Two consequences of the GTR2 proposal

The Yang-Mills fields - The concept of unique chronology

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In this document, I analyze two new consequences of the GTR2 proposal. The main purpose is to connect that approach with usual concepts in physics. First, I prove the existence of a formal link with Yang-Mills electromagnetic fields. Second, I reconsider the notion of time, especially the conditions allowing a unique chronology. At the end, I prove that a link with Lorentz transformations can be realized for a spherically symmetric and diagonal Friedmann Robertson Walker metric.

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1 Connecting the GTR2 and the GTR more explicitly

1.1 The initial formula and the goals

After having proposed [a], I have intensively explored the plausible links between the E. Cartan, B. Riemann and GTR2 approaches on metrics in the document [b]; see section 4 . I came to a very peculiar formula linking the coefficients of the GTR2 T-cubes and the Christoffel's cubes containing the symbols of the second kind [b; § 3.11, pp. 24-25], my *initial formula*:

$$
\begin{gathered}
{}_{\alpha\beta}F_{\lambda\mu}(GTR2) \\
= \\
\frac{1}{2}\,.\,(T^{\chi}_{\alpha\lambda}\,.\,T^{\phi}_{\beta\mu} \,-\, T^{\chi}_{\beta\lambda}\,.\,T^{\phi}_{\alpha\mu})\,.\,g_{\chi\phi} \\
= \\
-\frac{1}{2}\,.\,(q^{\alpha\beta} \,+\, q^{\beta\alpha})\,.\,g_{\alpha\phi}\,.\,\{\frac{\partial\Gamma^{\phi}_{\beta\lambda}}{\partial x^{\mu}} \,-\, \frac{\partial\Gamma^{\phi}_{\beta\mu}}{\partial x^{\lambda}} \,+\, (\Gamma^{\rho}_{\beta\lambda}\,.\,\Gamma^{\phi}_{\rho\mu} \,-\, \Gamma^{\rho}_{\beta\mu}\,.\,\Gamma^{\phi}_{\rho\lambda})\} \\
= \\
-t^{\alpha\beta}\,.\,R_{\beta\alpha\lambda\mu}(Riemann)
\end{gathered}
\tag{1}
$$

There is no sum on α and β; the invariant coefficients q are only depending on the pair (α, β).

For concrete applications, I give me two goals:

1. First, to connect the *initial formula* with some elements loaned to a Yang-Mills theory.
2. Second, isolate a function F such that:

$$T^{\chi}_{\alpha\lambda} \,=\, F(\Gamma^{\chi}_{\alpha\lambda},\, \frac{\partial\Gamma^{\chi}_{\alpha\lambda}}{\partial x^{\mu}}) \tag{2}$$

   In the next paragraphs, I shall explore some intuitive paths going into that direction. In general, the fact that the pseudo-curvature tensor of the GTR2 approach is formally similar to the Riemann's curvature tensor but not equal to it, makes it complicated to find the function F.

1.2 A formal link with a Yang-Mills theory

The greatest attention must be paid to the positions of indices and subscripts. For example: the part of the formula containing the components of the T-cube is compatible with the Yang-Mills terms; this is not true for the part of the formula involving the Christoffel's cube. This first encouraging compatibility is not a sufficient condition. Concretely, I must propose the following relations if I want to reconnect the GTR2 with a standard YM theory:

$${}_{\alpha\beta}f_{\lambda\mu}(GTR2) \,=\, \frac{1}{2}\,.\,T^{\chi}_{\alpha\lambda}\,.\,T^{\phi}_{\beta\mu}\,.\,g_{\chi\phi} \,=\, \frac{1}{2}\,.\,{}_{\alpha\beta}F^{a}_{\lambda\mu}\,.\,\tau_a \,=\, \{\frac{\partial({}_{\alpha\beta}A^{a}_{\lambda})}{\partial x^{\mu}} \,+\, c^{a}_{bc}\,.\,{}_{\alpha\beta}A^{b}_{\lambda}\,.\,{}_{\alpha\beta}A^{c}_{\mu}\}\,.\,E_a \tag{3}$$

Here, the $c_{bc}{}^{a}$ must be the structure constants of some ad hoc (but, at this stage, unknown) Lie algebra, L, of which the {... $1/2.\tau_a$ = E_a ...} form an orthogonal and normed basis.

Remark 1.1. ***An over-simplified model***

For a better understanding, let start with an oversimplified notation where the pairs ($\alpha\beta$) do not appear:

$$f_{\lambda\mu}(GTR2) \,=\, \frac{1}{2}\,.\,T^{\chi}_{\lambda}\,.\,T^{\phi}_{\mu}\,.\,g_{\chi\phi} \,=\, \frac{1}{2}\,.\,F^{a}_{\lambda\mu}\,.\,\tau_a \,=\, \{\frac{\partial(A^{a}_{\lambda})}{\partial x^{\mu}} \,+\, c^{a}_{bc}\,.\,A^{b}_{\lambda}\,.\,A^{c}_{\mu}\}\,.\,E_a \tag{4}$$

If I now suppose the existence of some cube D such that:

$$\frac{1}{2} \cdot g_{\chi\phi} = \sum_{a=1}^{a=dim L} d^a_{\chi\phi} \,.\, E_a \tag{5}$$

I get:

$$\forall\, a \in \{1,\, 2,\, ...,\, dim\, L\} :\; T^{\chi}_{\lambda} \,.\, T^{\phi}_{\mu} \,.\, d^a_{\chi\phi} = \frac{\partial(A^a_{\lambda})}{\partial x^{\mu}} + c^a_{bc} \,.\, A^b_{\lambda} \,.\, A^c_{\mu} \tag{6}$$

Here, the discussion involves four EM-potential vectors:

$$\lambda = 0,\, 1,\, 2,\, 3 :\; \mathbf{A}_{\lambda} = \sum_{a=1}^{a=dim L} A^a_{\lambda} \,.\, E_a \in E(dim\, L,\, R) \tag{7}$$

These vectors are commonly induced by the existence of electrical charges; for example: an electron, a quark, etc. They can be calculated [[01], German edition, p. 200, (65,6)]; at a distance r from a given charge q moving with a spatial speed $\mathbf{v}$:

$$\mathbf{A} = \frac{q}{2\,.\,c\,.\,r} \,.\, \{\mathbf{v} + < \mathbf{v},\, \mathbf{n} >_{Id_3} .\, \mathbf{n}\}$$

$$^{(3)}\mathbf{r} = r\,.\,^{(3)}\mathbf{n};\; r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \neq 0;\; ^{(4)}\mathbf{r} :\; (x^0 = c\,.\,t,\, x^1,\, x^2,\, x^3).$$

There are two manners to justify the existence of four EM potentials:

1. A given charge q is considered at four different positions in space and that charge is characterized by the pairs $(\mathbf{r}_{\lambda},\, \mathbf{v}_{\lambda})$.

   Pro: the appearance of these pairs would facilitate the integration of these EM-potential vectors into a theory including the Heisenberg's uncertainty principle (HUP).

   Contra: "Why only four positions?"

2. The field of four different charges is studied at a given position in space (the origin of the spatial frame) and each of them contributes with:

   $$\mathbf{A}_{\lambda} = \frac{q_{\lambda}}{2\,.\,c\,.\,r_{\lambda}} \,.\, \{\mathbf{v}_{\lambda} + < \mathbf{v}_{\lambda},\, \mathbf{n}_{\lambda} >_{Id_3} .\, \mathbf{n}_{\lambda}\};\; \mathbf{r}_{\lambda} = r_{\lambda} \,.\, \mathbf{n}_{\lambda},\; r_{\lambda} \neq 0.$$

   Contra: "Why only four charges?"

Furthermore, let suppose that an observer is following some geodesic (free fall) in an unknown connection U and that (s)he is studying the spatial variations of these EM-potential vectors. Concerning that point, recall that the *initial formula* has been obtained in a Fermi's frame; that means in a frame moving with one given particle. Then: the option 1 is meaningless since it will always give a vanishing potential; that observer can only study three charges in the option 2 in considering (s)he is always situated at one of the four summits of some virtual tetrahedron.

Let come back to the free-fall question. If I would have only one potential $\mathbf{A}$ at hand, I would write in a "usual" four-dimensional spacetime [[02]; p. 26, (1.7.2)]:

$$\nabla_{\mathbf{r}}\mathbf{A} = \sum_{a=0}^{a=3} \{ \sum_{\mu=0}^{\mu=3} \frac{\partial A^a}{\partial x^{\mu}} \,.\, \delta x_{\mu} + \sum_{c=0}^{c=3} \sum_{\mu=0}^{\mu=3} u^a_{c\mu} \,.\, x^{\mu} \,.\, A^c \} \,.\, \mathbf{e}_a = \mathbf{0} \tag{8}$$

or, in a more condensed manner:

$$\frac{\partial A^a}{\partial x^{\mu}} \,.\, \delta x_{\mu} + u^a_{b\mu} \,.\, x^{\mu} \,.\, A^b = 0 \tag{9}$$

My immediate technical problem consists in isolating the variations of the components of the EM-potential. Now, since:

$$A^a(t,\, ^{(3)}\mathbf{r},\, ^{(3)}\mathbf{v}) = \frac{q}{2\,.\,c\,.\,r} \,.\, \{v^a + <^{(3)} \mathbf{v},\, ^{(3)}\mathbf{n} >_{Id_3} .\, n^a\}$$

note that the variations of the EM-potentials by respect for the spacetime coordinates will have the following formalism:

$$\frac{\partial A^a(t,\, {}^{(3)}\mathbf{r},\, {}^{(3)}\mathbf{v})}{\partial x^\mu} \;=\; \frac{q}{2\,.\,c}\,.\,\{\frac{\partial(\frac{v^a}{r})}{\partial x^\mu} \;+\; \frac{\partial(<\mathbf{v},\,\mathbf{n}>_{Id_3}\,.\,\frac{n^a}{r})}{\partial x^\mu}\}$$

There exists a peculiar set of initial conditions. It is realized when, at instant $x^0 = 0$, a charge (i) moves with the spatial speed $\mathbf{v}$ inside a linear device and (ii) is studied by an observer situated in a plane orthogonal to the device. The relations are simplified in:

$$A^a(0,\, {}^{(3)}\mathbf{r},\, {}^{(3)}\mathbf{v}) \;=\; \frac{q}{2\,.\,c\,.\,r}\,.\,v^a;\; \frac{\partial A^a(0,\, {}^{(3)}\mathbf{r},\, {}^{(3)}\mathbf{v})}{\partial x^\mu} \;=\; \frac{q}{2\,.\,c\,.\,r}\,.\,\frac{\partial v^a}{\partial x^\mu}$$

In these conditions, I can then continue with:

$$dA^a \;=\; \frac{\partial A^a}{\partial x^\mu}\,.\,\delta x_\mu \;=\; \frac{q}{2\,.\,c\,.\,r}\,.\,\frac{\partial v^a}{\partial x^\mu}\,.\,\delta x_\mu \;=\; \frac{q}{2\,.\,c\,.\,r}\,.\,dv^a \;=\; \frac{q}{2\,.\,c\,.\,r}\,.\,g^a\,.\,dt$$

or, in a more condensed manner:

$$\frac{d\mathbf{A}(0,\, {}^{(3)}\mathbf{r},\, {}^{(3)}\mathbf{v})}{dt} \;=\; \frac{q}{2\,.\,c\,.\,r}\,.\,\mathbf{g}(0,\, {}^{(3)}\mathbf{r},\, {}^{(3)}\mathbf{v})$$

Here, $\mathbf{g}$ denotes the eventual acceleration of the charge. But: "What could be the cause for such an acceleration? And, when it exits, into which direction does this acceleration point?"

