Mixed Fractional Integration In Mixed Weighted Generalized Hölder Spaces

Author Details: Tulkin Mamatov-
Bukhara Technological Institut of Engineering, Department of Higher mathematics. st. K. Murtazaev 15, Bukhara, Republic of Uzbekistan mamatov.tulkin@mail.ru

Abstract: *We consider operators of mixed fractional integration in weighted generalized Hölder spaces of a function of two variables defined by a mixed modulus of continuity.*
***Keywords**: functions of two variables, mixed fractional integral, mixed difference, generalized Hölder spaces, weighted spaces, modulus, mixed modulus of continuity.*

1. Introduction

One of the most important problems in the theory of integral operators in space is the problem of elucidating the dependence of the smoothness of the image on the smoothness of the preimage. The solution to such a problem plays an important role in the solvability of integral equations, their stability, and so on. The
concept of smoothness can be formulated in a variety of terms. One of the ways of sufficiently fine-grabbing the smoothness of functions is the notion of generalized Hölderness, formulated in terms of the behavior of the modulus of continuity.
Thus, one of the important questions in the theory of operators is as follows: Let be A an operator acting in a Banach space X and let be the modulus of continuity $\omega(f;h)=\sup\limits_{|t|\le h}\left\|f(x+h)-f(x)\right\|_X$ of X. How can the behavior of the modulus of continuity be characterized $\omega(A\varphi,h)$ if the behavior of the modulus of continuity of a function $\omega(\varphi;h)$: $\omega(\varphi;h)\le C\psi(h)$ for all is known $\varphi\in X$, where is $\psi(x)$ a given continuous function, $\psi(0)=0$.
A similar problem can be considered completely solved for different spaces, and also for the Hölder space of functions of one variable and power weights, when $\left(A_+^\alpha\varphi\right)(x)=\Gamma^{-1}(\alpha)\left(t_+^{\alpha-1}*\varphi\right)$, $0<\alpha<1$ ([2] - [6], [8] – [13]). A detailed review of these and some other close results can be found in [10]. The assertion for multidimensional cases on the property of mapping in the usual Hölder and in
the Hölder spaces defined by mixed differences are known [7]. Also, f generalized Hölder space is known for the Riesz fractional integral [13] (see also [12], Theorem 25.5).
Mixed fractional Riemann-Liouville integrals of order (α,β)

$$\left(I_{0+,\,0+}^{\alpha,\,\beta}\varphi\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y\frac{\varphi(t,\tau)\,dt\,d\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}},\qquad(1.1)$$

where $x,\,y>0,\ \alpha,\beta\in(0,1)$ have not been studied.
This paper is devoted to the study of certain properties of the mixed fractional integral (1.1) in weighed generalized Hölder spaces of a function of two variables defined by a mixed modulus of continuity.
We consider the operator (1.1) in $Q=\{(x,y):\ 0<x<b,\ 0<y<d\}$.

2. Preliminary information and notations

When studying the properties of continuous functions of several variables, in particular, two variables, the following classes of functions arise:

$$H^{\omega_1,\omega_2,\omega_{1,1}}=\Big\{\varphi(x,y)\in\mathbf{C}_Q:\ \overset{1,0}{\omega}(\varphi;\delta,0)=O(\omega_1(\delta)),\ \overset{0,1}{\omega}(\varphi;0,\sigma)=O(\omega_2(\sigma)),\ \overset{1,1}{\omega}(\varphi;\delta,\sigma)=O(\omega_{1,1}(\delta,\sigma))\Big\},$$

$$H^{\omega_1,\omega_2}=\Big\{\varphi(x,y)\in\mathbf{C}_Q:\overset{1,0}{\omega}(\varphi;\delta,0)=O(\omega_1(\delta)),\ \overset{0,1}{\omega}(\varphi;0,\sigma)=O(\omega_2(\sigma))\Big\},$$

where $\overset{1,0}{\omega}(\varphi;\delta,0)=\sup\limits_{y}\sup\limits_{h\in(0,\delta]}\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|$,

$\overset{0,1}{\omega}(\varphi;0,\sigma)=\sup\limits_{x}\sup\limits_{\eta\in(0,\sigma]}\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|$ - are the partial modulus of continuity of the first order, and a

$\overset{1,1}{\omega}(\varphi;\delta,\sigma)=\sup\limits_{x,y}\sup\limits_{\substack{0<h\le\delta\\0<\eta\le\sigma}}\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|$ is a mixed modulus of continuity of order (1,1);

$$\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y);$$

$$\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y);$$

$$\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x+h,y)-\varphi(x,y+\eta)+\varphi(x,y),$$

$\omega_1,\,\omega_2\in\Phi^1$, $\omega_{1,1}\in\Phi^{1,1}$ (definition of classes Φ^1 and $\Phi^{1,1}$ see below).
The following identity is valid

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)+\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)+\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)+\varphi(x,y).\qquad(2.1)$$

Definition 2.1. Let function $\varphi(x)$ is a bounded on $[a,b]$. The modulus of continuity of $\varphi(x)$ is the expression

$$\omega(\varphi;\delta)=\sup\limits_{x_1,\,x_2\in[a,\,b]\ |x_1-x_2|\le\delta}|\varphi(x_1)-\varphi(x_2)|,$$

is defined for all δ that satisfy the condition $0<\delta\le b-a$.
Definition 2.2. A function $\omega(\delta)$ $(0<\delta\le b-a)$ is called a modulus of continuity if it satisfies conditions
1) $\lim\limits_{\delta\to0}\omega(\delta)=0$;
2) $\omega(\delta)$ is almost increasing on $(0,b-a]$;
3) $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$;

4) $\omega(\delta)$ is function continuous in δ on $(0,b-a]$

Definition 2.3. We denote by Φ^1 the class of functions $\omega(\delta)$ defined on $(0,b-a]$, and satisfying conditions

a) $\omega(\delta)$ is a modulus of continuity

b) $\int_0^{\delta}\frac{\omega(t)}{t}dt \le C\omega(\delta)$;

c) $\delta\int_{\delta}^{b-a}\frac{\omega(t)}{t^2}dt \le C\omega(\delta)$;

d) $\omega'(\delta)\sim\frac{\omega(\delta)}{\delta}$.

