U_1 \geq U_2]}\, dx$$

$$= \int_{\mathbb{R}^N} \rho_0 e^{-U_1} \chi_{[U_1 < U_2]}\, dx \cdot \int_{\mathbb{R}^N} \rho_0 e^{-U_2} \chi_{[U_1 \geq U_2]}\, dx\,.$$

But meas$[U_1 \geq U_2$ and $\rho_0 > 0] > 0$:

$$\int_{\mathbb{R}^N} \rho_0 e^{-U_1} \chi_{[U_1 \geq U_2]}\, dx \leq \int_{\mathbb{R}^N} \rho_0 e^{-U_2} \chi_{[U_1 \geq U_2]}\, dx$$

which implies

$$\int_{\mathbb{R}^N} \rho_0 e^{-U_2} \chi_{[U_1 < U_2]}\, dx \geq \int_{\mathbb{R}^N} \rho_0 e^{-U_1} \chi_{[U_1 < U_2]}\, dx$$

and therefore

$$\text{meas}[U_1 < U_2 \text{ and } \rho_0 > 0] = 0\,,$$

$$U_1 \geq U_2 \quad \text{a.e. on } [\rho_0 > 0]\,.$$

Using

$$0 = \left(\frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_1} dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_2} dy} \right) \chi_{[U_1 \geq U_2]}$$

$$= \frac{\rho_0 e^{-U_1}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_1} dy} - \frac{\rho_0 e^{-U_2}}{\int_{\mathbb{R}^N} \rho_0 e^{-U_2} dy},$$

we have

$$U_2 = U_1 + \ln \left(\frac{\int_{\mathbb{R}^N} \rho_0 e^{-U_1} dy}{\int_{\mathbb{R}^N} \rho_0 e^{-U_2} dy} \right) \text{ a.e. on } [\rho_0 > 0]$$

and therefore

$$\Delta U_1 = \Delta U_2 \qquad \text{on } \mathbb{R}^N .$$

Using a regularization method and the Harnack inequality, it is easy to prove that $U_1 - U_2$ is constant. According to the condition

$$\lim_{r \to +\infty} \Big(\sup_{x \in B(0, r+1)^c} \|U_1\|_{L^1(B(x, 1))} \Big) = \lim_{r \to +\infty} \Big(\sup_{x \in B(0, r+1)^c} \|U_2\|_{L^1(B(x, 1))} \Big) ,$$

we have

$$U_2 = U_1 \qquad \text{a.e. on } \mathbb{R}^N .$$

The solution of Eq. (1a) is unique up to an additive constant.

Remark 1. Physically, ρ_0 is the asymptotic density of the plasma when the temperature is going to infinity (let us forget the way we obtained ρ_0 and assume that it does not depend on the temperature). Multiplying Eq. (0b) by V and integrating over $\mathbb{R}^N$, we successively get (here, we assume that ρ_0 belongs to $L^1 \cap L^{2N/(N-2)}(\mathbb{R}^N)$):

$$T \|\nabla V\|^2_{L^2(\mathbb{R}^N)} = \frac{\int_{\mathbb{R}^N} \rho_0 V e^{-V} dx}{\int_{\mathbb{R}^N} \rho_0 e^{-V} dx},$$

$$T \|\nabla V\|^2_{L^2(\mathbb{R}^N)} \leq \frac{\int_{V \leq \bar{V}} \rho_0 e^{-V} \bar{V} dx + \int_{\mathbb{R}^N} \rho_0 V e^{-\bar{V}} dx}{\int_{\mathbb{R}^N} \rho_0 e^{-V} dx} .$$

If $\bar{V} = -\ln \left(\int_{\mathbb{R}^N} \rho_0 e^{-V} dx \right)$,

$$T \|\nabla V\|^2_{L^2(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \rho_0 V dx - \ln \left(\int_{\mathbb{R}^N} \rho_0 e^{-V} dx \right).$$