- *Proposition 1*: the electrical current flowing inside a linear device induces a field which is known to allow the appearance of a Lenz force. But that force cannot appear here as long as the observer is not him (her-) self carrying an electrical charge;
- *Proposition 2*: the charge is not only constrained by the fact that it must stay inside the cable. Its acceleration is induced by a difference of potential at the extremities of the cable. Therefore, that acceleration can only be parallel to the principal axis of that cable and to the speed $\mathbf{v}$. In that peculiar case, there exists a scalar k such that:

$$\mathbf{g} \;=\; k\,.\,\mathbf{v}$$

I have four such potentials and I should write:

$$0 \;=\; \frac{\partial(A^a_\lambda)}{\partial x^\mu}\,.\,\delta x_\mu \;+\; u^a_{b\mu}\,.\,x^\mu\,.\,A^b_\lambda \tag{10}$$

These relations can not be directly injected into the equation (6). But, I can multiply that equation and write:

$$\forall\, a \,\in\, \{1,\, 2,\, ...,\, dim\, L\}:\; T^\chi_\lambda\,.\,T^\phi_\mu\,.\,d^a_{\chi\phi}\,.\,\delta x_\lambda\,.\,\delta x_\mu \;=\; \frac{\partial(A^a_\lambda)}{\partial x^\mu}\,.\,\delta x_\lambda\,.\,\delta x_\mu + c^a_{bc}\,.\,A^b_\lambda\,.\,A^c_\mu\,.\,\delta x_\lambda\,.\,\delta x_\mu \tag{11}$$

As a consequence, for a free-falling set of EM-potentials:

$$\forall\, a \,\in\, \{1,\, 2,\, ...,\, dim\, L\}: \tag{12}$$

$$T^\chi_\lambda\,.\,T^\phi_\mu\,.\,d^a_{\chi\phi}\,.\,\delta x_\lambda\,.\,\delta x_\mu \;=\; -u^a_{b\mu}\,.\,x^\mu\,.\,(A^b_\lambda\,.\,\delta x_\lambda) \;+\; c^a_{bc}\,.\,A^b_\lambda\,.\,A^c_\mu\,.\,\delta x_\lambda\,.\,\delta x_\mu$$

Let consider the second option again, imagine that the observer travels with one charge (Fermi's walker) and suppose that (s)he can find three EM-potentials of which the characteristics give him (her) the sensation that they are "parallel transported". The Lie algebra at hand may then eventually be SU(2); hence: a = 1, 2, 3.

The cube $[d_{\chi\phi}{}^a] = D$ is a (4-4-3) cube and the l.h.t contains one of the three components of the $\otimes_D(\mathbf{T}_\lambda,\, \mathbf{T}_\mu)$ deformed tensor product between two vectors in E(4, R). More exactly, let introduce the vector:

$$\mathbf{T} \;=\; \sum_{\lambda\,=\,0}^{\lambda\,=\,3} \delta x^\lambda\,.\,\mathbf{T}_\lambda \tag{13}$$

The l.h.t is nothing else, but the a-th component of:

$$\otimes_D(\mathbf{T},\, \mathbf{T}) \,\in\, E(3, R)$$

Let try to decode the r.h.t in following the same vein. That term introduce a set of three Pfaff's forms:

$$b \,=\, 1,\, 2,\, 3 :\, l^b \,=\, \sum_{\lambda\,=\,0}^{\lambda\,=\,3} A^b_\lambda \,.\, \delta x^\lambda \tag{14}$$

These forms are well-known in physics since they represent a part of the integrand of the action (i.e.: a Lagrange function) [[04], German edition, p. 54, (16,1)]. With this writing, the r.h.t is such that:

$$a \,=\, 1,\, 2,\, 3 :\, (\otimes_D(\mathbf{T},\, \mathbf{T}))^a \,=\, -u^a_{b\mu} \,.\, x^\mu \,.\, l^b \,+\, c^a_{bc} \,.\, l^b \,.\, l^c \tag{15}$$

With the notations of my theory, it is easy to recognize:

$$|\otimes_D\,(\mathbf{T},\, \mathbf{T}) >= \,-_U\Phi(\mathbf{l}) \,.\, |\mathbf{x} > \,+ |\otimes_C\,(\mathbf{l},\, \mathbf{l}) > \tag{16}$$

Here, **l** is a vector in E(3, R) of which the components are the three Lagrange functions related to each of the three EM-potentials. This model can be seen as a "spherical cow" offering a first link between the T-vectors and the EM potentials.

Remark 1.2. ***Generalization***

Well, I must now repeat and extand that way of doing in taking care of the $(\alpha\beta)$ pairs. Following the spirit of that track, it is evident that I need three types of cubes: (i) cubes allowing an alternative expression for the metrics - see equation (–); (ii) cubes corresponding to the structure constants of an ad hoc Lie algebra, L, and (iii) cubes related to physical connections describing the plausible evolutions of EM potentials. The extrapolation is a relatively easy task for what concerns the l.h.t:

$$T^\chi_\lambda \,.\, T^\phi_\mu \,.\, d^a_{\chi\phi} \,\rightarrow\, T^\chi_{\alpha\lambda} \,.\, T^\phi_{\beta\mu} \,.\, d^a_{\chi\phi}$$

In that case, the l.h.t is representing one of the dim$L$ components of the $\otimes_D(\mathbf{T}_{\alpha\,\lambda},\, \mathbf{T}_{\beta\,\mu})$ deformed tensor product between two vectors in E(4, R); there are just quite more pairs of vectors than in the over-simplified version of that approach.

For the r.h.t, the question is: "Must I involve more cubes or more EM potentials?" Ad absurdo, if I would introduce more cubes, I would also introduce more physical connections and a priori suppose that the diverse EM potentials are following diverse geodesics simultaneously... this is not a convenient hypothesis. Therefore, I shall suppose the existence of more EM potentials. Starting again from equation (6):

$$T^\chi_{\alpha\lambda} \,.\, T^\phi_{\beta\mu} \,.\, d^a_{\chi\phi} \,=\, \frac{\partial({}_{\alpha\beta}A^a_\lambda)}{\partial x^\mu} \,+\, c^a_{bc} \,.\, {}_{\alpha\beta}A^b_\lambda \,.\, {}_{\alpha\beta}A^c_\mu \tag{17}$$

Multiplying it:

$$T^\chi_{\alpha\lambda} \,.\, T^\phi_{\beta\mu} \,.\, d^a_{\chi\phi} \,.\, \delta x_\lambda \,.\, \delta x_\mu \,=\, \frac{\partial({}_{\alpha\beta}A^a_\lambda)}{\partial x^\mu} \,.\, \delta x_\lambda \,.\, \delta x_\mu \,+\, c^a_{bc} \,.\, {}_{\alpha\beta}A^b_\lambda \,.\, {}_{\alpha\beta}A^c_\mu \,.\, \delta x_\lambda \,.\, \delta x_\mu \tag{18}$$

Supposing that:

$$\frac{\partial({}_{\alpha\beta}A^a_\lambda)}{\partial x^\mu} \,.\, \delta x_\mu \,+\, u^a_{b\mu} \,.\, x^\mu \,.\, {}_{\alpha\beta}A^b_\lambda \,=\, 0 \tag{19}$$

Injecting that hypothesis:

$$\forall\, a \,\in\, \{1,\, 2,\, ...,\, dim\, L\} : \tag{20}$$

$$T^\chi_{\alpha\lambda} \,.\, T^\phi_{\beta\mu} \,.\, d^a_{\chi\phi} \,.\, \delta x_\lambda \,.\, \delta x_\mu \,=\, -u^a_{b\mu} \,.\, x^\mu \,.\, ({}_{\alpha\beta}A^b_\lambda \,.\, \delta x_\lambda) \,+\, c^a_{bc} \,.\, {}_{\alpha\beta}A^b_\lambda \,.\, {}_{\alpha\beta}A^c_\mu \,.\, \delta x_\lambda \,.\, \delta x_\mu$$

The cube $[\mathrm{d}_{\chi\phi}{}^a] = \mathrm{D}$ is a (4-4-dim$L$) cube and the l.h.t contains one of the dim$L$ components of the $\otimes_D(\mathbf{T}_{\alpha\,\lambda},\, \mathbf{T}_{\beta\,\mu})$ deformed tensor product between two vectors in E(4, R). Let introduce the vectors:

$$\alpha \,=\, 0,\, 1,\, 2,\, 3 :\, \mathbf{T}_\alpha \,=\, \sum_{\lambda\,=\,0}^{\lambda\,=\,3} \delta x^\lambda \,.\, \mathbf{T}_{\alpha\lambda} \tag{21}$$