It follows from the definition $\overset{1,1}{\omega}(\varphi;\delta,\sigma)$ that this function belongs to Φ^1 each of the variables. In addition, we note the inequality

$$\overset{1,1}{\omega}(\varphi;\delta,\sigma)\le 2\min\left\{\overset{1,0}{\omega}(\varphi;\delta,0),\ \overset{0,1}{\omega}(\varphi;0,\sigma)\right\} \qquad (2.2)$$

Definition 2.4. We denote by $\Phi^{1,1}(Q)$ the class of functions of two variables $\omega(\delta,\sigma)$ satisfying conditions:

1) $\omega(\delta,\sigma)$ in δ for any fixed σ;

2) $\omega(\delta,\sigma)$ in σ for any fixed δ.

We call this class the class of mixed modulus of continuity of the first order of continuous functions of two variables.

In [1] was shown that the properties 1) and 2) are characteristic for continuity modulus in the sense that for every $\omega\in\Phi^{1,1}$ there exist such a function $\varphi\in\mathbf{C}_Q$, that

$$\overset{1,1}{\omega}(\varphi;\,\delta,\sigma)\sim\omega_{1,1}(\delta,\sigma),\ \ \overset{1,0}{\omega}(\varphi;\,\delta,0)\sim\omega_1(\delta),\ \ \overset{0,1}{\omega}(\varphi;\,0,\sigma)\sim\omega_2(\sigma).$$

Definition 2.5. Let us denote Φ the set of satisfying $(\omega_{1,1},\omega_1,\omega_2)$

1) $\omega_1(\delta),\ \omega_2(\sigma)\in\Phi^1$;

2) $\omega_{1,1}(\delta,\sigma)\in\Phi^{1,1}$;

3) $\omega_{1,1}(\delta,\sigma)\le C\min\{\omega_1(\delta),\ \omega_2(\sigma)\}$,

where C - is not envy from $\omega_1,\ \omega_2,\ \omega_{1,1}$.

Let $\omega=(\omega_1,\omega_2,\omega_{1,1})\in\Phi=\Phi^1\times\Phi^1\times\Phi^{1,1}$. We have introduced a norm in $\tilde H^{\omega}\overset{\mathbf{def}}{=}H^{(\omega_1,\omega_2,\omega_{1,1})}$ space

$$\|\varphi\|_{\tilde H^{\omega}}=\|\varphi\|_{H^{(\omega_1,\omega_2,\omega_{1,1})}}=\max\left\{C_{\varphi}^{1,0},\ C_{\varphi}^{0,1},\ C_{\varphi}^{1,1}\right\},$$

where

$$C_{\varphi}^{1,0}=\sup_{\delta>0}\frac{\overset{1,0}{\omega}(\varphi;\,\delta,0)}{\omega_1(\delta)},\qquad C_{\varphi}^{0,1}=\sup_{\sigma>0}\frac{\overset{0,1}{\omega}(\varphi;\,0,\sigma)}{\omega_2(\sigma)},$$

$$C_{\varphi}^{1,1}=\sup_{\delta>0,\ \sigma>0}\frac{\overset{1,1}{\omega}(\varphi;\,\delta,\sigma)}{\omega_{1,1}(\delta,\sigma)},\qquad \|\varphi\|_{\mathbf{C}(Q)}=\max_{(x,\,y)\in Q}|\varphi(x,\,y)|.$$

Definition 2.6. We say that $\varphi(x,y)\in\tilde H_0^{\omega}(Q)$, if $\varphi(x,y)\in\tilde H^{\omega}(Q)$ and $\varphi(x,y)\big|_{x=0,\,y=0}=\varphi(x,y)\big|_{x=b,\,y=d}\equiv 0$.

We will also make use of the following weighted spaces. Let $\rho(x,y)$ be a non-negative function on Q (we will only deal with degenerate weights $\rho(x,y)=\rho(x)\rho(y)$.

Definition 2.7. By $\tilde H^{\omega}(Q,\rho)=\tilde H^{\omega}(\rho)$ we denote the space of functions $\varphi(x,y)$ such that $\rho\varphi\in\tilde H^{\omega}$, respectively, equipped with the norm

$$\|\varphi\|_{\tilde H^{\omega}(\rho)}=\|\varphi\rho\|_{\tilde H^{\omega}}.$$

By $\tilde H_0^{\omega}(\rho)$ we denote the corresponding subspaces of functions $\varphi(x,y)$ such that

$$\varphi(x,y)\rho(x,y)\big|_{x=0,\,y=0}=\varphi(x,y)\rho(x,y)\big|_{x=b,\,y=d}\equiv 0.$$

Below we follow some technical estimations suggested in [11] for the case of one-dimensional Riemann - Liouville fractional integrals. We denote

$$B(x,y;t,\tau)=\frac{\rho(x,y)-\rho(t,\tau)}{\rho(t,\tau)(x-t)^{1-\alpha}(y-\tau)^{1-\beta}},$$

where $0<\alpha,\beta<1;\ \ 0<t<x<b,\ 0<\tau<y<d$. In the case $\rho(x,y)=\rho(x)\rho(y)$ we have

$$B(x,y;\,t,\tau)=B_1(x,t)B_2(y,\tau)+\frac{B_1(x,t)}{(y-\tau)^{1-\beta}}+\frac{B_2(y,\tau)}{(x-t)^{1-\alpha}}, \qquad (2.3)$$

where

$$B_1(x,t)=\frac{\rho_1(x)-\rho_1(t)}{\rho_1(t)(x-t)^{1-\alpha}},\quad B_2(y,\tau)=\frac{\rho_2(y)-\rho_2(\tau)}{\rho_2(\tau)(y-\tau)^{1-\beta}}.$$

Let also

$$D_1(x,h,t)=B_1(x+h,\,t)-B_1(x,t),\ \ t,x,x+h\in[0,b],\ h>0;$$
$$D_2(y,\,\eta,\,t)=B_2(y+\eta,\,\tau)-B_2(y,\,\tau),\ \ \tau,\ y,y+\eta\in[0,d],\ \eta>0.$$

Lemma 2.1. ([3]) Let $\rho_1(x)=x^{\mu},\ \mu\in\mathbf{R}^1,\ 0<\alpha<1$. Then

$$|B_1(x,\,t)|\le C\left(\frac{x}{t}\right)^{\max(\mu-1,0)}\frac{(x-t)^{\alpha}}{t}, \qquad (2.4)$$

$$|D_1(x,\,h,\,t)|\le C\left(\frac{x+h}{t}\right)^{\max(\mu-1,0)}\frac{h}{t(x+h-t)^{1-\alpha}}. \qquad (2.5)$$

Similar estimates hold for $B_2(y,\,\tau)$ and $D_2(y,\,\eta,\,\tau)$ with $\rho_2(y)=y^{\nu}$.