But

$$\ln\left(\int_{\mathbb{R}^N}\rho_0 e^{-V}dx\right) \geq -\int_{\mathbb{R}^N}\rho_0 V\,dx \geq -C\|\nabla V\|_{L^2(\mathbb{R}^N)}$$

and therefore

$$\|\nabla V\|_{L^2(\mathbb{R}^N)} \leq \frac{2C}{T},$$

$$e^{-2C^2/T} \leq \int_{\mathbb{R}^N}\rho_0 e^{-V}dx \leq 1.$$

Now, for all $T>0$, let us denote $V^{(T)}$ the solution of Eq. (0b) $\rho^{(T)}$ the corresponding density

$$\rho^{(T)} = \frac{\rho_0 e^{-V^{(T)}}}{\int_{\mathbb{R}^N}\rho_0 e^{-V^{(T)}}dx}.$$

Finally

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\rho^{(T)}(x) - \rho_0(x)|dx \\
&\leq \int_{\mathbb{R}^N}\rho_0\,(1 - e^{-V^{(T)}})\,dx + \int_{\mathbb{R}^N}\rho_0 e^{-V^{(T)}}\left(\Big(\int_{\mathbb{R}^N}\rho_0 e^{-V^{(T)}}dx\Big)^{-1} - 1\right)dx \\
&\leq 2(1 - \int_{\mathbb{R}^N}\rho_0 e^{-V^{(T)}}dx) \\
&\leq 2(1 - e^{-2C^2/T}),
\end{aligned}$$

which proves that $\rho^{(T)} = \rho_0 e^{-V^{(T)}} \to \rho_0$ in $L^1(\mathbb{R}^N)$ when T goes to $+\infty$.

Remark 2. If we do not impose any normalization on $\rho(x) = \int_{\mathbb{R}^N} m(x,\xi)\,d\xi$, Eq. (0b) is replaced by

$$-T\Delta V = \rho_0 e^{-V} \qquad (x\in\Omega),$$

and the conclusions of the theorem hold (the proof is the same). But in this case the mass of the system is $\|\rho\|_{L^1(\mathbb{R}^N)}$ and

$$\|\rho\|_{L^1(\mathbb{R}^N)} < \|\rho_0\|_{L^1(\mathbb{R}^N)},$$

because $\rho = \rho_0 e^{-V}$ and $V > 0$ a.e.. Moreover, like in remark 1, $\rho^{(T)} = \rho_0 e^{-V^{(T)}} \to \rho_0$ in $L^1(\mathbb{R}^N)$ when T goes to $+\infty$.

3. Some Complementary Results

3.1. *Ω is an open set of $\mathbb{R}^N (N \geq 3)$, but is different from $\mathbb{R}^N$*

Let us assume that ρ_0 is non-negative, belongs to $L^1 \cap L^{2N/(N+2)}(\Omega)$ with $N \geq 3$, and is not identically equal to 0. We have to solve the following problem:

$$-\Delta U = \frac{\rho_0 e^{-U}}{\int_\Omega \rho_0 e^{-U} dx} \qquad \text{on } \Omega \tag{1b}$$

with the boundary condition

$$U = \bar{U} \qquad \text{on } \partial\Omega.$$

If Ω and $\bar{U}$ are such that the equation

$$-\Delta W = 0 \qquad \text{on } \Omega \tag{4}$$

with the boundary condition

$$W = \bar{U} \qquad \text{on } \partial\Omega$$

has a solution in

$$D^{1,2}(\Omega) = \{V \in L^{2N/(N-2)}(\Omega) | \nabla V \in L^2(\Omega)\}$$

and if $\bar{U}$ is bounded from below ($(\bar{U})^-$ belongs to $L^\infty(\partial\Omega)$ or equivalently $e^{-\bar{U}}$ belongs to $L^\infty(\partial\Omega)$), then Eq. (1b) has a unique solution in $D^{1,2}(\Omega)$. Indeed, by the maximum principle, the solution W of Eq. (4) is unique and satisfies

$$W \geq \inf_{x \in \partial\Omega} \bar{U}(x).$$

Let us define V and ρ_1 by setting

$$V = U - W,$$

$$\rho_1 = \rho_0 e^{-W}.$$

ρ_1 belongs to $L^1 \cap L^{2N/(N+2)}(\Omega)$ and we now have to solve

$$-\Delta V = \frac{\rho_1 e^{-V}}{\int_\Omega \rho_1 e^{-V} dx}.$$

The same method as in part 2 applies.