The l.h.t is nothing else, but the a-th component of:

$$\otimes_D(\mathbf{T}_\alpha,\, \mathbf{T}_\beta) \,\in\, E(dim\, L, R)$$

Let try to decode the r.h.t in following the same vein. Here again, that term introduces a set of Pfaff's forms:

$$\alpha,\, \beta \,=\, 0,\, 1,\, 2,\, 3;\, b \,=\, 1,\, 2,\, ...,\, dim\, L : {}_{\alpha\beta}l^b \,=\, \sum_{\lambda\,=\,0}^{\lambda\,=\,3} {}_{\alpha\beta}A^b_\lambda \,.\, \delta x^\lambda \tag{22}$$

With this writing, the r.h.t is such that:

$$\alpha,\, \beta \,=\, 0,\, 1,\, 2,\, 3;\, a \,=\, 1,\, 2,\, ...,\, dimL : (\otimes_D(\mathbf{T}_\alpha,\, \mathbf{T}_\beta))^a \,=\, -u^a_{b\mu} \,.\, x^\mu \,.\, {}_{\alpha\beta}l^b + c^a_{bc} \,.\, {}_{\alpha\beta}l^b \,.\, {}_{\alpha\beta}l^c \tag{23}$$

With the notations of my theory, it is easy to recognize:

$$|\otimes_D\, (\mathbf{T}_\alpha,\, \mathbf{T}_\beta) >= \, -_U\Phi({}_{\alpha\beta}\mathbf{l}) \,.\, |\mathbf{x} > \, + |\otimes_C\, ({}_{\alpha\beta}\mathbf{l},\, {}_{\alpha\beta}\mathbf{l}) > \tag{24}$$

The cube U is a (dim$L$-4-dim$L$) cube. The vectors:

$$\alpha,\, \beta \,=\, 0,\, 1,\, 2,\, 3 : {}_{\alpha\beta}\mathbf{l} : (..., {}_{\alpha\beta}l^b, ...) \,\in\, E(dim\, L,\, R)$$

have components which are parts of a Lagrange function. The cube C is a (dim$L$-dim$L$-dim$L$) cube and the following vectors are in E(dim$L$, R):

$$\otimes_C({}_{\alpha\beta}\mathbf{l},\, {}_{\alpha\beta}\mathbf{l}) \,\in\, E(dim\, L,\, R)$$

The identity of the Lie algebra can be guessed if its dimension is known. Within the GTR2, there are six pairs ($\alpha\beta$). Since the discussion takes place in E(4, R), there are four values for λ. These two facts are only concerning the subscripts and they suggest the existence of at least twenty-four potentials times the dimension of the Lie algebra I am looking for. If the indice "a" would not have been there, I would have proposed to work with SU(5); i.e.: with a = 1, 2, ..., 24.

1.3 A formal link with 1928 Einstein's work

In an approach focusing on the search for the function F (see the goals of this document in subsection 1.1), a first step is eventually done in separating the *initial formula* into two parts:

$$T^\chi_{\alpha\lambda} \,.\, T^\phi_{\beta\mu} \,.\, g_{\chi\phi} \,=\, (q^{\alpha\beta} \,+\, q^{\beta\alpha}) \,.\, g_{\alpha\phi} \,.\, (\frac{\partial\Gamma^\phi_{\beta\mu}}{\partial x^\lambda} \,+\, \Gamma^\rho_{\beta\mu} \,.\, \Gamma^\phi_{\rho\lambda}) \tag{25}$$

Amazingly and unexpectidly, the *initial formula* has a very similar formalism than the one of an equation appearing in [[03]; p. 4, no number, 2019] and which the authors claim that it has been demonstrated in [[04]; 1976].

It is amazing because the latter is a prospect refering to E. Cartan's theory; as it is the case in my exploration [b]. And it is unexpected because the *initial formula* has been obtained in a totally different context than the one which I have exposed in [a] and [b].

For the pedagogy, it may also be remarked the *formal similitude* between the left hand side (the GTR2 side) and the Lagrange density form 2 which has been proposed by Einstein himself in his 1928 manuscript [[05]; p. 16, (144a) - and references there in; among them:] [[06]]. If a direct identification between the GTR2 cubes T and Einstein's Λ cubes (see definition in: [[05]; p. 15, (139)]) is acceptable, must yet be tested.

For now, and at a technical level, the Λ cubes can simply be related to a hundred years old tool (Pfaff's forms and the extern differentiation; see section 4 [b; § 3.3, pp. 12-14]). The GTR can be formulated in the tetrad (vier-bein) language and the concept of extern differentiation can be translated into the modern language of deformed exterior products.

Another intuitive development of that way of thinking is perhaps to consider that the most important actor in that formula is not the Christoffel's cube containing the symbols of the second kind but, one of its allowed factorizations (see, e.g.: either my work in section 4 [f] or academic sources, like, e.g.: MTW and [[11]]).

1.4 A hint when we work with Riemann's normal coordinates

The initial formula can also be written:

$$g_{\chi\phi}\,.\,(T^{\chi}_{\alpha\lambda}\,.\,T^{\phi}_{\beta\mu}\;-\;T^{\chi}_{\beta\lambda}\,.\,T^{\phi}_{\alpha\mu})\;=\;-(q^{\alpha\beta}\;+\;q^{\beta\alpha})\,.\,R_{\beta\alpha\lambda\mu}(Riemann) \tag{26}$$

In mathematics, if a theory is true in general, then it is automatically true for all peculiar cases which are contained in the domain of validity of that theory; of course. This is a trivial affirmation based on an elementary and old principle: one exception is enougth to invalid the rule.

As direct application of that principle, if I want to confront the GTR2 theory with previous knowledge which have been accumulated in mathematics and in physics, I must first recall its domain of validity.

The *initial formula* has been discovered in a mathematical context involving the Fermi-Walker coordinates. Hence, if I want to examine what happens, for example, with Riemann's normal coordinates, I must verify if and when both types of coordinates are compatible. The question is formally open since 1922, the year of the introduction of Fermi's coordinates into the discussions. The answer to that interrogation is known and described, e.g. in [[07]; p. 2, fourth §]: "(citation) *if the worldline is a geodesic, (the Fermi coordinates) display the same property as the Riemann's coordinates*[1] *all along the worldline ...* (end of the citation)". Conclusion: in restricting the analysis to a worldline, the *initial formula* can be studied in involving the Riemann's normal coordinates (RNC).

The next technical point is: "What is the Riemann's normal coordinates definition?" Apart from Riemann's initial work [[08]; 1872] to which I have no access, it is possible to get serious indications in [[09]; pp. 27-32, especially in § 26, equ. (40); 1925]. Due to [[09]; p. 23, (31)], these coordinates are such that in the vicinity of the origin:

$$\forall\,\mathbf{x}\,\sim\,\mathbf{0}:\;\Gamma^{\rho}_{\beta\gamma}.\Gamma^{\alpha}_{\rho\kappa}\;-\;\Gamma^{\rho}_{\beta\kappa}.\Gamma^{\alpha}_{\rho\gamma}\;=\;0 \tag{27}$$

$$\forall\,\mathbf{x}\,\sim\,\mathbf{0}:\;R^{\alpha}{}_{\beta\gamma\kappa}\;=\;\frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\kappa}}\;-\;\frac{\partial\Gamma^{\alpha}_{\beta\kappa}}{\partial x^{\gamma}} \tag{28}$$

More recent and precize information can be read for example in [[10]; pp. 2-4, especially (1), (2), (3) and (4); 2000].

Pro:

- The first equation, (–), insures the existence of a multiplicative morphism between {E(4, R), $\otimes_{\Gamma}$} and {Φ(4, R), "."} where Φ(4, R) denotes the set of all trivial decompositions for the deformed tensor product $\otimes_{\Gamma}$, whilst "." denotes the customized product of two elements in M(4, R); see my demonstration in [-].
- The second equation can be developped further with the help of, e.g., [[01]-a; § 92, p. 312, (92.1)]:

$$\forall\,\mathbf{x}\,\sim\,\mathbf{0}:\;R_{\alpha\beta\lambda\mu}(Riemann)\;=\;\frac{1}{2}\,.\,\{(\frac{\partial^2 g_{\alpha\mu}}{\partial x^{\beta}\partial x^{\lambda}}+\frac{\partial^2 g_{\beta\lambda}}{\partial x^{\alpha}\partial x^{\mu}})-(\frac{\partial^2 g_{\alpha\lambda}}{\partial x^{\beta}\partial x^{\mu}}+\frac{\partial^2 g_{\beta\mu}}{\partial x^{\alpha}\partial x^{\lambda}})\} \tag{29}$$

- This equation allows another writing for the initial formula when $\mathbf{x}\sim\mathbf{0}$ and justifies the two following new propositions:
  - *First proposition*

$$g_{\chi\phi}\,.\,T^{\chi}_{\alpha\lambda}\,.\,T^{\phi}_{\beta\mu}\;=\;\frac{1}{2}\,.\,q^{\alpha\beta}\,.\,(\frac{\partial^2 g_{\alpha\mu}}{\partial x^{\beta}\partial x^{\lambda}}\;+\;\frac{\partial^2 g_{\beta\lambda}}{\partial x^{\alpha}\partial x^{\mu}}):\;no \tag{30}$$

  - *Second proposition*

$$g_{\chi\phi}\,.\,T^{\chi}_{\alpha\lambda}\,.\,T^{\phi}_{\beta\mu}\;=\;\frac{1}{2}\,.\,q^{\alpha\beta}\,.\,(\frac{\partial^2 g_{\alpha\mu}}{\partial x^{\beta}\partial x^{\lambda}}\;-\;\frac{\partial^2 g_{\alpha\lambda}}{\partial x^{\beta}\partial x^{\mu}}):\;yes \tag{31}$$

[1] The authors certainly mean: Riemann's *normal* coordinates.

Since the left hand side is representing a component (λ, μ) of a GTR2-EM field, that side must be anti-symmetric when $\lambda \leftrightarrow \mu$; therefore, the first proposition must be discarded.