Remark 2.1. All the weighted estimations of fractional integrals in the sequel are based on inequalities (2.4)-(2.5). Note that the right - hand sides of these inequalities have the exponent $\max(\mu-1,0)$, which means that in the proof it suffices to consider only the case $\mu\ge 1$, evaluations of $\mu<1$ being the same as for $\mu=1$.

The following statements are known, begin first proved in (see also [12], p. 197). However, here we give a sketch of the proof of this lemma, in order to compose the representation of lightness for the two-dimensional case. Consider the one-dimensional fractional Riemann-Liouville integral

$$\left(I_{0+}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt,\ x>0,0<\alpha<1. \qquad (2.6)$$

Theorem 2.1. Let $\varphi(x)$ be continuous on $[0,\,b]$ and let $\varphi(0)=0$. For the fractional integral (2.6), the estimate

$$\omega\left(I_{0+}^{\alpha}\varphi,\,h\right)\le Ch\int_h^b\frac{\omega(\varphi,\,t)}{t^{2-\alpha}}dt \qquad (2.7)$$

is valid.

Proof. Representing (2.6) as

$$\left(I_{0+}^{\alpha}\varphi\right)(x)=\frac{\varphi(0)}{\Gamma(\alpha)}\int_0^x\frac{dt}{(x-t)^{1-\alpha}}+\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)-\varphi(0)}{(x-t)^{1-\alpha}}dt=$$
$$=A_1(x)+A_2(x)$$

Let $h>0;\ x,x+h\in[0,\,b]$. We have

$$A_2(x+h)-A_2(x)=\frac{\varphi(x)-\varphi(0)}{\Gamma(1+\alpha)}\left[(x+h)^\alpha-x^\alpha\right]+$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^h\frac{\varphi(x+t)-\varphi(t)}{(h-t)^{1-\alpha}}dt+$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^x[\varphi(x-t)-\varphi(t)]\left[(h+t)^{\alpha-1}-t^{\alpha-1}\right]dt=\Delta_1+\Delta_2+\Delta_3.$$

We have: $|\Delta_1|\le C\omega(\varphi;x)\left|(x+h)^\alpha-x^\alpha\right|$. In the case $x\le h$ we have $|\Delta_1|\le Ch^\alpha\omega(\varphi;h)$. Let $x\ge h$. Then

$$|\Delta_1|\le C\omega(\varphi;x)x^\alpha\left[\left(1+\frac{h}{x}\right)^\alpha-1\right]\le C\frac{\omega(\varphi;x)}{x^{1-\alpha}}h. \qquad (2.8)$$

Since

$$Cx^{\alpha-1}\omega(\varphi;x)\le\omega(\varphi;x)\int_x^b t^{\alpha-2}dx\le\int_x^b\frac{\omega(\varphi;t)}{t^{2-\alpha}}dt\le\int_h^b\frac{\omega(\varphi;t)}{t^{2-\alpha}}dt.$$

It follows from (2.8) that

$$|\Delta_1|\le Ch\int_h^b\frac{\omega(\varphi;t)}{t^{2-\alpha}}dt.$$

Further,

$$|\Delta_2|\le\int_0^h\frac{\omega(\varphi;t)}{(h-t)^{1-\alpha}}dt=h^\alpha\int_0^1\frac{\omega(\varphi;h\xi)}{(1-\xi)^{1-\alpha}}d\xi\le Ch^\alpha\omega(\varphi;h),$$

with $C=\int_0^1(1-\xi)^{\alpha-1}d\xi$. To estimate Δ_3 we distinguish the case 1) $x\ge h$ and 2) $x\le h$. In the first case

$$|\Delta_3|\le C\left[\int_0^h\omega(f,t)\left[t^{\alpha-1}-(h+t)^{\alpha-1}\right]dt+\right.$$

$$\left.+\int_h^x\omega(f,t)\left[t^{\alpha-1}-(h+t)^{\alpha-1}\right]dt\right]\le$$

$$\le C_2\left[h^\alpha\omega(f,h)+h\int_h^b\frac{\omega(f,t)}{t^{2-\alpha}}dt\right].$$

Obviously in the second case $|\Delta_3|\le C_1h^\alpha\omega(f;h)$.

Estimates for $\Delta_1,\Delta_2,\Delta_3$ the lead to (2.7) if we take into account the fact that $h^\alpha\omega(\varphi;h)$ is dominated by the right-hand side of (2.7). The latter is easily obtained in view of the monotonicity of the function $\omega(\varphi;t)$.

To obtain estimates of the Zygmund type in the weighted case, we use the notation and the proof scheme from [2] and [6].

Theorem 2.2. Let $\rho(x)=x^\mu, 0\le\mu<2-\alpha$. If the function $f(x),\ x\in[0,b]$ satisfies the condition:

1) $\rho(x)f(x)\in\mathbf{C}_{[0,b]}$ and $\rho(x)f(x)|_{x=0}=0$;

2) the integral $\int_0^b\frac{\omega(\rho f,t)}{t^\gamma}dt$ converges for $\gamma=\max(1,\mu)$.

Then estimates of the Zygmund type

$$\omega(\rho I_{0+}^\alpha f,h)\le C\left(h^{\alpha+\gamma-1}\int_0^h\frac{\omega(\rho f,t)}{t^\gamma}dt+h\int_h^b\frac{\omega(\rho f,t)}{t^{2-\alpha}}dt\right). \qquad (2.9)$$

Proof. We denote this $g(x)=\rho(x)f(x)$. We have

$$(\rho I_{0+}^\alpha f)(x)=\left(I_{0+}^\alpha g\right)(x)+\left(J_{0+}^\alpha g\right)(x),\quad \left(J_{0+}^\alpha g\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x B(x,t)g(t)dt.$$

Here the estimates for $\left(I_{0+}^\alpha g\right)(x)$ are solved in Theorem 2.1.

