The Generalized Hybrid Weighted Average Operator Based on Interval Neutrosophic Hesitant Set and Its Application to Multiple Attribute Decision Making

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Abstract: The Neutrosophic set and Hesitant set are the important and effective tools to describe the uncertain information, in this paper, we combine the interval neutrosophic sets (INSs) and interval valued hesitant fuzzy sets (IVHFSs), and propose the concept of the interval neutrosophic hesitant fuzzy set (INHFS) in order to use the advantages of them. Then we present the operations and comparison method of INHFS, and develop some new aggregation operators for the interval neutrosophic hesitant fuzzy information, including interval neutrosophic hesitant fuzzy generalized weighted (INHFGWA) operator, interval neutrosophic hesitant fuzzy generalized ordered weighted (INHFGOWA) operator, and interval neutrosophic hesitant fuzzy generalized hybrid weighted (INHFGHWA) operator, and discuss some properties. Furthermore, we propose the decision making method for multiple attribute group decision making (MAGDM) with interval neutrosophic hesitant fuzzy information, and give the detail decision steps. Finally, we give an illustrate example to show the process of decision making.

Keywords: interval neutrosophic hesitant fuzzy set; neutrosophic set; generalized weighted aggregation (GWA) operator; multiple attribute decision making

1. Introduction

Decision making has the wide application requirements in the business, service, economic, military and the other aspects. But in real life, the decision information is often incomplete, indeterminate and inconsistent, how to express the decision information is the first task of making decision. The fuzzy set (FS) theory, presented by Zadeh [1], is an effective tool to process fuzzy information. Since FS was established, it has been attracted wide attentions, and it is extended from two directions.

One direction, FS only has a membership, and it cannot express some complex fuzzy information. For example, in a voting process, there are 10 persons to vote a matter, three of them give the opinion “agree”, four of them give the opinion “disagree”, and the others give up the voting. Obviously, it is difficult to express the voting information by FS. Based on FS and real applications, Atanassov [2, 3] presented the intuitionistic fuzzy set (IFS) by adding a non-membership function on the basis of FS, i.e., IFS is with membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$. The example above can be expressed by membership 0.3 and non-membership 0.4. Because membership and non-membership in IFS are crisp numbers, sometimes, it is difficult to use in real decision making problems. Further, Atanassov and Gargov [4], Atanassov [5] proposed the interval-valued intuitionistic fuzzy set (IVIFS) by extending the membership and

non-membership functions to interval numbers. Zhang and Liu [6] proposed the triangular intuitionistic fuzzy number by extending the membership and the non-membership functions to triangular fuzzy numbers. Wang [7] proposed the intuitionistic trapezoidal fuzzy number. However, IFSs and IVIFSs can only handle incomplete information, and cannot deal with the indeterminate and inconsistent information. In IFSs, there is a default indeterminacy degree $1-T_A(x)-F_A(x)$. In some complex decision-making environment, IFS has also some limitations. For example, when we ask an expert for the opinion about a statement, he/she may think the right possibility of the statement is 0.5 and the false possibility of the statement is 0.6 and the degree that he or she is not sure is 0.2 [8]. In this case, IFS doesn't process this type of information. In order to solve this class of decision making problems, based on IFS, Smarandache [9] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership function. Obviously, NS is a generalization of the existing fuzzy sets, such as fuzzy set, intuitionistic fuzzy set, paraconsistent set, paradoxist set and so on. In NS, the indeterminacy degree is an independent part, and truth-membership, false-membership and indeterminacy membership are completely independent. About the research on NS, some achievements have been made. Wang et al. [10, 11] proposed an interval neutrosophic set (INS) and a single valued neutrosophic set (SVNS), which are some instances of the neutrosophic set. Ye [12, 13] proposed the correlation coefficient and the cross-entropy measure of SVNSs, and then applied them to single valued neutrosophic decision-making problems.