Following that vein and reconsidering my exploration in section 4 [b; § 3.9, p. 22; § 3.6, p. 19], I am pushed to propose the next identifications in Fermi's coordinates (only when they coincide with the Riemann's normal coordinates):

$$ {}_{\alpha\beta}d_{\lambda\mu} \;=\; \frac{\partial^2 g_{\alpha\mu}}{\partial x^\beta \partial x^\lambda} $$

$$ {}_{\alpha\beta}d_{\lambda\mu} \;+\; {}_{\alpha\beta}d_{\mu\lambda} \;=\; \frac{\partial^2 g_{\alpha\mu}}{\partial x^\beta \partial x^\lambda} \;+\; \frac{\partial^2 g_{\beta\lambda}}{\partial x^\alpha \partial x^\mu} \;=\; \frac{\partial_{\alpha\beta} d_\lambda}{\partial x^\mu} \;=\; \frac{\partial_{\alpha\beta} d_\mu}{\partial x^\lambda} $$

Contra: This complementary analysis holds only when the Riemann's normal coordinates are compatible with Fermi's coordinates.

1.5 When the GTR2-T cubes are equivalent to the Christoffel's cubes

I have explained, also in section 4 [b; remark 2.1, pp. –, especially (24)], the conditions allowing a coincidence between the E. Cartan's F-cubes, the GTR2-T cubes and the Christoffel's cubes:

$$ \frac{\partial T^\alpha_{\beta\gamma}}{\partial x^\kappa} \;-\; \frac{\partial T^\alpha_{\beta\kappa}}{\partial x^\gamma} \;=\; \frac{\partial F^{\beta\gamma\alpha}}{\partial x^\kappa} \;-\; \frac{\partial F^{\beta\kappa\alpha}}{\partial x^\gamma} \;=\; \frac{\partial^2 (g\,.\,g^{\beta\gamma})}{\partial u^\alpha \partial x^\kappa} \;-\; \frac{\partial^2 (g\,.\,g^{\beta\kappa})}{\partial u^\alpha \partial x^\gamma} \;=\; 0 $$

that can be also rewritten as:

$$ \frac{\partial (g\,.\,g^{\beta\gamma})}{\partial x^\kappa} \;-\; \frac{\partial (g\,.\,g^{\beta\kappa})}{\partial x^\gamma} \;=\; \Psi(\mathbf{x}),\, \forall\, \mathbf{u} \tag{32} $$

In extenso: the value Ψ does not depend on $\mathbf{u}$; it only depends on $\mathbf{x}$.

Pro: This situation facilitates the calculations; see examples below: subsections 2.2 and 2.3 .

Contra: There is only a small number of situations for which the necessary conditions hold.

2 Consequences of a unique lapse of time

2.1 General considerations

The GTR2 proposal, [a], tells a fundamental question as soon as one is thinking about its implication on the notion of time. This can be easily proved in considering a ADM-like partition of spacetime: i.e., a 1D (time: label 0) + 3D (space: latin labels 1, 2 and 3) decomposition. Concerning the variations of the basis vectors, the theory introduces the following equations:

$$ \forall\, \alpha \;=\; 0, 1, 2, 3 :\; d\mathbf{e}_\alpha \;=\; \{T^\beta_{\alpha\lambda}\,.\,\delta x^\lambda_1 \;+\; \frac{1}{2}\,.\,T^\beta_{\alpha\lambda\mu}\,.\,\delta x^\lambda_1\,.\,\delta x^\mu_1 \;+\; 0(3)\}\,.\,\mathbf{e}_\beta \tag{33} $$

They immediately yield:

$$ \omega^\beta{}_\alpha \;=\; T^\beta_{\alpha\lambda}\,.\,\delta x^\lambda_1 \;+\; \frac{1}{2}\,.\,T^\beta_{\alpha\lambda\mu}\,.\,\delta x^\lambda_1\,.\,\delta x^\mu_1 \;+\; 0(3) \tag{34} $$

The ω's can de facto be understood as 16 quasi-vanishing polynomials of degree two depending only on the spatial components of $\delta\mathbf{x}$; precisely one for each possible value of the generic pair (λ, θ):

$$ 0 \;=\; {}^\theta_\lambda c_{kp}\,.\,\delta x^k\,.\,\delta x^p \;+\; {}^\theta_\lambda c_k\,.\,\delta x^k \;+\; {}^\theta_\lambda c \tag{35} $$

As a matter of facts, each polynomial, P(λ, θ), is characterized by its coefficients; respectively:

- of degree two

$$ {}^{\theta}_{\lambda}c_{kp} \,=\, \frac{1}{2}\,.\,T^{\theta}_{kp\lambda} \tag{36} $$

- of degree one

$$ {}^{\theta}_{\lambda}c_{k} \,=\, T^{\theta}_{k\lambda} \,+\, \frac{1}{2}\,.\,(T^{\theta}_{0k\lambda} \,+\, T^{\theta}_{k0\lambda})\,.\,\delta x^{0} \tag{37} $$

- of degree zero

$$ {}^{\theta}_{\lambda}c \,=\, -\omega^{\theta}{}_{\lambda} \,+\, T^{\theta}_{0\lambda}\,.\,\delta x^{0} \,+\, \frac{1}{2}\,.\,T^{\theta}_{00\lambda}\,.\,(\delta x^{0})^{2} \tag{38} $$

The lapse of time δx^0 plays a role for the evolution of the coefficients of degree zero and of degree one whilst it has no importance for those of degree two. Anyway, this 1D + 3D decomposition allows the introduction of an important concept called "evolution along the time" into the theory. Two problematics must now be addressed:

- the continuity of the polynomials;
- the unicity of the lapse of time.

Continuity: Without entering in any details, the continuity is obtained for a given pair (λ, θ) (and I shall admit without demonstration that it is the case) when:

$$ {}^{\theta}_{\lambda}c_{kp} \,=\, \frac{1}{2}\,.\,T^{\theta}_{kp\lambda} \,=\, \frac{1}{2}\,.\,T^{\theta}_{pk\lambda} \,=\, {}^{\theta}_{\lambda}c_{pk} \tag{39} $$

Definition 2.1. ***Continuous story***

Per definition, within the GTR2 theory (that theory), a story is continuous when the continuity is simultaneously realized for all polynomials P(λ, θ).

Unicity of the lapse of time: Is is easy to understand why this unicity is important in physics: "When it is realized, then there is only one possible story between a situation in space-time and the next one". This unicity is obtained for a given pair (λ, θ) when the discriminant of the coefficient of degree zero is null:

$$ (T^{\theta}_{0\lambda})^{2} \,+\, 2\,.\,(\omega^{\theta}{}_{\lambda} \,+\, {}^{\theta}_{\lambda}c)\,.\,T^{\theta}_{00\lambda} \,=\, 0 \tag{40} $$

Definition 2.2. ***Unique story***

Per definition, within the GTR2 theory (that theory), a story is unique when the unicity of the lapse of time is realized simultaneously for all pairs (λ, θ).

Within the GTR2 approach, we must write (see section 4 [a]):

$$ \frac{\partial \mathbf{e}_{\alpha}}{\partial x^{0}} \,=\, T^{\beta}_{0\alpha}\,.\,\mathbf{e}_{\beta};\; \frac{\partial^{2}\mathbf{e}_{\alpha}}{\partial^{2}x^{0}} \,=\, T^{\beta}_{00\alpha}\,.\,\mathbf{e}_{\beta} \tag{41} $$

But provided the lapse of time between an initial and a final situation is unique, I must write:

$$ \delta x^{0} \,=\, -\frac{T^{\beta}_{0\alpha}}{T^{\beta}_{00\alpha}} \tag{42} $$

This obligation has an immediate consequence:

$$ \frac{\partial \mathbf{e}_{\alpha}}{\partial x^{0}} \,=\, -\delta x^{0}\,.\,T^{\beta}_{00\alpha}\,.\,\mathbf{e}_{\beta} \,=\, -\delta x^{0}\,.\,\frac{\partial^{2}\mathbf{e}_{\alpha}}{\partial^{2}x^{0}} \tag{43} $$

Following the same vein, I can also write (because of the consequences of that unicity on the relation giving the coefficients of degree one):

$$ \frac{\partial \mathbf{e}_{\alpha}}{\partial x^{k}} \,=\, T^{\beta}_{k\alpha}\,.\,\mathbf{e}_{\beta} \,=\, \{{}^{\beta}_{\alpha}c_{k} \,-\, \delta x^{0}\,.\,T^{\beta}_{0k\alpha}\}\,.\,\mathbf{e}_{\beta} \,=\, {}^{\beta}_{\alpha}c_{k}\,.\,\mathbf{e}_{\beta} \,-\, \delta x^{0}\,.\,\frac{\partial^{2}\mathbf{e}_{\alpha}}{\partial x^{0}\partial x^{k}} $$

This yields for a non-vanishing lapse of time:

$$ \frac{\delta x^{k}}{\delta x^{0}}\,.\,\frac{\partial \mathbf{e}_{\alpha}}{\partial x^{k}} \,=\, \frac{\delta x^{k}}{\delta x^{0}}\,.\,{}^{\beta}_{\alpha}c_{k}\,.\,\mathbf{e}_{\beta} \,-\, \delta x^{k}\,.\,\frac{\partial^{2}\mathbf{e}_{\alpha}}{\partial x^{0}\partial x^{k}} \tag{44} $$

Considering both results simultaneously, this is also:

$$\frac{\delta \mathbf{e}_\alpha}{\delta x^0} \;=\; \frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \,.\, \mathbf{e}_\beta \;-\; \frac{\delta \frac{\partial \mathbf{e}_\alpha}{\partial x^0}}{\delta x^0}$$

or:

$$\frac{\delta \mathbf{e}_\alpha}{\delta x^0} \;=\; \frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \,.\, \mathbf{e}_\beta \;-\; \frac{\delta (T^{\beta}_{0\alpha} \,.\, \mathbf{e}_\beta)}{\delta x^0}$$

or:

$$\frac{\delta \mathbf{e}_\alpha}{\delta x^0} \;=\; \frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \,.\, \mathbf{e}_\beta \;-\; \frac{\delta T^{\beta}_{0\alpha}}{\delta x^0} \,.\, \mathbf{e}_\beta \;-\; T^{\beta}_{0\alpha} \cdot \frac{\delta \mathbf{e}_\beta}{\delta x^0}$$

or:

$$\frac{\delta \mathbf{e}_\alpha}{\delta x^0} \;+\; T^{\beta}_{0\alpha} \cdot \frac{\delta \mathbf{e}_\beta}{\delta x^0} \;=\; (\frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \;-\; \frac{\delta T^{\beta}_{0\alpha}}{\delta x^0}) \,.\, \mathbf{e}_\beta$$