Now consider the difference

$$\left(J_{0+}^\alpha g\right)(x+h)-\left(J_{0+}^\alpha g\right)(x)=F_1(x,h)+F_2(x,h),$$

where

$$F_1(x,h)=\int_x^{x+h}B(x+h,t)g(t)dt,\quad F_2(x,h)=\int_a^x D(x,h,t)g(t)dt.$$

Taking into account Remark 1.1, we consider only the case $1\le\mu<2-\alpha$. From (2.4) we have

$$|F_1|\le C\int_x^{x+h}\left(\frac{x+h}{t}\right)^{\mu-1}\frac{(x+h-t)^\alpha}{t}\omega(g,t)dt.$$

If $x\le h$, then

$$|F_1|\le Ch^{\mu+\alpha-1}\int_x^{x+h}\frac{\omega(g;t)}{t^\mu}dt.$$

Using the property of almost decreasing $\frac{\omega(g;t)}{t}$, we obtain

$$|F_1|\le Ch^{\mu+\alpha-1}\int_x^{x+h}\frac{\omega(g;t-x)}{(t-x)^\mu}dt=Ch^{\mu+\alpha-1}\int_0^h\frac{\omega(g;t)}{t^\mu}dt.$$

If $x>h$, then

$$|F_1|\le Ch^\alpha(x+h)^{\mu-1}\int_x^{x+h}\frac{\omega(g;t)}{t^\mu}dt=C\frac{h^\alpha}{(x+h)^{\mu-1}}\int_0^h\frac{\omega(g;x+t)}{(x+t)^\mu}dt\le$$

$$\le Ch^\alpha x^{\mu-1}\int_0^h\frac{\omega(g;x+t)}{(x+t)^{\mu-1}}\frac{dt}{x+t}\le Ch^\alpha\int_0^h\frac{\omega(g;x+t)}{x+t}dt.$$

Further, it is clear that

$$|F_1|\le Ch^\alpha\int_0^h\frac{\omega(g;t)}{t}dt.$$

Collecting the estimates F_1, we obtain the inequality for $0\le\mu<2-\alpha$

$$|F_1|\le Ch^{\alpha+\gamma-1}\int_0^h\frac{\omega(g;t)}{t^\gamma}dt,\quad \gamma=\max(1,\mu).$$

We pass to the estimate F_2. Using the estimate (2.5), we obtain

$$|F_2|\le Ch\int_0^x\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}t}dt. \qquad (2.10)$$

When $h\ge x$,

$$|F_2|\le Ch^{\alpha+\mu-1}\int_0^x\frac{\omega(g;t)}{t^\mu}dt\le Ch^{\alpha+\mu-1}\int_0^h\frac{\omega(g;t)}{t^\mu}dt.$$

If $h<x$, then, we represent the right-hand side of (2.10) as a sum of three terms:

$$F_2'=Ch\int_0^h\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}}\frac{dt}{t},$$

$$F_2''=Ch\int_h^{\frac{1}{2}(x+h)}\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}}\frac{dt}{t},$$

$$F_2'''=Ch\int_{\frac{1}{2}(x+h)}^x\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}}\frac{dt}{t}.$$

Then $|F_2|\le F_2'+F_2''+F_2'''$.

For the term F_2' the relations are valid $x+h\le 2(x+h-t)$, therefore

$$F_2'\le Ch\int_0^h\frac{\omega(g;t)dt}{t^\mu(x+h-t)^{2-\alpha-\mu}}\le Ch^{\alpha+\mu-1}\int_0^h\frac{\omega(g;t)}{t^\mu}dt.$$

For the summand, F_2'' we have $2t\le x+h$, so $1\le\mu<2-\alpha$ we obtain the estimate

$$F_2'' \le Ch \int_h^{\frac{1}{2}(x+h)} \frac{\omega(g;t)}{t^{\mu}\left(\frac{x+h}{2}\right)^{2-\mu-\alpha}} dt \le Ch\int_h^b \frac{\omega(g;t)}{t^{2-\alpha}} dt.$$

We estimate the term F_2'''. Here $t \ge x+h-t$, therefore $\frac{\omega(g;t)}{t} \le C\frac{\omega(g;x+h-t)}{x+h-t}$, it follows that

$$F_2''' \le Ch \int_{\frac{1}{2}(x+h)}^{x} \frac{\omega(g;x+h-t)}{(x+h-t)^{2-\alpha}} dt.$$

Because $x+h \le 2t$. Having made the change $\xi = x+h-t$ and going back to the variable t, we get

$$F_2''' \le Ch\int_h^b \frac{\omega(g;t)}{t^{2-\alpha}} dt.$$

From the estimates F_2', F_2'', F_2''' follows when $h < x$

$$| F_2 | \le C\left(h^{\mu+\alpha-1}\int_0^h \frac{\omega(g;t)}{t^{\mu}} dt + h\int_h^b \frac{\omega(g;t)}{t^{2-\alpha}} dt\right).$$

Thus, when $0 \le \mu < 2-\alpha$

$$| F_2 | \le C\left(h^{\gamma+\alpha-1}\int_0^h \frac{\omega(g;t)}{t^{\gamma}} dt + h\int_h^b \frac{\omega(g;t)}{t^{2-\alpha}} dt\right), \quad \gamma = \max(1,\mu),$$

which completes the proof.

3. Zygmund type estimates for the mixed fractional integral

Theorem 3.1. Let $\varphi \in \mathbf{C}(Q)$ and $\varphi(x, y) =|_{x=0,\, y=0} = 0$. Then for (1.1), we have estimates of the Zygmund type

$$\overset{1,1}{\omega}(f;\, h,\eta) \le C_1 h\eta \int_h^b \int_\eta^d \frac{\overset{1,1}{\omega}(\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}} dt d\tau. \tag{3.1}$$

$$\overset{1,0}{\omega}(f;\, h,0) \le C_2 h\int_h^b \frac{\overset{1,1}{\omega}(\varphi;\, t,d)}{t^{2-\alpha}} dt,$$

$$\overset{0,1}{\omega}(f;\, 0,\eta) \le C_3 \eta\int_\eta^d \frac{\overset{1,1}{\omega}(\varphi;\, b,\tau)}{\tau^{2-\beta}} d\tau. \tag{3.2}$$

Proof. Using the identity (2.1), we represent the integral (1.1) in the form

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y) = \frac{\varphi(0,0)x^{\alpha}y^{\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)} + \frac{x^{\alpha}\psi_2(y)}{\Gamma(1+\alpha)} + \frac{y^{\beta}\psi_1(x)}{\Gamma(1+\beta)} + \psi(x,y),$$

where

$$\psi_1(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t,0)-\varphi(0,0)}{(x-t)^{1-\alpha}} dt, \quad \psi_2(y) = \frac{1}{\Gamma(\beta)}\int_0^y \frac{\varphi(0,\tau)-\varphi(0,0)}{(y-\tau)^{1-\beta}} d\tau,$$

$$\psi(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y \frac{\left(\overset{1,1}{\Delta}_{t,\tau}\varphi\right)(0,0)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} dt d\tau.$$

