DETECTION AND ESTIMATION OF CHANGES IN A POLYNOMIAL-PHASE SIGNAL USING THE DPPT

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ABSTRACT

This paper is concerned with on-line detection and estimation of changes in the parameters of a noisy polynomial-phase signal. This problem arises in vibration monitoring where the measured signals reflect both the nonstationarities due to the surrounding excitation, modelled by a polynomial-phase and the nonstationarities due to changes in the eigen structure, modelled by a break in the polynomial parameters. Development of a likelihood ratio test to detect and estimate changes in a polynomial-phase signal requires accurate estimation of the parameters vector after change, $\boldsymbol{\theta}_1$. Use of the Maximum Likelihood Estimate (MLE) of $\boldsymbol{\theta}_1$ is not practically useful since it involves the optimization of a multi-variable cost function. We propose to estimate $\boldsymbol{\theta}_1$ by using the Discrete Polynomial-Phase Transform (DPPT) in order to derive a detector having asymptotically the same properties than the GLR one for a much lower computational cost. Experimental performances, mean delay to the detection as a function of mean time between false alarms, will be studied.

1 PROBLEM STATEMENT

The signal model herein is:

$$z_n = A\exp(j\sum_{p=0}^{P} a_p n^p) + w_n, \qquad (1)$$

for $0 \le n \le N-1$, A is assumed to be a constant, w_n is a white Gaussian noise with variance σ_w^2 and the parameters vector under study is $\boldsymbol{\theta} = (a_0, a_1, \ldots, a_P)^T$. The problem of sequential detection of changes in $\boldsymbol{\theta}$ is: given the measurements $z_0, z_1, \ldots, z_n$, decide at instant k, $0 \le k \le N-1$, between the two hypotheses:

$$H_0 : \quad z_n = A\exp(j\boldsymbol{v}_n^T\boldsymbol{\theta}^0) + w_n \quad n = 0\ldots k, \qquad (2)$$

$$H_1 : \begin{cases} z_n = A\exp(j\boldsymbol{v}_n^T\boldsymbol{\theta}^0) + w_n & n = 0\ldots r-1 \\ z_n = A\exp(j\boldsymbol{v}_n^T\boldsymbol{\theta}^1) + w_n & n = r\ldots k, \end{cases} \qquad (3)$$

where $\boldsymbol{v}_n = (1, n, n^2, \ldots, n^P)^T$, $\boldsymbol{\theta}^0 = (a_0^0, a_1^0, \ldots, a_P^0)^T$, $\boldsymbol{\theta}^1 = (a_0^1, a_1^1, \ldots, a_P^1)^T$.

H_0 is the hypothesis that no change has occurred between samples 0 and k and H_1 is the hypothesis that a change has occurred at instant r unknown, $0 \le r \le k$.

The log-likelihood ratio between these two hypotheses is:

$$L(k, r, \boldsymbol{\theta}_1) = \frac{1}{\sigma_w^2}[(\boldsymbol{z}_r - \boldsymbol{s}_0)^H(\boldsymbol{z}_r - \boldsymbol{s}_0) - (\boldsymbol{z}_r - \boldsymbol{s}_1)^H(\boldsymbol{z}_r - \boldsymbol{s}_1)], \qquad (4)$$

$\boldsymbol{z}_r = (z_r, \ldots, z_k)^T$,
$\boldsymbol{s}_i = (A\exp(j\boldsymbol{v}_r^T\boldsymbol{\theta}^i), \ldots, A\exp(j\boldsymbol{v}_k^T\boldsymbol{\theta}^i))^T$, $i = 0, 1$.

In the case where the parameters vector after change is unknown, two possible solutions exist, [4]. The first one consists of weighting the likelihood ratio with respect to all possible values of $\boldsymbol{\theta}_1$ using a priori known cumulative distribution function. In the second solution, the unknown parameters vector $\boldsymbol{\theta}_1$ is replaced by its MLE, resulting in the GLR algorithm.

We take place in the general and realistic case where no a priori information on $\boldsymbol{\theta}_1$ is available.