This is:

$$(\delta^{\gamma}_{\alpha} \;+\; T^{\gamma}_{0\alpha}) \,.\, \frac{\delta \mathbf{e}_\gamma}{\delta x^0} \;=\; (\frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \;-\; \frac{\delta T^{\beta}_{0\alpha}}{\delta x^0}) \,.\, \mathbf{e}_\beta$$

This can be condensed in:

$$[\delta^{\gamma}_{\alpha} \;+\; T^{\gamma}_{0\alpha}] \,.\, |\frac{\delta \mathbf{e}_\gamma}{\delta x^0} >= [\frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \;-\; \frac{\delta T^{\beta}_{0\alpha}}{\delta x^0}] \,.\, |\mathbf{e}_\beta > \tag{45}$$

Or, in extenso:

$$\begin{bmatrix} 1 + T^0_{00} & T^1_{00} & T^2_{00} & T^3_{00} \\ T^0_{01} & 1 + T^1_{01} & T^2_{01} & T^3_{01} \\ T^0_{02} & T^1_{02} & 1 + T^2_{02} & T^3_{02} \\ T^0_{03} & T^1_{03} & T^2_{03} & 1 + T^3_{03} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta \mathbf{e}_0}{\delta x^0} \\ \frac{\delta \mathbf{e}_1}{\delta x^0} \\ \frac{\delta \mathbf{e}_2}{\delta x^0} \\ \frac{\delta \mathbf{e}_3}{\delta x^0} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \frac{\delta x^k}{\delta x^0} \cdot {}^{0}_{0}c_k - \frac{\delta T^0_{00}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{1}_{0}c_k - \frac{\delta T^1_{00}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{2}_{0}c_k - \frac{\delta T^2_{00}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{3}_{0}c_k - \frac{\delta T^3_{00}}{\delta x^0} \\ \frac{\delta x^k}{\delta x^0} \cdot {}^{0}_{1}c_k - \frac{\delta T^0_{01}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{1}_{1}c_k - \frac{\delta T^1_{01}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{2}_{1}c_k - \frac{\delta T^2_{01}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{3}_{1}c_k - \frac{\delta T^3_{01}}{\delta x^0} \\ \frac{\delta x^k}{\delta x^0} \cdot {}^{0}_{2}c_k - \frac{\delta T^0_{02}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{1}_{2}c_k - \frac{\delta T^1_{02}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{2}_{2}c_k - \frac{\delta T^2_{02}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{3}_{2}c_k - \frac{\delta T^3_{02}}{\delta x^0} \\ \frac{\delta x^k}{\delta x^0} \cdot {}^{0}_{3}c_k - \frac{\delta T^0_{03}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{1}_{3}c_k - \frac{\delta T^1_{03}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{2}_{3}c_k - \frac{\delta T^2_{03}}{\delta x^0} & \frac{\delta x^k}{\delta x^0} \cdot {}^{3}_{3}c_k - \frac{\delta T^3_{03}}{\delta x^0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

If, as usual, we need to know how the variations along the time depend on the initial frame, then we must also verify that the following (4-4) matrix is non-degenerated:

$$[\delta^{\beta}_{\alpha} \;+\; T^{\beta}_{0\alpha}] \;=\; \begin{bmatrix} 1 + T^0_{00} & T^1_{00} & T^2_{00} & T^3_{00} \\ T^0_{01} & 1 + T^1_{01} & T^2_{01} & T^3_{01} \\ T^0_{02} & T^1_{02} & 1 + T^2_{02} & T^3_{02} \\ T^0_{03} & T^1_{03} & T^2_{03} & 1 + T^3_{03} \end{bmatrix}$$

2.2 Example: variations along the time in a *static* FRW metric

When the conditions which have been recalled in subsection 1.5 hold, the GTR2 proposal is seemingly a possible limit case of the GTR universe. Prior general results are yielding:

$$[\delta^{\gamma}_{\alpha} \;+\; \Gamma^{\gamma}_{0\alpha}] \,.\, |\frac{\delta \mathbf{e}_\gamma}{\delta x^0} >= [\frac{\delta x^k}{\delta x^0} \cdot {}^{\beta}_{\alpha}c_k \;-\; \frac{\delta \Gamma^{\beta}_{0\alpha}}{\delta x^0}] \,.\, |\mathbf{e}_\beta > \tag{46}$$

In case of a *static* FRW metric, the non-vanishing Christoffel's symbols of interest are [[11]; p. 130, (23.5) and p. 132, (23.10)]:

$$(...)' \;=\; \frac{d...}{dr};\; A \;=\; 1 \;+\; \frac{r_S}{r};\; B \;=\; 1 \;-\; \frac{r_S}{r} \tag{47}$$

More precisely:

$$B' \;=\; \frac{d(1 \;-\; \frac{r_S}{r})}{dr} \;=\; \frac{r_S}{r^2}$$

$$\Gamma^0_{01} \;=\; \frac{B'}{2B} \;=\; \frac{r_S}{r^2} \cdot \frac{1}{2 \,.\, (1 \;-\; \frac{r_S}{r})};\; \Gamma^1_{00} \;=\; \frac{B'}{2A} \;=\; \frac{r_S}{r^2} \cdot \frac{1}{2 \,.\, (1 \;+\; \frac{r_S}{r})}$$

As usual, the introduction of the variable:

$$X = \frac{r_S}{r} \tag{48}$$

is certainly a pertinent choice for future numerical explorations.

$$\Gamma^0_{01} = \frac{1}{2\,.\,r_S}\cdot\frac{X^2}{1-X};\ \Gamma^1_{00} = \frac{B'}{2A} = \frac{1}{2\,.\,r_S}\cdot\frac{X^2}{1+X} \tag{49}$$

For now, I consider the (4-4) matrix:

$$[\delta^\beta_\alpha + \Gamma^\beta_{0\alpha}] = \begin{bmatrix} 1 & \frac{B'}{2A} & 0 & 0 \\ \frac{B'}{2B} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{50}$$

Its determinant is:

$$|\delta^\beta_\alpha + \Gamma^\beta_{0\alpha}| = 1 - \frac{B'^2}{4\,.\,A\,.\,B} \tag{51}$$

Remark 2.1. ***Degeneracy***

This determinant is the discriminant of a linear system. It always exists, except when B = 0 (*nota bene*: the term "A" can never vanish); this mathematical forbidden situation is realized in nature when:

$$r = 2\,.\,\frac{G\,.\,M}{c^2} = r_{Schwarzschild} \tag{52}$$

This determinant doesn't vanish except when:

$$4\,.\,A\,.\,B = B'^2$$

This is corresponding to (recall that r is strictly real positive in what follows):

$$4\,.\,(1 + 2\,.\,\frac{G\,.\,M}{c^2\,.\,r})\,.\,(1 - 2\,.\,\frac{G\,.\,M}{c^2\,.\,r}) = (2\,.\,\frac{G\,.\,M}{c^2\,.\,r^2})^2$$

or:

$$4\,.\,.\,(1 - 4\,.\,\frac{G^2\,.\,M^2}{c^4\,.\,r^2}) = 4\,.\,\frac{G^2\,.\,M^2}{c^4\,.\,r^4} \leftrightarrow 4\,.\,(1 - \frac{r_S^2}{r^2}) = \frac{r_S^2}{r^4}$$

The degeneracy is thus corresponding to the solutions of:

$$r^4 - r_S^2\,.\,r^2 - \frac{r_S^2}{4} = 0$$

I must consider:

$$\Delta = r_S^4 + r_S^2 = r_S^2\,.\,(r_S^2 + 1) > 0$$

This quantity furnishes:

$$r^2 = \frac{1}{2}\,.\,\{r_S^2 \pm r_S\,.\,(r_S^2 + 1)^{1/2}\}$$

More precisely, the degeneracy is realized for the unique real positive value:

$$r = \frac{r_S}{\sqrt{2}}\,.\,\{1 + \{1 + \frac{1}{r_S^2}\}^{1/2}\}^{1/2} \geq r_S \tag{53}$$

Examples: If the Schwarzschild's radius becomes infinite, the degeneracy is realized at infinity when r = r_S which corresponds to a forbidden mathematical situation (see above).

At the other extremity of the length scale, let suppose that the Schwarzschild radius is approximately 10^{-58} m (this would be approximately the Schwarzschild radius for an electron). Since 10^{58} is quite greater than 1, it follows that r(degeneracy) is about 10^{29} greater than r_S, hence about 10^{-29} meter (Planck scale). This radius of degeneracy is inside the radius of the electron itself (about 10^{-15} m)!

Conclusion: A coincidence between the GTR2 limit case and the GTR is compatible with:

1. the existence of a unique lapse of time (the topic of that subsection),
2. a *static* FRW metric outside of the Schwarzschild's radius (r greater than r_S) and
3. the resolution of a linear system yielding a precise and unique description of the evolutions of the frame along the time.