Let $h > 0$, $x, x+h \in [0,b]$. Consider the difference

$$\left(\overset{1,0}{\Delta}_h f\right)(x,y) = \frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y \frac{g(x,y-\tau)}{\tau^{1-\beta}} d\tau +$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_0^y \frac{g(x-t,\, y-\tau)-g(x,\, y-\tau)}{\tau^{1-\beta}(t+h)^{1-\alpha}} dt d\tau +$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y \frac{g(x-t,\, y-\tau)-g(x,\, y-\tau)}{\tau^{1-\beta}}\left|(t+h)^{\alpha-1}-t^{\alpha-1}\right| dt d\tau.$$

The following inequality is valid

$$\left|\left(\overset{1,0}{\Delta}_h f\right)(x,y)\right| \le C\left(|(x+h)^{\alpha}-x^{\alpha}| \int_0^y \frac{\overset{1,1}{\omega}(\varphi;\, x,\, y-\tau)}{\tau^{1-\beta}} d\tau + \right.$$

$$+\int_{-h}^0\int_0^y \frac{\overset{1,1}{\omega}(\varphi;\, t,\, y-\tau)}{(h+t)^{1-\alpha}\tau^{1-\beta}} dt d\tau +$$

$$\left. +\int_0^x\int_0^y \overset{1,1}{\omega}(\varphi;\, t,\, y-\tau)\,|(h+t)^{\alpha-1}-t^{\alpha-1}|\,\tau^{\beta-1} dt d\tau\right).$$

We make use of (2.2) and obtain

$$\left|\left(\overset{1,0}{\Delta}_h f\right)(x,y)\right| \le C_1\left(|(x+h)^{\alpha}-x^{\alpha}|\, \overset{1,1}{\omega}(\varphi;\, x,\, d) + \int_{-h}^0 \frac{\overset{1,1}{\omega}(\varphi;\, t,\, d)}{(h+t)^{1-\alpha}} dt + \right.$$

$$\left. +\int_0^x \overset{1,1}{\omega}(\varphi;\, t,\, d)\,|(h+t)^{\alpha-1}-t^{\alpha-1}|\, dt\right).$$

Using the estimates $\Delta_1, \Delta_2, \Delta_3$ in the proof of Theorem 2.1, it is easy to obtain

$$| f(x,y+\eta) - f(x,y)| \le C_2\eta\int_\eta^{d-c} \frac{\overset{1,1}{\omega}(\varphi;\, b-a,\, \tau)}{\tau^{2-\beta}} d\tau. \tag{3.3}$$

Similarly, we can obtain the estimate

$$| f(x,y+\eta) - f(x,y)| \le C_2\eta\int_\eta^d \frac{\overset{1,1}{\omega}(\varphi;\, b,\, \tau)}{\tau^{2-\beta}} d\tau. \tag{3.4}$$

From (3.3) and (3.4) follows the inequalities (3.2).

Let $h,\eta > 0$ and $x, x+h \in [a,b]$, $y, y+\eta \in [c,d]$. Consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y) = \left(\overset{1,1}{\Delta}_{h,\eta}\psi\right)(x,y) = \sum_{k=1}^{9} T_k :=$$

$$:= \frac{g(x,y)}{\Gamma(1+\alpha)\Gamma(1+\beta)}\left[(x+h)^{\alpha}-x^{\alpha}\right][(y+\eta)^{\beta}-y^{\beta}] +$$

$$+\frac{(y+\eta)^{\beta}-y^{\beta}}{\Gamma(\alpha)\Gamma(1+\beta)}\int_{-h}^0 \frac{g(x-t,y)-g(x,y)}{(t+h)^{1-\alpha}} dt +$$

$$+\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(1+\alpha)\Gamma(\beta)}\int_{-\eta}^0 \frac{g(x,y-\tau)-g(x,y)}{(\tau+\eta)^{1-\beta}} d\tau +$$

$$+\frac{(y+\eta)^{\beta}-y^{\beta}}{\Gamma(\alpha)\Gamma(1+\beta)}\int_0^x [g(x-t,y)-g(x,y)]\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right] dt +$$

$$+\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(1+\alpha)\Gamma(\beta)}\int_0^y [g(x,y-\tau)-g(x,y)]\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right] d\tau +$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_{-\eta}^0 \frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)}{(h+t)^{1-\alpha}(\eta+\tau)^{1-\beta}} dt d\tau +$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_0^y \frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)}{(h+t)^{1-\alpha}}\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right] dt d\tau +$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_{-\eta}^0 \frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)}{(\eta+\tau)^{1-\beta}}\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right] dt d\tau +$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\left(\overset{1,1}{\Delta}_{-t,-\tau}\, g\right)(x,y)\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]\times$$
$$\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]dtd\tau .$$

The inequality is valid

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)\right|\le C\Bigg(\overset{1,1}{\omega}(\varphi;x,y)\,|(x+h)^{\alpha}-x^{\alpha}\,\|\,(y+\eta)^{\beta}-y^{\beta}|+$$
$$+|(y+\eta)^{\beta}-y^{\beta}|\,\left|\int_0^h\frac{\overset{1,1}{\omega}(\varphi;t,y)}{(h-t)^{1-\alpha}}dt\right|+|(x+h)^{\alpha}-x^{\alpha}|\,\left|\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi;x,\tau)}{(\eta-\tau)^{1-\beta}}d\tau\right|+$$
$$+|(y+\eta)^{\beta}-y^{\beta}|\,\left|\int_0^x\overset{1,1}{\omega}(\varphi;t,y)\,|(h+t)^{\alpha-1}-t^{\alpha-1}|\,dt\right|+$$
$$+|(x+h-a)^{\alpha}-(x-a)^{\alpha}|\,\left|\int_0^{y-c}\overset{1,1}{\omega}(\varphi;x-a,\tau)\,|(\eta+\tau)^{\beta-1}-\tau^{\beta-1}|\,d\tau\right|+$$
$$+\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi;t,\tau)}{(h-t)^{1-\alpha}(\eta-\tau)^{1-\beta}}dtd\tau+$$
$$+\int_0^h\int_0^y\overset{1,1}{\omega}(\varphi;t,\tau)(h-t)^{\alpha-1}\,|(\eta+\tau)^{\beta-1}-\tau^{\beta-1}|\,dtd\tau+$$
$$+\int_0^x\int_0^\eta\overset{1,1}{\omega}(\varphi;t,\tau)\,|(h+t)^{\alpha-1}-t^{\alpha-1}|\,(\eta-\tau)^{\beta-1}dtd\tau+$$
$$+\int_0^x\int_0^y\overset{1,1}{\omega}(\varphi;t,\tau)\,|(h+t)^{\alpha-1}-t^{\alpha-1}\,\|\,(\eta+\tau)^{\beta-1}-\tau^{\beta-1}|\,dtd\tau\Bigg).$$

Each term of this inequality is estimated in the standard way, and one can obtain

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)\right|\le C_3 h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau,$$

from which inequality (3.1) follows.