For the GLR algorithm, decision of a change is taken following:

$$t_a = arg\min_k(g_k \ge \lambda) \qquad (5)$$

$$g_k = \max_{k-M \le r \le k} \max_{\boldsymbol{\theta}_1} L(k, r, \boldsymbol{\theta}_1) \qquad (6)$$

In hypothesis (3), r can take all values between 0 and k, leading to growing arrays. In practice the search over r is reduced to a fixed length window $[k-M, k]$.

At every instant k, $0 \le k \le N-1$ and every instant r in the window $[k-M, k]$, $\boldsymbol{\theta}_1$ must be replaced by its MLE to compute g_k. If g_k is greater than a fixed threshold λ, decision of a change is taken, the corresponding $\hat{r}$ and $\hat{\boldsymbol{\theta}}_1$ are the estimated values of r and $\boldsymbol{\theta}_1$ at corresponding time k.

Main problem is the estimation of $\boldsymbol{\theta}_1$ since its MLE requires a large amount of computations, involving the optimization of a multi-variable cost function:

$$\hat{\boldsymbol{\theta}}_1 = arg\max_{\boldsymbol{\theta}_1} |\sum_{n=r}^{k} z_n \exp(-j\boldsymbol{v}_n^T\boldsymbol{\theta}_1)|. \qquad (7)$$

An alternative solution will consist of developing a GLR test from the phase of z_n itself, [2]. In effect, the approximation $z_n \approx A\exp(j\sum_{p=0}^{P} a_p n^p + u_n)$ with u_n white Gaussian and variance $\sigma^2/2A^2$ is available for a large snr, [3]. However, this algorithm requires phase unwrapping which is delicate operation, for noisy data.

We propose, in this communication, to estimate $\boldsymbol{\theta}_1$ by using the Discrete Polynomial-Phase Transform (DPT), in order to derive a detector from the exact model of the signal, having asymptotically the same properties than the GLR test for a much lower computational cost.

2 ESTIMATION OF θ_1 USING THE DPT

Let s_n be a complex-valued function of a real discrete variable n, let τ and M be positive integers. The operators $DP_2(s_n, \tau)$ and $DP_M(s_n, \tau)$ are defined by

$$DP_2(s_n, \tau) := s_n s_{n-\tau}^*, \tag{8}$$

$$DP_M(s_n, \tau) := DP_2[DP_{M-1}(s_n, \tau)]. \tag{9}$$

If $s_n = A\exp(j\sum_{p=0}^{P} a_p^1 n^p)$ (s_n under hypothesis H_1) , it has been proved that $DP_P(s_n, \tau) = A^{2^{P-1}}\exp(j(\omega_0 n + \phi_0)$ where, for $(P-1)\tau \le n \le N-1$:

$$\omega_0 = P!\tau^{P-1} a_P^1. \tag{10}$$

Applying the operator of order P to s_n, transforms this broadband signal into a single tone with frequency ω_0 related to a_P^1. Then, if we define the Discrete-Polynomial-Phase Transform of order P (DPT) as the discrete time Fourier transform of $DP_P(s_n, \tau)$ and by applying it to the DP_P of s_n, we get an estimation of the highest order polynomial coefficient, (10).

Once a_P^1 has been estimated , the order can be reduced by multiplying s_n with $\exp(-j\hat{a}_P^1 n^P)$, $r \le n \le k$. If the estimate is accurate, the highest term is removed and we can proceed to use the DPT to estimate $a_{P-1}^1, a_{P-2}^1, \ldots a_0^1$.

The main simplifications with respect to the MLE is the replacement of M P-dim search by M*P 1-dim searches for each instant k. The white Gaussian noise w_n added on s_n is no more Gaussian and no more white on $DP_P(s_n, \tau)$ but it has been proved that for high snr, $\hat{a}_P$ is asymptotically unbiased and its mean square-error (MSE) achieves a minimum for $\tau = N/P$, see [1].