Nota bene: The points 1 and 3, together, define a "chronology" for the frame in a given metric.The evolutions of that frame are described with the following system:

$$\begin{bmatrix} 1 & \frac{B'}{2A} & 0 & 0 \\ \frac{B'}{2B} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta \mathbf{e}_0}{\delta x^0} \\ \frac{\delta \mathbf{e}_1}{\delta x^0} \\ \frac{\delta \mathbf{e}_2}{\delta x^0} \\ \frac{\delta \mathbf{e}_3}{\delta x^0} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \frac{\delta x^k}{\delta x^0} \cdot {}^0_0 c_k & (\frac{\delta x^k}{\delta x^0} \cdot {}^1_0 c_k - \frac{\delta \Gamma^1_{00}}{\delta x^0}) & \frac{\delta x^k}{\delta x^0} \cdot {}^2_0 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^3_0 c_k \\ (\frac{\delta x^k}{\delta x^0} \cdot {}^0_1 c_k - \frac{\delta \Gamma^0_{01}}{\delta x^0}) & \frac{\delta x^k}{\delta x^0} \cdot {}^1_1 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^2_1 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^3_1 c_k \\ \frac{\delta x^k}{\delta x^0} \cdot {}^0_2 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^1_2 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^2_2 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^3_2 c_k \\ \frac{\delta x^k}{\delta x^0} \cdot {}^0_3 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^1_3 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^2_3 c_k & \frac{\delta x^k}{\delta x^0} \cdot {}^3_3 c_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

Remark 2.2. ***In general, in a standardized FRW metric***

The system can be rewriten as:

$$\frac{\delta \mathbf{e}_0}{\delta x^0} + \frac{B'}{2A} \cdot \frac{\delta \mathbf{e}_1}{\delta x^0} = \frac{\delta x^k}{\delta x^0} \cdot {}^\alpha_0 c_k \,.\, \mathbf{e}_\alpha - \frac{\delta \Gamma^1_{00}}{\delta x^0} \,.\, \mathbf{e}_1 \tag{54}$$

$$\frac{B'}{2B} \cdot \frac{\delta \mathbf{e}_0}{\delta x^0} + \frac{\delta \mathbf{e}_1}{\delta x^0} = \frac{\delta x^k}{\delta x^0} \cdot {}^\alpha_1 c_k \,.\, \mathbf{e}_\alpha - \frac{\delta \Gamma^0_{01}}{\delta x^0} \,.\, \mathbf{e}_0$$

$$\frac{\delta \mathbf{e}_2}{\delta x^0} = \frac{\delta x^k}{\delta x^0} \cdot {}^\alpha_2 c_k \,.\, \mathbf{e}_\alpha$$

$$\frac{\delta \mathbf{e}_3}{\delta x^0} = \frac{\delta x^k}{\delta x^0} \cdot {}^\alpha_3 c_k \,.\, \mathbf{e}_\alpha$$

But, in a *static* FRW metric:

$$\frac{\delta X}{\delta x^0} = -\frac{r_S}{r^2} \cdot \frac{\delta r}{\delta x^0} = -\frac{X^2}{r_S} \,.\, v \tag{55}$$

$$v = \frac{\delta r}{\delta x^0} = \frac{\delta \{\sum_k (x^k)^2\}^{1/2}}{\delta x^0} = \frac{1}{r} \cdot <^{(3)} \mathbf{x}, \frac{\delta^{(3)} \mathbf{x}}{\delta x^0} >_{Id_3}$$

$$\frac{\delta \Gamma^0_{01}}{\delta x^0} = \frac{1}{2\,.\,r_S} \cdot \frac{\delta \frac{X^2}{1-X}}{\delta x^0} = \frac{1}{2\,.\,r_S^2} \cdot \frac{X^3\,.\,(X-2)}{(1-X)^2} \,.\, v$$

$$\frac{\delta \Gamma^1_{00}}{\delta x^0} = \frac{1}{2\,.\,r_S} \cdot \frac{\delta \frac{X^2}{1+X}}{\delta x^0} = -\frac{1}{2\,.\,r_S^2} \cdot \frac{X^3\,.\,(X+2)}{(1+X)^2} \,.\, v$$

$$\Gamma^0_{01}\,.\,\Gamma^1_{00} = \frac{B'}{2A} \cdot \frac{B'}{2B} = \frac{1}{4\,.\,r_S^2} \cdot \frac{X^4}{1-X^2}$$

$$1 - \frac{B'}{2A} \cdot \frac{B'}{2B} = 1 - \frac{1}{4\,.\,r_S^2} \cdot \frac{X^4}{1-X^2}$$

$$\frac{B'}{2A} \cdot \frac{\delta \Gamma^0_{01}}{\delta x^0} = \frac{1}{4\,.\,r_S^3} \cdot \frac{X^5\,.\,(X-2)}{(1-X)\,.\,(1-X^2)} \,.\, v$$

$$\frac{B'}{2B} \cdot \frac{\delta \Gamma^1_{00}}{\delta x^0} = -\frac{1}{4\,.\,r_S^3} \cdot \frac{X^5\,.\,(X+2)}{(1+X)\,.\,(1-X^2)} \,.\, v$$

$$-\frac{\delta \Gamma^0_{01}}{\delta x^0} \cdot \frac{\delta \Gamma^1_{00}}{\delta x^0} = \frac{1}{4\,.\,r_S^4} \cdot \frac{X^6\,.\,(X^2-4)}{(1-X^2)^2} \,.\, v^2$$

Remark 2.3. ***Stationary zone***

Within that specific GTR2 approach, the absence of spatial motion corresponds to situations such that:

$$k = 1, 2, 3 : \frac{\delta x^k}{\delta x^0} = 0 \Rightarrow \frac{\delta\, {}^{(3)}\mathbf{x}}{\delta x^0} = \mathbf{0} \Rightarrow v = 0 \tag{56}$$

They also drastically reduce the system at hand:

$$\frac{\delta \mathbf{e}_0}{\delta x^0} + \Gamma^1_{00} \cdot \frac{\delta \mathbf{e}_1}{\delta x^0} = \mathbf{0} \tag{57}$$

$$\frac{B'}{2B} \cdot \frac{\delta \mathbf{e}_0}{\delta x^0} + \frac{\delta \mathbf{e}_1}{\delta x^0} = \mathbf{0}$$

$$\frac{\delta \mathbf{e}_2}{\delta x^0} = \mathbf{0}$$

$$\frac{\delta \mathbf{e}_3}{\delta x^0} = \mathbf{0}$$

Proposition: The system is degenerated.

Proof: Whilst the axis 2 and 3 are invariant along the time, the variations of axis 0 and 1 are, a priori, possible. But, for the mathematical coherence:

$$1 - \frac{B'}{2A} \cdot \frac{B'}{2B} = 0$$

This relation signs a vanishing discriminant (see above), hence a degenerated system. Therefore, the proposition is true. □

Conclusion: In the GTR2 inner logic, when (i) the T-cubes are identified with the Γ cubes, (ii) the metric is FRW static and (iii) the chronology is unique, a stationary zone exists at a distance r of the source given by the equation (–) and the frame has two invariant axis whilst the two remaining ones have a strongly corelated evolution which is described with the equations (–) and (–).

Example: Let consider our Sun as an example; $r_S = 3\ 000$ m [[11]; p. 136, (24.38)]. With the help of equation (–), I find that the "stationary zone" is a sphere of radius 3 030 m (about thirty meter bigger than the Schwarzschild radius).

Remark 2.4. ***Vanishing coefficients of degree one***

When the coefficients of degree one vanish in the 16 polynomials, see equation (37), then:

$$\overset{\theta}{\lambda} c_k = T^\theta_{k\lambda} + \frac{1}{2} \cdot (T^\theta_{0k\lambda} + T^\theta_{k0\lambda}) \cdot \delta x^0 = 0 \tag{58}$$

The system describing the evolution of a four-dimensional frame with a unique chronology is:

$$[\delta^\gamma_\alpha + T^\gamma_{0\alpha}] \cdot |\frac{\delta \mathbf{e}_\gamma}{\delta x^0} >= [-\frac{\delta T^\beta_{0\alpha}}{\delta x^0}] \cdot |\mathbf{e}_\beta > \tag{59}$$

Especially here, when the T-cubes can be identified with the Christoffel's cubes:

$$[\delta^\gamma_\alpha + \Gamma^\gamma_{0\alpha}] \cdot |\frac{\delta \mathbf{e}_\gamma}{\delta x^0} >= [-\frac{\delta \Gamma^\beta_{0\alpha}}{\delta x^0}] \cdot |\mathbf{e}_\beta > \tag{60}$$

And more precisely for *any standardized* formulation of some FRW metric:

$$\frac{\delta \mathbf{e}_0}{\delta x^0} + \Gamma^1_{00} \cdot \frac{\delta \mathbf{e}_1}{\delta x^0} = -\frac{\delta \Gamma^1_{00}}{\delta x^0} \cdot \mathbf{e}_1 \tag{61}$$

$$\Gamma^0_{01} \cdot \frac{\delta \mathbf{e}_0}{\delta x^0} + \frac{\delta \mathbf{e}_1}{\delta x^0} = -\frac{\delta \Gamma^0_{01}}{\delta x^0} \cdot \mathbf{e}_0$$

$$\frac{\delta \mathbf{e}_2}{\delta x^0} = \mathbf{0}$$

$$\frac{\delta \mathbf{e}_3}{\delta x^0} = \mathbf{0}$$

Outside of the *inertial zone*, the evolution of axis 0 and 1 can be calculated and is unique.

$$(1 - \Gamma^1_{00} \,.\, \Gamma^0_{01}) \,.\, \frac{\delta \mathbf{e}_0}{\delta x^0} = \Gamma^1_{00} \,.\, \frac{\delta \Gamma^0_{01}}{\delta x^0} \,.\, \mathbf{e}_0 - \frac{\delta \Gamma^1_{00}}{\delta x^0} \,.\, \mathbf{e}_1 \tag{62}$$

$$(1 - \Gamma^1_{00} \,.\, \Gamma^0_{01}) \,.\, \frac{\delta \mathbf{e}_1}{\delta x^0} = -\frac{\delta \Gamma^0_{01}}{\delta x^0} \,.\, \mathbf{e}_0 + \Gamma^0_{01} \,.\, \frac{\delta \Gamma^1_{00}}{\delta x^0} \,.\, \mathbf{e}_1$$

For a non-degenrated system, these relations correspond to a matrix:

$$[\Lambda] = \frac{1}{1 - \Gamma^1_{00} \,.\, \Gamma^0_{01}} \,.\, \begin{bmatrix} \Gamma^1_{00} \,.\, \frac{\delta \Gamma^0_{01}}{\delta x^0} & -\frac{\delta \Gamma^1_{00}}{\delta x^0} \\ -\frac{\delta \Gamma^0_{01}}{\delta x^0} & \Gamma^0_{01} \,.\, \frac{\delta \Gamma^1_{00}}{\delta x^0} \end{bmatrix} \tag{63}$$

Proposition 2.1. ***A link with the Lorentz transformations?***

For the coherence of that approach, let suppose that this matrix is mimicking a Lorentz transformation within a restricted formulation of the theory of relativity. It should have the generic formalism [[13]; annex 9.b]:

$$[\Lambda] = \begin{bmatrix} sh\psi & ch\psi \\ ch\psi & sh\psi \end{bmatrix} \tag{64}$$

Here, this hypothesis imposes two constraints:

$$\Gamma^1_{00} = \Gamma^0_{01}; \; \frac{\delta \Gamma^0_{01}}{\delta x^0} = \frac{\delta \Gamma^1_{00}}{\delta x^0} \tag{65}$$

1. In a *static* FRW metric [[11]; p. 130, (23.5)], the first constraint can be rewriten as:

   $$\Gamma^1_{00} = \Gamma^0_{01} \Rightarrow A - B = 0 \Rightarrow (\gamma + 1) \,.\, X + \frac{1}{2} \,.\, (\gamma - \beta) \,.\, X^2 + ... = 0$$

   Within Newton's theory, $\beta = \gamma = 0$ [[11]; p. 131, (23.6)] and the constraint is equivalent to: X = 0. Within Einstein's theory, $\beta = \gamma = 1$ and this constraint is approximately again X = 0 but, suggests reconsidering the development of the FRW metric in including at least the third order for a better precision. Within any theory of the gravitation based on a metric, the coefficients of the FRW metric must be recalculated in taking care of the effective physics like, e.g. in [[12]].