Theorem 3.2. Let $\rho(x,y)=\rho(x)\rho(y)=x^{\mu}y^{\nu}$, $0\le\mu<2-\alpha$, $0\le\nu<2-\beta$. If the function $\varphi(x,y)\in Q$ satisfies the following conditions:

1) $\varphi_0(x,y)=\rho(x,y)\varphi(x,y)\in \mathbf{C}(Q)$ and $\varphi_0(x,y)|_{x=0,\ y=0}=0$;

2) $\int_0^{b-a}\int_0^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{t^{\gamma}\tau^{\lambda}}dtd\tau$ the integral converges for $\gamma=\max\{1,\mu\}$, $\lambda=\max\{1,\nu\}$. Then the following estimates of Zygmund type are valid

$$\overset{1,0}{\omega}(\rho f;h,0)\le C_1\left[h^{\alpha+\gamma-1}\int_0^h\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^{\gamma}}dt+\right.$$
$$\left.+h\int_h^b\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^{2-\alpha}}dt\right],\qquad(3.5)$$

$$\overset{0,1}{\omega}(\rho f;0,\eta)\le C_2\left[\eta^{\beta+\lambda-1}\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^{\lambda}}d\tau+\right.$$
$$\left.+\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^{2-\beta}}d\tau\right],\qquad(3.4)$$

$$\overset{1,1}{\omega}(\rho f;h,\eta)\le C_3\left[h^{\alpha+\gamma-1}\eta^{\beta+\lambda-1}\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{\gamma}\tau^{\lambda}}dtd\tau+\right.$$
$$+h\eta^{\beta+\lambda-1}\int_h^b\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{\lambda}}dtd\tau+h^{\alpha+\gamma-1}\eta\int_0^h\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{\gamma}\tau^{2-\beta}}dtd\tau+$$
$$\left.+h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\right].\qquad(3.5)$$

Proof. By Remark 2.1, it suffices to deal with the case $\mu,\nu\ge 1$. Let $\varphi\in\tilde H_0^{\omega}(\rho)$, so that $\varphi_0(x,y)=\varphi(x,y)\rho(x,y)$, where $\varphi_0(x,y)\in\tilde H_0^{\omega}(\rho)$ and $\varphi_0(x,y)\big|_{x=0,\ y=0}=0$. For

$$G(x,y):=\int_0^x\int_0^y\frac{\rho(x,y)\varphi_0(t,\tau)dtd\tau}{\rho(t,\tau)(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}.$$

We represent $G(x,y)$ in the form

$$G(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left(\int_0^x\int_0^y\frac{\varphi_0(t,\tau)dtd\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}+\right.$$
$$\left.+\int_0^x\int_0^y B(x,y;t,\tau)\varphi_0(t,\tau)dtd\tau\right)=G_1(x,y)+G_2(x,y).$$

Here the question of the estimation of the modulus of continuity for the first term is solved by us in Theorem 3.1. Therefore, inequalities

$$\overset{1,0}{\omega}(G_1;h,0)\le C_1\left(h^{\alpha}\,\overset{1,1}{\omega}(\varphi_0;h,d)+h\int_h^b\frac{\overset{1,1}{\omega}(\varphi_0;t,d)}{t^{2-\alpha}}dt\right),\qquad(3.6)$$

$$\overset{0,1}{\omega}(G_1;0,\eta)\le C_2\left(\eta^{\beta}\,\overset{1,1}{\omega}(\varphi_0;b,\eta)+\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi_0;b,\tau)}{\tau^{2-\beta}}d\tau\right),\qquad(3.7)$$

$$\overset{1,1}{\omega}(G_1;h,\eta)\le C_3\left(h^{\alpha}\eta^{\beta}\,\overset{1,1}{\omega}(\varphi_0;h,\eta)+h\eta^{\beta}\int_h^b\frac{\overset{1,1}{\omega}(\varphi_0;t,\eta)}{t^{2-\alpha}}dt+\right.$$
$$\left.+h^{\alpha}\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi_0;h,\tau)}{\tau^{2-\beta}}d\tau+h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\right).\qquad(3.8)$$

To estimate the term $G_2(x,y)$, we note that the weight being degenerate, we have

$$\rho(x,y)-\rho(t,\tau)=[\rho(x)-\rho(t)][\rho(y)-\rho(\tau)]+\rho(\tau)[\rho(x)-\rho(t)]+$$
$$+\rho(t)[\rho(y)-\rho(\tau)],$$

which leads to the following representation

$$G_2(x,y)=\int_0^x\int_0^y B_1(x,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau+$$
$$+\int_0^x\int_0^y B_1(x,t)\varphi_0(t,\tau)(y-\tau)^{\beta-1}dtd\tau+$$
$$+\int_0^x\int_0^y B_2(y,\tau)\varphi_0(t,\tau)(x-t)^{\alpha-1}dtd\tau.$$

where the notation (2.3) has been used. For the difference $\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)$ with $h>0$ and $x,\ x+h\in(0,b)$, we have

$$\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)=\int_x^{x+h}\int_0^y B_1(x+h,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+\int_x^{x+h}\int_0^y\frac{\varphi_0(t,\tau)B_2(y,\tau)}{(x+h-t)^{1-\alpha}}dtd\tau+$$

$$+\int_0^x\int_0^y\varphi_0(t,\tau)\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]B_2(y,\tau)dtd\tau.$$

Since $\varphi_0(x,0)=0$ then the inequality

$$\left|\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)\right|\le\int_x^{x+h}\int_0^y|B_1(x+h,t)||B_2(y,\tau)|\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y|D_1(x,h,t)||B_2(y,\tau)|\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$\int_x^{x+h}\int_0^y|B_1(x+h,t)|\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+$$

$$+\int_0^x\int_0^y|D_1(x,h,t)|\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+$$

$$\int_x^{x+h}\int_0^y\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{(x+h-t)^{1-\alpha}}|B_2(y,\tau)|dtd\tau+$$

$$+\int_0^x\int_0^y\overset{1,1}{\omega}(\varphi_0;t,\tau)|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}||B_2(y,\tau)|dtd\tau.$$