3 ALGORITHM

Assuming $\boldsymbol{\theta}_0$ known or estimated before the test, the sequential test detection algorithm is summarized as follows:

1. Initialization: $k = M$, $r = k - M$

   (a)
   $$\tau_{k,P} = \frac{k-P-1}{P}, \tag{11}$$
   estimate of $a_P^1, a_{P-1}^1, \ldots, a_0^1$ on samples $z_r, z_{r+1}, \ldots, z_{k-P-1}$ by DPT, eqs. (9, 10) and compute $L(k, r, \hat{\boldsymbol{\theta}}_1)$, (eq. 4),

   (b) substitute $r = r+1$. If $r \le k-P-1$, go back to step (a),

   (c) search the maximum of L over r gives g_k.

2. Compare g_k to λ: if $g_k \ge \lambda$, decision of a break is taken $t_a = k$; else $k = k+1$, go back to step 1.

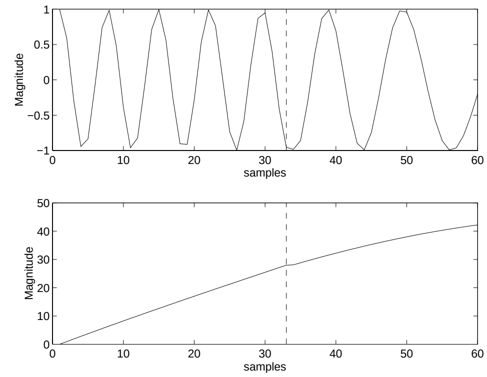

Figure 1: Real part and argument of z_n, $snr = 50dB$.

It is important to notice that besides a sequential algorithm gives a result at each instant k, it requires a $t_a(M-P)$ total number of $\boldsymbol{\theta}_1$ estimations instead of $N(M-P)$ for a global algorithm. Probability of $t_a = N$ is non zero for high threshold but we will see further that for an average given mean time between false alarms, test ended for $k < N$. The choice of the search window M must result from a tradeoff between the precision required of the estimation of r and the computational cost.

4 NUMERICAL EXAMPLE

In order to illustrate the proposed algorithm, an example of a 60 samples polynomial-phase signal of order $P = 3$, which parameters vector jumps from $\boldsymbol{\theta}_0 = (0, 0.3\pi, -\pi 10^{-3}, \pi 10^{-5})^T$ to $\boldsymbol{\theta}_1 = (-0.5\pi, 0.33\pi, -\pi 10^{-3}, -\pi 10^{-5})^T$ at instant $r = 33$ is given.

Fig. 1 shows the real part of the signal s_n and its argument: instantaneous phase. Fig. 2 represents g_k with $M = 10$ samples and corresponding $\hat{r}(k)$.

Fig. 3 depicts the behavior of the $\hat{a}_0^1$, $\hat{a}_1^1$, $\hat{a}_2^1$ and $\hat{a}_3^1$; exact values are plotted in dotted lines. For this experiment, snr has been fixed to $50dB$ in order to put in evidence the behavior of the algorithm.

Initialization of the algorithm lasts $M = 10$ samples, over which three areas (depicted by 1, 2 and 3 on the figures) can be distinguished on the results.

A first one, for $k \le r-1$ where $\hat{\boldsymbol{\theta}}_1 \approx \boldsymbol{\theta}_0$ and $L(k) \approx 0$; a second one for $r \le k \le r+M$ where $\hat{\boldsymbol{\theta}}_1 \ne \boldsymbol{\theta}_0 \ne \boldsymbol{\theta}_1$ since estimation is proceeded on each side of the break, and a third one for $k > M$ where $\hat{\boldsymbol{\theta}}_1 \approx \boldsymbol{\theta}_1$ and $L(k) \approx \sum_{n=r}^{k}(|A\exp(j\boldsymbol{v}_r^T(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0)) + w_n|^2 - |w_n|^2)$.

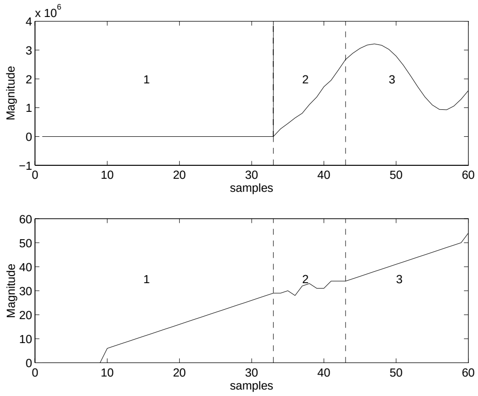

Figure 2: $L_1(k)$, $\hat{r}(k)$, $snr = 50dB$.