   For now, I shall limit myself to the approximation X = 0. It is reached either for a vanishing Schwarzschild radius ($r_S = 0$) or for any non-vanishing Schwarzschild radius when r is at infinity. A vanishing Schwarzschild radius is associated with massless physical objects; among them: the photons. This specific GTR2 conclusion is indirectly confirming the conclusion which has been obtained in [[12]]: "(citation) ... *The spacetime of a perfect fluid in cosmology may be correctly written as FRW only when its equation of state is $\rho + 3.p = 0$, in terms of the total pressure p and total energy density ρ...* (end of citation)". This is what the author calls the FRW zero effective mass cosmology. The conclusion is claimed to be coherent with the lattest model-independant BAO data.

   There is an inconvenient when X = 0 exactly; that case imposes precisely:

   $$\Gamma^1_{00} = \Gamma^0_{01} = 1$$

   and the system is degenerated. This fact exhibits clearly the advantage that would be obtained if a more precise development of the FRW metric could be written.

2. For the second constraint, the following limits must be calculated:

   $$Lim_{X \to 0} \frac{\delta \Gamma^0_{01}}{\delta x^0} = Lim_{X \to 0} \frac{1}{2 \,.\, r_S^2} \,.\, \frac{X^3 \,.\, (X - 2)}{(1 - X)^2} \,.\, v$$

   $$Lim_{X \to 0} \frac{\delta \Gamma^1_{00}}{\delta x^0} = Lim_{X \to 0} \frac{1}{2 \,.\, r_S^2} \,.\, \frac{X^3 \,.\, (X + 2)}{(1 + X)^2} \,.\, v$$

   As consequence of the first constraint, there are two eventualities:

- $r_S = 0$, $r \neq 0$:

$$Lim_{r_S \to 0} \frac{\delta \Gamma^0_{01}}{\delta x^0} = Lim_{r_S \to 0} \frac{1}{2 \,.\, r^2} \cdot \frac{X^3 \,.\, (X - 2)}{X^2 \,.\, (1 - X)^2} \,.\, v = 0$$

$$Lim_{r_S \to 0} \frac{\delta \Gamma^1_{00}}{\delta x^0} = Lim_{r_S \to 0} \frac{1}{2 \,.\, r^2} \cdot \frac{X^3 \,.\, (X + 2)}{X^2 \,.\, (1 + X)^2} \,.\, v = 0$$

Since the second constraint writes:

$$\frac{\delta \Gamma^0_{01}}{\delta x^0} = \frac{\delta \Gamma^1_{00}}{\delta x^0}$$

I can consider that it is in peculiar true for the limit $r_S \to 0$.

- $r_S \neq 0$, r at infinity:

$$Lim_{\frac{1}{r} \to 0} \frac{\delta \Gamma^0_{01}}{\delta x^0} = 0^-$$

$$Lim_{\frac{1}{r} \to 0} \frac{\delta \Gamma^1_{00}}{\delta x^0} = 0^+$$

Here again, I can consider that the second constraint holds true for the limit $1/r \to 0$.

Lemma 2.1.

When the coefficients of degree one of the polynomials vanish (see equ. 37), if the transformation of the four-dimensional frame proposed by the GTR2 proposal (equ. 61) is a representation of the Lorentz transformation, then -for a *static* FRW metric- this representation is unfortunately undeterminated.

2.3 Example: variations along the time in a *spherically symmetric and diagonal form* of the metric

Remark 2.5. ***Properties of a spherically symmetric and diagonal metric***

I now consider the following spherically symmetric and diagonal metric [[12]; p. 2, (4)]:

$$(ds)^2 = e^{2.\Phi/c^2} \,.\, c^2 \,.\, (dt)^2 - e^{\lambda} \,.\, (dr)^2 - R^2 \,.\, (d\Omega)^2 \tag{66}$$

where Φ, λ and R are functions of r and t. The non-vanishing coefficients of that metric are:

$$A = e^{\lambda} = g_{00};\ B = e^{2.\Phi/c^2} = g_{rr} = g_{11};\ R^2 = g_{\theta\theta} = g_{22};\ R^2 \,.\, sin^2\theta = g_{\phi\phi} = g_{33}$$

That metric has the FRW standardized formalism proposed in [[11]; p. 130, (23.3)]. An important technical question is: "Do the equations [[11]; p. 132, (23.10)] hold yet?" If it is the case, then:

$$B' = \frac{dB}{dr} = \frac{de^{2.\Phi/c^2}}{dr} = \frac{2}{c^2} \cdot \frac{d\Phi}{dr} \,.\, B \tag{67}$$

And:

$$\Gamma^1_{00} = \frac{B'}{2.A} = \frac{1}{c^2} \cdot \frac{d\Phi}{dr} \cdot \frac{B}{A};\ \Gamma^0_{01} = \frac{B'}{2.B} = \frac{1}{c^2} \cdot \frac{d\Phi}{dr}$$

These relations are also yielding:

$$\frac{\delta \Gamma^1_{00}}{\delta x^0} = \frac{1}{c^2} \cdot \frac{\delta(\frac{d\Phi}{dr} \cdot \frac{B}{A})}{\delta x^0} \tag{68}$$

$$\frac{\delta \Gamma^0_{01}}{\delta x^0} = \frac{1}{c^2} \cdot \frac{\delta \frac{d\Phi}{dr}}{\delta x^0}$$

There is an abondant litterature concerning the FRW metrics. The formalisms proposed in [[11]; p. 130, (23.3)] and [[12]; p. 2, (4)] do not correspond to the standardized expressions that appear in the academic courses presenting this topic because most of them are only focusing on the case of a vanishing potential Φ (and they do not systematically explain that choice). This is why an academic treatment often starts with the formalisms [[11]; p. 129, (23.1)] or [[14]; p. 72, (2.21.28)]. The latter is a Lemaitre-Robertson metric and imposes a set of Christoffel's symbols which differs from the set [[11]; p. 132, (23.10)].

Remark 2.6. ***Confrontation with a special Lorentz transformation***

When: (i) the conditions which have been recalled in subsection 1.5 hold; (ii) the coefficients of degree one vanish, see equation (37), the system which is imposed by the GTR2 proposal is unchanged; see equ. (61). Therefore, a confrontation with a Lorentz transformation (equ. 64) is yielding the same two constraints than previously; recall (65):

$$\Gamma^1_{00} = \Gamma^0_{01};\ \frac{\delta\Gamma^0_{01}}{\delta x^0} = \frac{\delta\Gamma^1_{00}}{\delta x^0}$$

The first one is exactly and again:

$$A = B. \tag{69}$$

This fact has two immediate consequences:

$$\lambda = \frac{2.\Phi}{c^2} \tag{70}$$

and, the discriminant of the system is not necessarily null since:

$$1 - \Gamma^1_{00}.\Gamma^0_{01} = 1 - \frac{B'^2}{4.A.B} = 1 - \frac{1}{c^4}.(\frac{d\Phi}{dr})^2 \tag{71}$$

In fact, the system is only degenerated when the potential Φ is varying in space in such a way that:

$$\frac{d\Phi}{dr} = \pm c^2 \tag{72}$$

The second constraint is now a tautology:

$$\frac{\delta\Gamma^0_{01}}{\delta x^0} = \frac{\delta\Gamma^1_{00}}{\delta x^0} \xrightarrow{A=B} \frac{\delta\frac{d\Phi}{dr}}{\delta x^0} = \frac{\delta\frac{d\Phi}{dr}}{\delta x^0}$$

and there is no specific condition which is imposed to the variation along the time; with other words: it does not systematically vanish. The confrontation with a Lorentz transformation can now be tested in writing:

$$[\Lambda] = \frac{1}{1 - \frac{1}{c^4}.(\frac{d\Phi}{dr})^2}.\begin{bmatrix} \frac{1}{c^4}.\frac{d\Phi}{dr}.\frac{\delta\frac{d\Phi}{dr}}{\delta x^0} & -\frac{1}{c^4}.\frac{\delta\frac{d\Phi}{dr}}{\delta x^0} \\ -\frac{1}{c^4}.\frac{\delta\frac{d\Phi}{dr}}{\delta x^0} & \frac{1}{c^4}.\frac{d\Phi}{dr}.\frac{\delta\frac{d\Phi}{dr}}{\delta x^0} \end{bmatrix} = \begin{bmatrix} sh\psi & ch\psi \\ ch\psi & sh\psi \end{bmatrix} \tag{73}$$

More exactly:

$$sh\psi = \frac{\frac{1}{c^4}.\frac{d\Phi}{dr}.\frac{\delta\frac{d\Phi}{dr}}{\delta x^0}}{1 - \frac{1}{c^4}.(\frac{d\Phi}{dr})^2} = \frac{th\psi}{(1 - th^2\psi)^{1/2}}$$

$$ch\psi = -\frac{\frac{1}{c^4}.\frac{d\Phi}{dr}}{1 - \frac{1}{c^4}.(\frac{d\Phi}{dr})^2} = \frac{1}{(1 - th^2\psi)^{1/2}}$$