Let us note that if $\partial\Omega$ is regular (of class C^1 by parts) and bounded, a necessary but not sufficient condition for the existence of solutions of Eq. (4) is the following condition:

$$\bar{U} \in L^{(2N-1)/(N-2)}(\partial\Omega).$$

If Ω is a regular bounded open set,

$$D^{1,2}(\Omega) = H^1(\Omega)$$

and therefore, if $\bar{U}$ belongs to $H^{1/2}(\Omega)$ and is bounded below, Eq. (4) has a unique solution in $H^1(\Omega)$. This result is an extension of the case

$$\rho_0 = 1 \qquad \forall x \in \Omega$$

which has already been treated in Ref. 17.

3.2. *Cases N = 1 and N = 2*

The previous results can be extended to the cases $N=1$ and $N=2$, provided that the hypotheses on ρ_0 are modified (see Ref. 2 for explicit expressions of the solution in the case $\rho_0 = 1$).

Case $N=1$

Let us assume for instance that $\Omega =]0,1[$, and that ρ_0 is non-negative, belongs to $L^1(]0,1[)$ and is not identically equal to zero. In this case, $D^{1,24}(]0,1[) = H^1(]0,1[)$ and Eq. (2) has to be changed into

$$\|V\|_{L^2(]0,1[)} \leq C(1) \cdot \|\nabla V\|_{L^2(]0,1[)},$$

where $C(1)$ is a strictly positive constant. The same method as in part 2 applies: Eq. (1b) has a unique solution in $H_0^1(]0,1[)$, and it is not difficult to extend this result to every bounded open set of $\mathbb{R}$.

Case $N=2$

Here Ω is even $\mathbb{R}^2$ or a bounded set of R^2 (in this case, we deal with the homogeneous problem $\bar{U}=0$ on $\partial\Omega$). Let us assume that ρ_0 is non-negative, belongs to $L^1 \cap L^{1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, and is not identically equal to 0. Then Eq. (1b) has a unique solution in

$$D^{1,2}(\Omega) = \{V \in L^{(1+\varepsilon)/\varepsilon} | \nabla V \in L^2(\Omega)\}.$$

Once again, it is enough to prove the following inequality:

$$\ln\left(\int_\Omega \rho_0 e^{-V} dx\right) \geq \ln\|\rho_0\|_{L^1(\Omega)} - C(2, \varepsilon) \frac{\|\rho_0\|_{L^{1+\varepsilon}(\Omega)}}{\|\rho_0\|_{L^1(\Omega)}} \|\nabla V\|_{L^2(\Omega)},$$

where $C(2, \varepsilon)$ is a strictly positive constant. In fact, it is enough to prove only this for functions of class C^1 with compact support. With Hölder's and Sobolev's inequalities, we obtain successively:

$$\int_\Omega \rho_0 V dx \leq \|\rho_0\|_{L^{1+\varepsilon}(\Omega)} \|V\|_{L^{(1+\varepsilon)/\varepsilon}(\Omega)},$$

$$\|V\|_{L^{1+\varepsilon/\varepsilon}(\Omega)} \leq C(2, \varepsilon) \|\nabla V\|_{L^2(\Omega)},$$

and therefore

$$\int_\Omega \rho_0 e^{-V} dx \geq \|\rho_0\|_{L^1(\Omega)} e^{-C(2,\varepsilon)(\|\rho_0\|_{L^{1+\varepsilon}(\Omega)})/\|\rho_0\|_{L^1(\Omega)}},$$

which proves the result.

3.3. *Positivity, regularity, symmetry*

Using the maximum principle and elliptic bootstraping arguments, one can prove results of strict positivity and regularity. If supplementary hypotheses about ρ_0, Ω and possibly U_0 are assumed, one can also prove results of symmetry, using the uniqueness of the solution (for example, if Ω is a ball, ρ_0 is radially symmetric, and $\bar{U}$ is constant, then the solution composed with a rotation is still a solution: the solution is therefore radially symmetric).

Appendix A: Why Maxwellian Solutions?

One can derive directly Eq. (1) (see Ref. 2). In this section, we give some motivations for studying Maxwellian solutions of the Vlasov–Poisson system as steady states of more complicated kinetic equations. Maxwellian solutions often appear in kinetic models which have a dissipation term. The dissipation term is even a collision integral like the Boltzmann collision term (see Refs. 1, 5, and 8) or a phenomenological collision term which contains some Laplacian in the velocity variable ξ (see Refs. 5, 14, 15 and 23) and is equal to 0 if and only if the solution is a Maxwellian (if there is an entropy dissipation term, it is also equal to 0).