We make use of (2.2) and obtain

$$|G_2(x+h,y)-G_2(x,y)|\le\int_x^{x+h}|B_1(x+h,t)|\overset{1,1}{\omega}(\varphi_0;t,d)dt+$$

$$+\int_0^x|D_1(x,h,t)|\overset{1,1}{\omega}(\varphi_0;t,d)dt+\int_x^{x+h}|B_1(x+h,t)|\overset{1,1}{\omega}(\varphi_0;t,d)dt+$$

$$+\int_0^x|D_1(x,h,t)|\overset{1,1}{\omega}(\varphi_0;t,d)dt+\int_x^{x+h}\frac{\overset{1,1}{\omega}(\varphi_0;t,d)}{(x+h-t)^{1-\alpha}}dt+$$

$$+\int_0^x\overset{1,1}{\omega}(\varphi_0;t,d)|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}|dt.$$

From the estimates $\Delta_1,\Delta_2,\Delta_3$ of Theorem 2.1 and from the estimates F_1,F_2 in Theorem 2.2, one can easily verify the validity of inequality

$$\left|\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)\right|\le C_1\left[h^{\alpha+\gamma-1}\int_0^h\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^\gamma}dt+h\int_h^{b-a}\frac{\overset{1,1}{\omega}(\rho\varphi;t,d-c)}{t^{2-\alpha}}dt\right], \qquad (3.9)$$

где $\gamma=\max(1,\mu)$.

The estimate

$$\left|\left(\overset{0,1}{\Delta}_\eta G_2\right)(x,y)\right|\le C_2\left[\eta^{\beta+\lambda-1}\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^\lambda}d\tau+\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^{2-\beta}}d\tau\right],$$

is symmetrically obtained, where $\lambda=\max(1,\nu)$.

For the mixed difference $\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)$ with $h,\eta>0$ and $x,x+h\in[0,b]$, $y,y+\eta\in[0,d]$ the appropriate representation leading to the separate evaluation in each variable without losses in another variable is as follows:

$$\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)=\int_x^{x+h}\int_y^{y+\eta}B_1(x+h,t)B_2(y+\eta,\tau)\varphi_0(t,\tau)d\tau dt+$$

$$+\int_0^x\int_0^y D_1(x,h,t)D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-\tau)^{1-\beta}}\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)(y+\eta-\tau)^{\beta-1}\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}(x+h-t)^{\alpha-1}B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y(x+h-t)^{\alpha-1}D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau.$$

The inequality is rightly

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)\right|\le C\left|\int_x^{x+h}\int_y^{y+\eta}B_1(x+h,t)B_2(y+\eta,\tau)\times\right.$$

$$\times\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+\int_0^x\int_0^y D_1(x,h,t)D_2(y,\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)D_2(y,\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)B_2(y+\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-\tau)^{1-\beta}}\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+ \qquad (3.10)$$

$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)(y+\eta-\tau)^{\beta-1}\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}(x+h-t)^{\alpha-1}B_2(y+\eta,\tau)\,\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]B_2(y+\eta,\tau)\,\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y(x+h-t)^{\alpha-1}D_2(y,\eta,\tau)\,\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]D_2(y,\eta,\tau)\,\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau\Bigg|.$$

We omit the details of evaluation of each term in the above representation; it is standard via Lemma 2.1 and yields

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)\right|\le C_3\Bigg[h^{\alpha+\gamma-1}\eta^{\beta+\lambda-1}\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^\lambda}dtd\tau+$$

$$+h\eta^{\beta+\lambda-1}\int_h^b\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^\lambda}dtd\tau+h^{\alpha+\gamma-1}\eta\int_0^h\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^{2-\beta}}dtd\tau+$$

$$+h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\Bigg],\qquad(3.11)$$

where $\gamma=\max(1,\mu)$ и $\lambda=\max(1,\nu)$.

From the inequalities (3.11), (3.10), (3.9) and (3.6), (3.7), (3.8), we obtain the corresponding estimates (3.3), (3.4) and (3.5).

4. Mapping properties of the mixed fractional integration operators in the space $\tilde{H}_0^\omega(\rho)$

In this section, we give a generalization of the theorem to the weighted.

Theorem 4.1. Let $0<\alpha,\beta<1$, $\rho(x,y)=(x-a)^\mu(y-c)^\nu$, $0\le\mu<2-\alpha$, $0\le\nu<2-\beta$. If $\omega(x,y)\in\Phi(Q)$ and assume that

1) $$\int_0^x\int_0^y\left(\frac{x}{t}\right)^\gamma\left(\frac{y}{\tau}\right)^\lambda\frac{\overset{1,1}{\omega}(t,\tau)}{t\tau}dtd\tau\le C\,\overset{1,1}{\omega}(x,y),$$

2) $$\int_x^b\int_y^d\left(\frac{x}{t}\right)^{1-\alpha}\left(\frac{y}{\tau}\right)^{1-\beta}\frac{\overset{1,1}{\omega}(t,\tau)}{t\tau}dtd\tau\le C\,\overset{1,1}{\omega}(x,y),$$

where $\gamma=\max(\mu-1,0)$, $\lambda=\max(\nu-1,0)$. Then the mixed fractional integral operator $I_{0+,0+}^{\alpha,\beta}$ is bounded from the weight space $\tilde{H}_0^\omega(\rho)$ to the space $\tilde{H}_0^{\omega_{\alpha,\beta}}(\rho)$ with the same weight and with the characteristic $\omega_{\alpha,\beta}(t,\tau)=t^\alpha\tau^\beta\omega(t,\tau)$.

Proof. Let $f=I_{0,0+}^{\alpha,\beta}\varphi$, where $\varphi\in\tilde{H}_0^\omega(\rho)$. We will show that $f\in\tilde{H}_0^{\omega_{\alpha,\beta}}(\rho)$. For this, it suffices to show that

$$\sup_{h>0}\frac{\overset{1,0}{\omega}(\rho f;h,0)}{h^\alpha\omega_1(h)}=C_1<\infty,\qquad\sup_{\eta>0}\frac{\overset{0,1}{\omega}(\rho f;0,\eta)}{\eta^\beta\omega_2(\eta)}=C_2<\infty,$$

$$\sup_{h>0,\eta>0}\frac{\overset{1,1}{\omega}(\rho f;h,\eta)}{h^\alpha\eta^\beta\omega_{1,1}(h,\eta)}=C_3<\infty.$$

From membership $\omega(t,\tau)$ in the class $\Phi(Q)$ and satisfaction of inequalities (4.1), (4.2) the convergence of the integrals follows

$$\int_0^b\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^\gamma}dt,\quad\int_0^d\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^\lambda}d\tau,\quad\int_0^b\int_0^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^\lambda}dtd\tau.$$