If that confrontation is meaningfull, then:

$$\frac{d(ch\psi)}{d\psi} = sh\psi = -\frac{\delta\frac{d\Phi}{dr}}{\delta x^0}.ch\psi$$

and it immediately furnishes:

$$th\psi = -\frac{\delta\frac{d\Phi}{dr}}{\delta x^0}$$

The special Lorentz transformation does not give any indication concerning the meaning of ψ. The coherence only imposes that its hyperbolic tangent represents the ratio v/c. Hence, here:

$$th\psi = -\frac{\delta\frac{d\Phi}{dr}}{\delta x^0} = \frac{v}{c} \tag{74}$$

Nevertheless, it can be remarked that if the argument ψ would be related to the fourth coordinates (the time) in the following manner:

$$\psi \sim x^0;\ d \sim \delta$$

then, the GTR2-Lorentz confrontation would roughly result in:

$$\frac{d(ch\psi)}{ch\psi} \sim -\delta\frac{d\Phi}{dr} \rightarrow Log(ch\psi) \sim -\frac{d\Phi}{dr}$$

Considering a Newtonian potential at this stage would yield:

$$Log(ch\psi) = Log(\frac{1}{(1-(v/c)^2)^{1/2}}) \sim \frac{G\,.\,M}{r^2}$$

which, for a given and invariant source M, would vanish at infinity and impose a vanishing spatial speed:

$$\frac{1}{r} \rightarrow 0 \Rightarrow Log(ch\psi) = 0 \Rightarrow ch\psi = 1 = \frac{1}{(1-(v/c)^2)^{1/2}} \Rightarrow v = 0$$

This would be an acceptable prediction.

Remark 2.7. ***The evolutions of the potential***

The coherence of that GTR2-Lorentz confrontation imposes a new condition on the evolution of Φ:

$$ch^2\psi - sh^2\psi = 1 \Rightarrow (\frac{1}{c^4}\cdot\frac{d\Phi}{dr})^2\,.\,\{1-(\frac{\delta\frac{d\Phi}{dr}}{\delta x^0})^2\} = \{1-\frac{1}{c^4}\,.\,(\frac{d\Phi}{dr})^2\}^2$$

Let write for convenience:

$$Y = \frac{d\Phi}{dr};\ Z = Y^2 \tag{75}$$

The new condition is:

$$\frac{1}{c^8}\,.\,Y^2\,.\,\{1-(\frac{\delta Y}{\delta x^0})^2\} = \{1-\frac{1}{c^4}\,.\,Y^2\}^2$$

or:

$$\frac{1}{c^8}\,.\,Y^2 - \{1-\frac{1}{c^4}\,.\,Y^2\}^2 = \frac{1}{4.c^8}\,.\,(\frac{\delta Y^2}{\delta x^0})^2$$

or:

$$\frac{1}{c^8}\,.\,Z - \{1-\frac{1}{c^4}\,.\,Z\}^2 = \frac{1}{4.c^8}\,.\,(\frac{\delta Z}{\delta x^0})^2$$

or:

$$Z - \{c^4 - Z\}^2 = -Z^2 + (2.c^4+1)\,.\,Z - c^8 = \frac{1}{4}\,.\,(\frac{\delta Z}{\delta x^0})^2 \tag{76}$$

Lemma 2.2. ***The allowed Lorentz transformations***

The square of the variation along the time of the square of the variation by respect for the distance r of the potential Φ depends on the latter. The dependance can not be any one if an identification between the transformation arising from the GTR2 proposal and the Lorentz transformation is needed.

3 Conclusions

The GTR2 proposal, see [a], sheds a totally new light on our understanding of the nature. It incorporates the variations of the canonical basis until the second order and comes to the very strange conclusion that electromagnetic fields and geometrical curvatures can not be independent concepts in our universe; see [b]. I would say that the most fundamental stuff in that universe is a field with two visages that can never be separated, a kind of Janus. The old concept of spacetime, understood as a pragmatical tool for the every day experiments and without link with the substrate that it is supposed to describe, must -*in fine*- be abandonned because the substrate is the spacetime itself.

This new paradigm forces the re-examination of very basic notions or problems. The re-analysis has already been started in [c] and [d]. Here, in that document, I have looked for a formal link with the Yang-Mills theory (see section 1). The introduction of a Lie algebra into the GTR2 proposal is a reachable purpose. Three cubes are needed: the first

one contains the structure constants of the algebra (cube C, see equ. 3), the second gives an expression of the coefficients of the metric in the basis of that algebra (cube D: see equ. 5) and the third one is a cube insuring a parallel transport of the components of the EM-potential vectors (cube U: see equ. 19). My exploration yields in fine a relation (equ. 24) linking the vectors which are implicitly contained in the T-cubes of the GTR2 proposal (equ. 21) and the set of Lagrangian functions attached with each EM-potential vector (equ. 22). This abstract result gives the hope to be able to coinnect surfaces and energies in the future. I hope that these first ingredients (see subsection 1.2) will bring usefull stones in the quest for solutions of the Millenium problem [[15]] related to the topic.

The concept of time occupies a peculiar situation within the GTR2 proposal (see section 2) . For example, because of the non-linearity, chronologies are not obligatorily unique. But, since nobody knows what would happen if a chronology could be freely chosen (science-fiction or multiple histories in quantum fields theories), I have considered the conditions insuring the existence of a unique chronology (see subsection 2.1) . They have been tested for a limit case of the GTR2 (the T-cubes can be identified with the Christoffel's cubes) when the metric is a *static* FRW one. Concerning that test, no definitive conclusion can be promoted and more work is needed but, a concept of *stationary zone* has emerged (see remark 2.3). It is related to a degeneracy of the system describing the evolutions of the local four-dimensional frame (see remark 2.1). Encouraged by the first results, I have then envisaged the same GTR2 proposal for any spherically symmetric and diagonal metric (see remark 2.5). An identification with a special Lorentz transformation is possible (see remark 2.6), provided the evolution along the time of the variation by respect for the distance r of the gravitational potential, Φ, follows an imposed relation of coherence (see equ. 75 and 76).

4 Personal contributions and bibliography

These texts are available on my website.

a. PERIAT, T.: Foundations of the GTR2, projective variations of the local tetrad until the second order and links with the A. Einstein's theory of relativity; ISBN 9782-36923-091-5, EAN 9782369230915, v-23 March 2016. There are several versions of the foundations (e.g.: one has been proposed to the journal "Mathematical Physics, Analysis and Geometry" (Springer) in July 2018.) but the text here above is refering to the version of March 2016.

b. PERIAT, T.: GTR2, part 02: electromagnetic fields and curvature tensors; ISBN-978-2-36923-131-8, EAN 9782369231318, 01 October 2019.

c. PERIAT, T.: GTR2, the flow of time; ISBN-978-2-36923-145-5, EAN 9782369231455, 05 February 2019.

d. PERIAT, T.: GTR2, Weak gravitational fields limit; ISBN-978-2-36923-132-5, EAN 9782369231325, 18 September 2019.

e. PERIAT, T.: GTR2, The terrific duality; ISBN-978-2-36923-149-3, EAN 9782369231493, 03 October 2019.

f. PERIAT, T.: Homogeneous 4D spaces; ISBN-978-2-36923-105-9, EAN 9782369231059, 14 October 2019.

References

[01] (a) Landau, L. D. und Lifschitz, E.M.: Lehrbuch der theoretischen Physik, Band II: Klassische Feldtheorie (German edition). (b) L. D. Landau and E. M. Lifshitz: The Classical Theory of Fields, Vol.2 of Course of Theoretical Physics, p. 281; transl. from the Russian. Pergamon, Oxford (1962).

[02] Stewart, J.: Advanced general relativity; Cambridge monographs on mathematical physics, 228 p.

[03] The Froehlich-Morchio-Strocchi mechanism and quantum gravity; arXiv: 1908.02140v1 [hep-th] 06 August 2019.

[04] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, "General Relativity with Spin and Torsion: Foundations and Prospects", Reviews of Modern Physics, Vol. 48, No. 3, 1976, pp. 393-416. doi:10.1103/RevModPhys.48.393.

[05] Einstein's vierbein field theory of curved space; arXiv: 1106.2037v1 [gr-qc] 10 June 2011.

[06] Einstein, A. (1928a). New possibility for a unified field theory of gravity and electricity. Sitzungsberichte der Preussischen Akademie der Wissenschaften. PhysikalischMathematische Klasse, pages 223-227; (1928b). Riemann geometry with preservation of the concept of distant parallelism. Sitzungsberichte der Preussischen Akademie der Wissenschaften. PhysikalischMathematische Klasse, pages 217-221.

[07] Extended Fermi coordinates; arXiv:0901.4465v2 [gr-qc], 12 July 2010.

[08] Riemann (B.). - Ueber die Hypothesen welche der Geometrie zu Grunde liegen (Habilitationschrift soutenue le 10 juin 1854 devant la Faculté de philosophie de Goettingen (Gesammelte Math. Werke, Leipzig, 1872, p. 254-269).

[09] Cartan, E.: La géométrie des espaces de Riemann; *Mémorial des sciences mathématiques*, fascicule 9 (1925), accessible sur numdam, ©Gauthier-Villars; 1925, tous droits réservés.

[10] Riemann's normal coordinates, Fermi reference system and the geodesic deviation equation; arXiv:gr-qc/0010096v1, 26 October 2000.

[11] Fliessbach, T.: Allgemeine Relativitaetstheorie, 4. Auflage, Spektrum Lehrbuch, © 2003, 1998, 1995, SpektrumAkademischer Verlag, Heidelberg, Berlin, ISBN 3-8274-1356-7.

[12] Physical basis for the symmetries in the FRW metric; arXiv:1601.04991v2 [physics.gen-ph], 7 March 2016.

[13] Lennuier, R., Gal, P.-Y. et Perrin, D. : Mécanique des particules, champs, Collection U; ©Librairie Armand Coln, Paris 1970.

[14] Catalogue of spacetimes; arXiv:0904.4184v3 [gr-qc], 04 November 2010.

[15] Yang-Mills Millennium problem; official site of the Clay Math Institute.