The Vlasov–Fokker–Planck model

The Vlasov–Fokker–Planck model describes a plasma in a thermal bath at temperature $T\,(T>0)$. The density f is supposed to obey

$$\partial_t f + \xi \cdot \partial_x f + (E(t,x) + E_0((x)) \cdot \partial_\xi f - \eta \operatorname{div}_\xi (\xi f + \frac{T}{2}\,\partial_\xi f) = 0\,, \qquad (VFP)$$

where E satisfies Poisson's equation:

$$\operatorname{div} E = \rho\,. \qquad (P)$$

The friction term $\operatorname{div}_\xi (\xi f + \frac{T}{2}\,\partial_\xi f)$ imposes only the stationary solutions to be Maxwellian. The friction parameter η $(\eta>0)$ has no influence on these solutions. To be more precise, we give here a result due to Dressler (see Ref. 15) for the derivation of the stationary solutions of (*VFP*) in $\mathbb{R}^N$. We assume that there exist potentials U and U_0 such that

$$E = -\partial_x U \qquad \text{and} \qquad E_0 = -\partial_x U_0\,.$$

Let $U_0 \in C^k(\mathbb{R}^N)$ for some $k \geq 1$, and be at least linearly growing at infinity (as a consequence, ρ_0 is a very rapidly decreasing function). Then every weak stationary solution $f \in L^1 \cap L^\infty(\mathbb{R}^N)$ of (*VFP*) can be written as a Maxwellian solution of the (*VP*) system. This solution exists and is unique for T big enough. The proof is based on a fixed-point method. Theorem 1 gives a generalization to all temperatures and to a larger class of ρ_0 of the existence and uniqueness result (it allows the study of condensation problems in the limit $T \to 0$).

The Vlasov–Poisson–Boltzmann model

The Vlasov–Poisson–Boltzmann system (see Ref. 6) describes a plasma interacting with an electric field $(E+E_0)$, where E is the field created by the particles themselves, but takes also the collisions between particles into account through a quadratic collision term:

$$\partial_t f + \xi \cdot \partial_x f + (E(x,t) + E_0((x)) \cdot \partial_\xi f = Q(f,f)\,, \qquad (VPB)$$

where the collision term has the classical form (see Ref. 3):

$$Q(f,f) = Q_+(f,f) - Q_-(f,f),$$

$$Q_+(f,f) = \int_{\omega \in S^{N-1}} \int_{\xi_* \in \mathbb{R}^N} q(\xi - \xi_*, \omega) f' f'_*\, d\xi_*\, d\omega\,,$$

$$Q_-(f,f) = \int_{\omega \in S^{N-1}} \int_{\xi_* \in \mathbb{R}^N} q(\xi - \xi_*, \omega) f f_* \, d\xi_* \, d\omega$$

with the notations:

$$f_* = f(t,x,\xi_*)\,,$$

$$f' = f(t,x,\xi')\,,$$

$$f'_* = f(t,x,\xi'_*)\,,$$

and

$$\xi' = \xi - (\xi - \xi_*) \cdot \omega\, \omega\,,$$

$$\xi'_* = \xi_* + (\xi - \xi_*) \cdot \omega\, \omega$$

and $q > 0$ a.e. One can show under technical conditions that a strong solution (see Ref. 6) of (*VPB*) in an open bounded set converges in large time towards a Maxwellian. At the limit, the collision term is null and theorem 2 applies: for all temperature T, there exists a unique solution.

The Vlasov–Maxwell–Boltzmann model

The Vlasov–Maxwell–Boltzmann system (see Refs. 12 and 13) corresponds to a more complicated case, the case when the magnetic field created by the particles themselves cannot be neglected anymore. The density f is supposed to obey

$$\partial_t f + \xi \cdot \partial_x f + F(x,t) \cdot \partial_\xi f = Q(f,f)\,, \qquad (VMB)$$

where F is the Lorentz's force:

$$F(x,t) = E(x,t) + E_0(x) + \xi \wedge B(x,t)$$

and the electromagnetic field $(E + E_0, B)$ is supposed to obey Maxwell's equations:

$$\partial_t E - \operatorname{curl} B = -j\,,$$

$$\partial_t B + \operatorname{curl} E = 0\,,$$

$$\operatorname{div} E = \rho\,,$$

$$\operatorname{div} B = 0\,,$$

where j is given by

$$j(t,x) = \int_{\mathbb{R}^N} \xi f(t,x,\xi)\, d\xi .$$

We will not give here a rigorous derivation of the form of the stationary states (see Ref. 25 for a more general heuristic method). Formally, if we are interested in searching stationary solutions like:

$$f(t,x,\xi) = m(x,\xi) \qquad \forall\, t \in \mathbb{R}, \qquad (x,\xi) \in \Omega \times \mathbb{R}^N \text{ a.e.},$$

$$E(t,x) = E(x), \qquad B(t,x) = B(x) \qquad \forall\, t \in \mathbb{R}, \qquad x \in \Omega \text{ a.e.},$$

we can assume that

$$0 = \frac{d}{dt} \int\int_{\Omega \times \mathbb{R}^N} (f \ln f - f)\, dx\, d\xi$$

$$= \int\int_{\Omega \times \mathbb{R}^N} Q(m,m) \ln m\, dx\, d\xi$$

$$= \frac{1}{4} \int\int\int\int_{\Omega \times (\mathbb{R}^N)^2 \times S^{N-1}} q(\xi - \xi_*, \omega)\, (m'm'_* - mm_*) \ln\left(\frac{m'm'_*}{mm_*}\right) dx\, d\xi\, d\xi_*\, d\omega .$$

One can prove (see Ref. 4) that if m is strictly positive, we have

$$\ln m(x,\xi) = a(x) + b(x)\xi + c(x)|\xi|^2$$

and a, b, c satisfy the following system:

$$(E + E_0) \cdot b = 0 ,$$

$$\partial_x a + 2c(E + E_0) + B \wedge b = 0 ,$$

$$\partial_x b = 0 ,$$

$$\partial_x c = 0 .$$

Using the fact that m belongs to L^1, we can write:

$$m(x,\xi) = \frac{1}{(2\pi T)^{N/2}} \rho(x) e^{-|\xi - \bar{\xi}|^2/(2T)}$$

with $T>0$ and where ρ is a non-negative L^1 function defined on Ω. But

$$\bar{\xi}\cdot(E+E_0)=0$$

and, if we impose some technical conditions (case $\Omega = \mathbb{R}^N$), or if we assume (see Ref. 5) some particular boundary conditions if Ω is bounded (specular reflexion, for example, if Ω is not a surface of revolution), then T is a strictly positive constant and $\bar{\xi}=0$. Finally, let us note that $(\xi\wedge B)\cdot\partial_\xi m=0$. The stationary solutions of (*VPB*) (case $B=0$) and the stationary solutions of (*VMB*) are Maxwellian solutions of the stationary Vlasov–Poisson system:

$$\xi\cdot\partial_x m+(E+E_0)\cdot\partial_\xi m=0$$

$$\operatorname{div} E=\rho$$

with

$$m(x,\xi)=\frac{1}{(2\pi T)^{N/2}}\,\rho(x)e^{-|\xi|^2/(2T)},$$

where ρ is a non-negative function of $L^1(\Omega)$ and T a strictly positive real number.

Other models

Let us mention here two other kinetic models whose stationary states are Maxwellian solutions.

The Vlasov–Maxwell–BGK model (see Refs. 6 and 23) which is a good approximation for the Vlasov–Maxwell–Boltzmann model when the solutions are approximatively Maxwellian:

$$\partial_t f+\xi\cdot\partial_x f+F(x,t)\cdot\partial_\xi f=f-M_f, \qquad (VM\text{-}BGK)$$

where F is the Lorentz's force and M_f is the Maxwellian having the same moments as f.

The Vlasov–Fokker–Planck–Landau model is given by:

$$\partial_t f+\xi\cdot\partial_x f+F(x,t)\cdot\partial_\xi f=C(f), \qquad (VFPL)$$

where F is the Lorentz's force and $C(f)$ is the Fokker–Planck–Landau kernel:

$$C(f)=\nabla_\xi\int_{\xi^*}\Theta(|\xi-\xi_*|)\left(\mathrm{Id}-\frac{(\xi-\xi_*)\times(\xi-\xi_*)}{|\xi-\xi_*|^2}\right.$$

$$\cdot\{f(\xi_*)\nabla_\xi f(\xi)-f(\xi)\,\nabla_{\xi_*}f(\xi_*)\}d\xi_*,$$

where Θ is a function going from $\mathbb{R}$ to $\mathbb{R}$.