Therefore, there are estimates of the Zygmund type from Theorem 3.2. Whence follows

$$\frac{\overset{1,0}{\omega}(\rho f;h,0)}{h^\alpha\omega_{1,1}(h,d)}\le C_1\left(\frac{h^{\gamma-1}}{\omega_{1,1}(h,d)}\int_0^h\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^\gamma}dt+\right.$$

$$\left.+\frac{h^{1-\alpha}}{\omega_{1,1}(h,d)}\int_h^b\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^{2-\alpha}}dt\right)\le$$

$$\le C_1\|\rho\varphi\|_{\tilde{H}_0^\omega}\left\{\frac{h^{\gamma-1}}{\omega_{1,1}(h,d)}\int_0^h\frac{\overset{1,1}{\omega}(t,d)}{t^\gamma}dt+\frac{h^{1-\alpha}}{\omega_{1,1}(h,d)}\int_h^b\frac{\overset{1,1}{\omega}(t,d)}{t^{2-\alpha}}dt\right\}.$$

$$\frac{\overset{0,1}{\omega}(\rho f;0,\eta)}{\eta^\beta\omega_{1,1}(b,\eta)}\le C_2\left(\frac{\eta^{\lambda-1}}{\omega_{1,1}(b,\eta)}\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^\lambda}d\tau+\right.$$

$$\left.+\frac{\eta^{1-\beta}}{\omega_{1,1}(b,\eta)}\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^{2-\beta}}d\tau\right)\le$$

$$\le C_2\|\rho\varphi\|_{\tilde{H}_0^\omega}\left\{\frac{\eta^{\lambda-1}}{\omega_{1,1}(b,\eta)}\int_0^\eta\frac{\overset{1,1}{\omega}(b,\tau)}{\tau^\lambda}d\tau+\frac{\eta^{1-\beta}}{\omega_{1,1}(b,\eta)}\int_\eta^d\frac{\overset{1,1}{\omega}(b,\tau)}{\tau^{2-\beta}}d\tau\right\}.$$

$$\frac{\overset{1,1}{\omega}(\rho f;h,\eta)}{h^\alpha\eta^\beta\omega_{1,1}(h,\eta)}\le C_3\left(\frac{h^{\gamma-1}\eta^{\lambda-1}}{\omega_{1,1}(h,\eta)}\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^\lambda}dtd\tau+\right.$$

$$+\frac{h^{\gamma-1}\eta^{1-\beta}}{\omega_{1,1}(h,\eta)}\int_0^h\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^{2-\beta}}dtd\tau+\frac{h^{1-\alpha}\eta^{\lambda-1}}{\omega_{1,1}(h,\eta)}\int_h^b\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^\lambda}dtd\tau+$$

$$\left.+\frac{h^{1-\alpha}\eta^{1-\beta}}{\omega_{1,1}(h,\eta)}\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\right)\le\qquad(4.1)$$

$$\le C_3\|\rho\varphi\|_{\tilde{H}_0^\omega}\left\{\frac{h^{\gamma-1}\eta^{\lambda-1}}{\omega_{1,1}(h,\eta)}\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(t,\tau)}{t^\gamma\tau^\lambda}dtd\tau+\right.\qquad(4.2)$$

$$+\frac{h^{\gamma-1}\eta^{1-\beta}}{\omega_{1,1}(h,\eta)}\int_0^h\int_\eta^d\frac{\overset{1,1}{\omega}(t,\tau)}{t^\gamma\tau^{2-\beta}}dtd\tau+\frac{h^{1-\alpha}\eta^{\lambda-1}}{\overset{1,1}{\omega}(h,\eta)}\int_h^b\int_0^\eta\frac{\overset{1,1}{\omega}(t,\tau)}{t^{2-\alpha}\tau^\lambda}dtd\tau+$$

$$\left.+\frac{h^{1-\alpha}\eta^{1-\beta}}{\overset{1,1}{\omega}(h,\eta)}\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\right\}.$$

It's obvious that

$$\frac{\overset{1,0}{\omega}(\rho f;h,0)}{h^\alpha\omega_{1,1}(h,d)}\le C_1\|\varphi\|_{\tilde{H}_0^\omega(\rho)},\quad\frac{\overset{0,1}{\omega}(\rho f;0,\eta)}{\eta^\beta\omega_{1,1}(b,\eta)}\le C_2\|\varphi\|_{\tilde{H}_0^\omega(\rho)},$$

$$\frac{\overset{1,1}{\omega}(\rho f;h,\eta)}{h^\alpha\eta^\beta\omega_{1,1}(h,\eta)}\le C_3\|\varphi\|_{\tilde{H}_0^\omega(\rho)}.\qquad(4.3)$$

We estimate $\|\rho f\|_{C(Q)}$. We have

$$\rho(x,y)f(x,y)=x^{\mu}y^{\nu}\int_0^x\int_0^y\frac{\varphi_0(x-t,y-\tau)dtd\tau}{t^{1-\alpha}\tau^{1-\beta}(x-t)^{\mu}(y-\tau)^{\nu}}=$$
$$=x^{\alpha}y^{\beta}\int_0^1\int_0^1\frac{\varphi_0[x-x\xi,y-ys]d\xi ds}{\xi^{1-\alpha}(1-\xi)^{\mu}s^{1-\beta}(1-s)^{\nu}}.$$

Since $\varphi_0(x,y)|_{x=0,y=0}=0$, then

$$\left|\varphi_0(x-x\xi,y-ys)-\varphi_0(0,y-ys)\right|\le C_1\overset{1,0}{\omega}(\varphi_0;1-\xi),$$
$$\left|\varphi_0(x-x\xi,y-ys)-\varphi_0(x-x\xi,0)\right|\le C_2\overset{0,1}{\omega}(\varphi_0;1-s),$$
$$\left|\left(\overset{1,1}{\Delta}_{x(1-\xi),y(1-s)}\varphi_0\right)(x,y)\right|\le C_3\overset{1,1}{\omega}(\varphi_0;1-\xi,1-s).$$

It follows that

$$|\rho(x,y)f(x,y)|\le C\|\varphi\|_{\tilde H_0^{\omega}(\rho)}\int_0^1\int_0^1\frac{\omega_{1,1}(t,\tau)dtd\tau}{t^{\mu}\tau^{\nu}(1-t)^{1-\alpha}(1-\tau)^{1-\beta}}.$$

Therefore

$$\|\rho f\|_{C(Q)}\le C\|\varphi\|_{\tilde H_0^{\omega}(\rho)}. \tag{4.4}$$

From the inequalities (4.3) and (4.4) follows the assertion of the theorem.

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