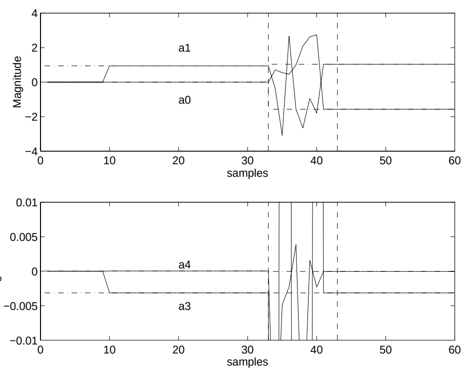

Figure 3: $\hat{\boldsymbol{\theta}}_1(k)$, $snr = 50dB$

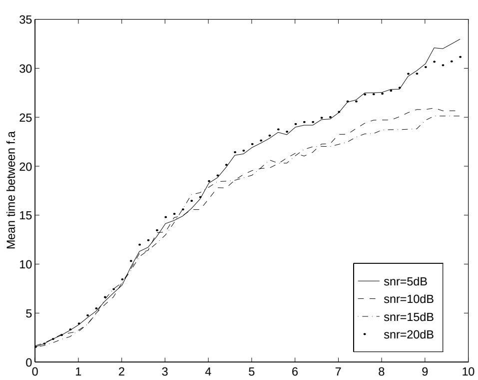

Figure 4: Mean time between false alarms as a function of the threeshold

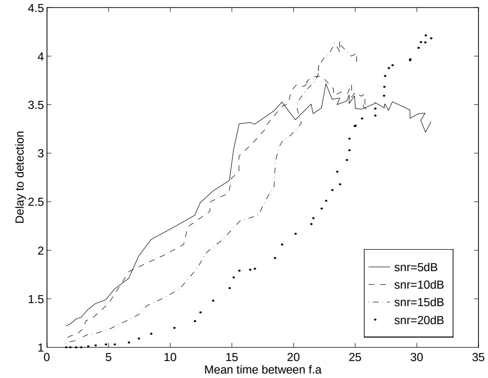

Figure 5: Delay to the detection as a function of mean time between false alarms

5 EXPERIMENTAL PERFORMANCES

In this section, some experimental results on on-line performances are given for different snr (polynomial order and magnitude of jump fixed) and for different polynomial orders (snr fixed).

In the on-line framework, the criteria are the *delay for detection* (T_D), characterizing the ability of an algorithm to set an alarm when a change actually occurs, and the *mean time between false alarms* (T_{FA}), which gives a limit for the possible mean time between successive jumps in the signal.

Theoretical general results on performances in multiple hypothesis test do not exist, bounds for T_{FA} and T_D are given in the case of a likelihood test, i.e $\boldsymbol{\theta}_1$ known, see [4].

Consequently we propose to fix the threshold in the adaptive following manner:

$$\lambda = \frac{1}{N} T \sum_{l=q+1}^{q+N} g_{k-l}, \qquad (12)$$

where T determines T_{FA} and $\frac{1}{N}\sum_{l=q+1}^{q+N} g_{k-l}$ is an estimation of the mean of g_k, over a window of fixed length: N, q are the number of "guard samples".

T_{FA} has been estimated as the mean time before the first alarm on 100 white and Gaussian noise sequences.

For the first experiment, T_D has been estimated on 100 realizations of the previous signal (see section 4).

Fig. (4) depicts T_{FA} as a function of T for snr going from $5dB$ to $20dB$. Note that a snr of $5dB$ is the limit under which the estimation algorithm does not operate properly.

A first and obvious result is that T_{FA} increases with T. The growing space between curves as T increases is probably due to the procedure itself. In fact, length of noise sequences limits the highest T_{FA}, above this maximum possible value for T_{FA}, a probability of no detection appears.