Remark 3. One can replace the classical Boltzmann collision term by a modified Boltzmann collision term, which takes some quantum effects (Pauli's exclusion principle) into account. The collision integral then has the following form (see Ref. 7):

$$C(f) = \int\int_{\Omega\times\mathbb{R}^N} q(\xi - \xi_*, \omega)\,(f'f'_*(1 - \varepsilon f)\,(1 - \varepsilon f_*)$$

$$- ff_*(1 - \varepsilon f'_*)\,(1 - \varepsilon f'_*))\,d\xi_* d\omega\,,$$

where ε is a parameter proportional to Planck's constant. Similar arguments as before ensure that stationary solutions are Planckians and can be written in the following form:

$$p(x, \xi) = \frac{m(x, \xi)}{1 + \varepsilon m(x, \xi)}\,,$$

where m is a Maxwellian. The Vlasov equation being linear, the problem reduces once again to solving the Vlasov–Poisson system for Maxwellian solution, but the equation for the potential has to be modified.

Appendix B: Vlasov–Poisson Model with Several Species of Particles

The previous results can easily be extended to models (see Refs. 17 and 19) which describes several species of particles (say n species of particles) interacting only through electromagnetic forces: f^i is the density of the particles of species i; it is a function defined on $\mathbb{R}\times\Omega\times\mathbb{R}^N$ such that:

$$\rho^i(t,x) = \int_{\mathbb{R}^N} f^i(t, x, \xi)\,d\xi$$

is the spatial density of the particles of the species i. Particles of species i are supposed of mass m^i and charge ϵ^i. Vlasov's equation must be rewritten in the following system:

$$\partial_t f^i + \xi\partial_x f^i + \frac{\epsilon^i}{m^i}(E(x,t) + E_0(x))\cdot\partial_\xi f^i = 0 \qquad (i = 1, 2, \ldots, n)\,,$$

where

$$E = \sum_{i=1}^{n} E^i$$

and

$$\operatorname{div} E^i = \varepsilon^i \rho^i .$$

If we assume that there exist potentials U and U_0 such that $E = -\partial_x U$ and $E_0 = -\partial_x U_0$, and if we look for solutions such that

$$f^i(t, x, \xi) = \frac{1}{(2\pi T^i)^{N/2}} \rho^i(x) e^{-|\xi|^2/(2T^i)},$$

we get

$$-\Delta U = \sum_{i=1}^{n} \varepsilon^i \mu^i \frac{e^{-\varepsilon^i \lambda^i (U_0 + U)}}{\int_\Omega e^{-\varepsilon^i \lambda^i (U_0 + U)} dx}$$

with

$$(\lambda^i)^{-1} = m^i T^i$$

and

$$\mu^i = \|\rho^i\|_{L^1(\Omega)},$$

or

$$-\Delta U = \sum_{i=1}^{n} \varepsilon^i \mu^i \frac{\rho_0^i e^{-\varepsilon^i \lambda^i U}}{\int_\Omega \rho_0^i e^{-\varepsilon^i \lambda^i U} dx},$$

where

$$\rho_0^i(x) = e^{-\varepsilon^i \lambda^i U_0},$$

and the same methods as above apply. One could think that other methods are needed to treat the case of negative charges, but this case can easily be reduced to the case of positive charges. Indeed, let us consider the following equation:

$$-\Delta U = -\frac{\rho_0 e^U}{\int_\Omega \rho_0 e^U dx}.$$

Changing U into $(-U)$, we note that it is equivalent to

$$-\Delta U = \frac{\rho_0 e^{-U}}{\int_\Omega \rho_0 e^{-U} dx},$$

and finally the case of several species of particles with charges of different signs do not provide new difficulties (see Ref. 17 for the bounded case without external electric field).

References

1. A. Arkeryd, *On the long time behaviour of the Boltzmann equation in a periodic box*, Technical report, Göteborg (1988).
2. F. Bavaud, *Equilibrium properties of the Vlasov functional: the generalized Poisson–Boltzmann–Emden equation*, preprint of the Heriot-Watt University, Edinburgh (1989).
3. C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer, 1988), 2nd edition.
4. S. Chapman and T. G. Cawlings, *the Mathematical Theory of Non Uniform Gases* (Cambridge Univ. Press, 1970), 3rd edition.
5. L. Desvillettes, *Convergence to equilibrium in large time for Boltzmann and BGK equation, Arch. Rational Mech. Anal.* **110** (1990) 73–91.
6. L. Desvillettes and J. Dolbeault, *On the long time asymptotics of the Vlasov–Poisson–Boltzmann equation, to appear in Comm. Partial Differential Equations.*
7. J. Dolbeault, *Kinetic models and quantum effects: a modified Boltzmann equation for Fermi–Dirac particles*, work in preparation.
8. R. DiPerna and P. L. Lions, *On the Cauchy problem for the Boltzmann equation: global existence and weak stability, Ann. Math.* **130** (1989) 321–366.
9. R. DiPerna and P. L. Lions, *Global solutions of Boltzmann equations and the entropy inequality, Arch. Rational Mech. Anal.* **105** (1989).
10. R. DiPerna and P. L. Lions, *Solutions globales de l' équation de Boltzmann, C.R. Acad. Sci. Paris Sér. I Math.* **306** (1988) 343–346.
11. R. DiPerna and P. L. Lions, *Solutions globales d' équations du type Vlasov-Poisson, C.R. Acad. Sci. Paris Sér. I Math.* **307** (1988) 655–658.
12. R. DiPerna and P. L. Lions, *Global solutions of Vlasov–Maxwell systems, Comm. Pure Appl. Math.* **42** (1989) 729–757.
13. R. DiPerna and P. L. Lions, *Global weak solutions of kinetic equations, Sem. Math. Torino* (1991).
14. R. DiPerna and P. L. Lions, *On the Fokker–Planck–Boltzmann equation, Comm. Math. Phys.* **120** (1988) 1–23.
15. K. Dressler, *Stationary solutions of the Vlasov–Fokker–Planck equation, Math. Appl. Sci.* **9** (1987) 169–176.
16. P. Gérard, *Solutions globales du problème de Cauchy pour l' équation de Boltzmann* (d' après R DiPerna et P. L. Lions), *Sém. Bourbaki* (1988) No. 699.
17. D. Gogny and P. L. Lions, *Sur les états d' équilibre pour les densités électroniques dans les plasmas, RAIRO Modél. Math. Anal. Numér.* **23** (1989) 137–153.
18. H. Grad, *Principles of the kinetic theory of gases, Flüge's Handbuch der Physik* **12** (Springer, 1958) 205–294.
19. Ju. A. Markov, G. A. Rudykh, N. A. Sidorov, and A. V. Sinicin, *Existence of stationary solutions for the Vlasov–Maxwell equation and some of them* (in Russian), *Matematisheskoe Modelirovanie* **1** (1989) 95–107.

20. H. Neunzert, *An introduction to the nonlinear Boltzmann–Vlasov equation*, Proceedings Montecatini (1981); *Lect. Notes Math.* **1048** (Springer, 1984) 60–110.
21. R. O'Neil, *Convolution operators and L(p, q) spaces, Duke Math, J.* **30** (1963) 129–142.
22. J. Peetre, *Espaces d' interpolation et théorème de Soboleff, Ann. Inst. Fourier* **16** (1966) 279 –317.
23. B. Perthame, *Global existence to the BGK model of Boltzmann equation*, to appear.
24. F. J. Pfreundt, *Zur stabilität von stationären Lösungen der Vlasov-Poisson Gleichung*, Dissertation, Kaiserslautern (1983).
25. G. A. Rudykh, N. A. Sidorov, and A. V. Sinicin, *On existence of classical stationary solutions for the Vlasov–Maxwell equation*, preprint of the Irkutsk computing center (1988).
26. H. Triebel, *Interpolation theory, function spaces, differential operators* (North Holland, 1978).
27. C. Truesdell and R. Muncaster, *Fundamentals of Maxwell kinetic theory of a simple monoatomic gas (treated as a branch of rational mechanics)* (Acad. Press, 1980).