P	a_0	a_1	a_2	a_3	a_4
2	0	.6912	.0125	0	0
3	0	$.3\pi$	$\pi 10^{-3}$	$\pi 10^{-5}$	0
4	0	$.3\pi$	$\pi 10^{-3}$	$\pi 10^{-5}$	$\pi 10^{-7}$

Table 1: Parameters before the change

P	a_0^1	a_1^1	a_2^1	a_3^1	a_4^1	dp
2	.0112	.6912	.0112	0	0	-2.43
3	$-.5\pi$	$.33\pi$	$\pi 10^{-3}$	$\pi 10^{-6}$	0	-0.26
4	$-.5\pi$	$.33\pi$	$\pi 10^{-3}$	$\pi 10^{-6}$	$\pi 10^{-7}$	-0.26

Table 2: Parameters after the change

Fig. (5) shows T_D as a function of T_{FA}. First, general remarks can be made.

T_D increases with T_{FA}, which is easily understandable since higher is the threshold and so T_{FA}, higher is the delay to the detection. T_D reaches a maximum of around 4 samples for a T_{FA} of 30 samples.

Whatever is the snr, two principal areas can be pointed out. A first area where T_D grows proportionaly with T_{FA} and a second one where T_D is equal to a maximum value whatever is T_{FA}.

Curves are shifted to the left as the snr decreases. For a fixed T_{FA} in the first area, T_D raises with the snr. If T_{FA} is in the second area, T_D decreases with the snr for the reason raised previously.

For the second experiment, 3 polynomial-phase signals of order P=2,3 and 4 have been used. $\boldsymbol{\theta}_0$, $\boldsymbol{\theta}_1$, magnitude of the instantaneous jump (dp) are given in the tables (1) and (2), dp is the same for order P=3 and P=4 and is 10 times greater for the order P=2.

T_{FA} as a function of T is depicted on fig. (6). As curves are nearly superimposed, it can be noticed that T_{FA} is independant of the order and of the magnitude of jump.

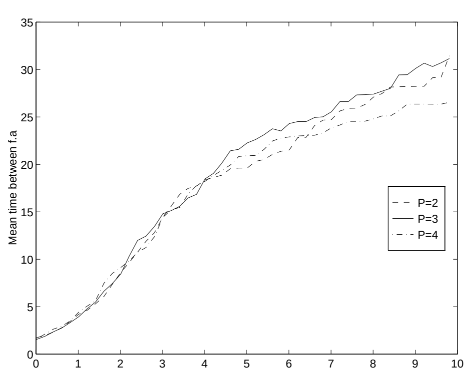

Figure 6: Mean time between false alarms as a function of the threshold

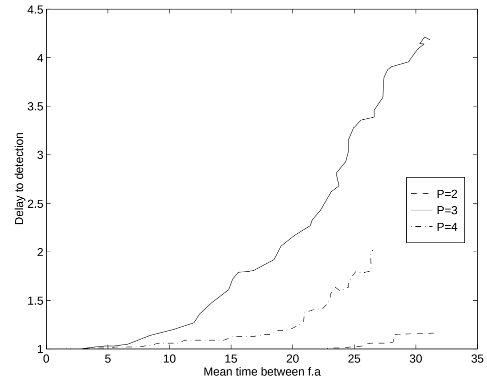

Figure 7: Delay to the detection as a function of mean time between false alarms

From curves of T_D as function of T_{FA}, fig. (7), a general result can be given: delay to the detection depends on the magnitude of jump and on the polynomial order.

Let us add some general comments about the tunning of the change detection algorithm. Minimum values of jumps must be close to the precision of the estimation algorithm and the threshold has to be chosen in such a way that the mean time between false alarms should not be too much less than the mean time between successive jumps in the signal.

6 CONCLUSION

A GLR algorithm to detect changes in the parameters of a polynomial-phase signal has been proposed where the unknown parameters vector after change is estimated by the discrete polynomial-phase transform, this estimator having asymptotically the same properties than the MLE one for a much lower computational cost. Time delay to the detection as a function of time between false alarm has been estimated following various polynomial orders and snr. For relatively high snr, mean time delay to the detection for a mean time between false alarms given is low and estimation of changes is accurate. Nevertheless, more experiments are necessary to go further into conclusions.

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