The other direction, FS has only one membership, this is a limitation for some decision making problems. Based on FS, Torra and Narukawa [14], Torra [15] proposed a new concept of hesitant fuzzy sets (HFSs). As a generalization of fuzzy sets, HFSs use several possible values of an element to replace the membership degree, which is an important tool to represent indefinite information in multiple attribute decision making. Then, Chen et al. [16] proposed interval valued hesitant fuzzy sets (IVHFSs) where each membership degree is extended to interval numbers. Zhao et al. [17] proposed hesitant triangular fuzzy set and developed some aggregation operators for the hesitant triangular fuzzy sets based on the Einstein operations. Meng et al. [18] proposed linguistic hesitant fuzzy sets (LHFSs), and developed some linguistic hesitant fuzzy hybrid weighted operators. Farhadinia [19] and Ye [20] proposed the dual hesitant fuzzy sets and dual interval hesitant fuzzy sets. Peng et al. [21] proposed the hesitant interval-valued intuitionistic fuzzy sets (HIVIFSs), and developed some hesitant interval intuitionistic fuzzy number weighted averaging operators based on t-conorms and t-norms. Ye [22] proposed a neutrosophic hesitant fuzzy set by combining the hesitant fuzzy sets with single-valued neutrosophic sets (SVNHFS), and then some weighted averaging and weighted geometric operators for SVNHFS are developed.

As mentioned above, HFS and NS can extend the FS from two directions, the HFS can allow the membership function having several possible values, which is a good tool to process uncertain information in real decision making by hesitant manners; however, it cannot handle indeterminate and inconsistent information, while the NS can easily represent uncertainty, incomplete, and inconsistent information. Obviously, each of them has its strengths and weaknesses. So, combining the IVHFS and INS, we further propose the concept of interval neutrosophic hesitant fuzzy sets (INHFSs), which extend truth-membership degree, indeterminacy-membership degree, and falsity-membership degree of an element to a given set to IVHFS, i.e., they may have a few different interval values. So, the INHFS is generalization of fuzzy set, IFS, IVIFS, NS, INS, HS, IVHFS, and so on. In addition, because the aggregation operators are the important tools to process the fuzzy decision making problems and research on aggregation operators has achieved fruitful results [17, 21-24]. This paper's aim is to

propose the concept, score function and comparison method of IVHFS, and then to develop some new generalization aggregation operators, including interval neutrosophic hesitant fuzzy generalized weighted average (INHFGWA) operator, an interval neutrosophic hesitant fuzzy generalized ordered weighted average (INHFGOWA) operator and an interval neutrosophic hesitant fuzzy generalized hybrid weighted average (INHFGHWA) operator, further to develop the decision method for the MADM problems under interval neutrosophic hesitant fuzzy environment.

To achieve the above purposes, the remainder of this paper is organized as follows. In the next section, we present some concepts of interval fuzzy numbers, HFSs, IVHFSs, generalized weighted average (GWA) operator, generalized ordered weighted average (GOWA) operator and generalized hybrid weighted average (GHWA) operator. In section 3, we propose the concept and operations of INHFSs. In section 4, we present some generalized aggregation operators based on INHFS, including INHFGWA, INHFGOWA and INHFGHWA, and introduce some properties and special cases of them. Section 5 establishes the procedure of the decision-making method based on the INHFWA operators. Section 6 gives a numerical example according to our approach. Section 7 summarizes the main conclusion of this paper.

2. Preliminaries

2.1 The interval fuzzy numbers

Definition 1 [25]. Let $\tilde{a} = [a^L, a^U] = \{x | a^L \leq x \leq a^U\}$, then $\tilde{a}$ is called an interval fuzzy number.

If $0 \leq a^L \leq a^U$, then $\tilde{a}$ is called a positive interval fuzzy number.

Definition 2 [25]. Let $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$ be two interval numbers and $l_{\tilde{a}} = a^U - a^L$ and $l_{\tilde{b}} = b^U - b^L$, then the degree of possibility of $\tilde{a} > \tilde{b}$ is formulated by

$$p(\tilde{a} > \tilde{b}) = \max \left[1 - \max \left(\frac{b^U - a^L}{l_{\tilde{a}} + l_{\tilde{b}}}, 0 \right), 0 \right]. \quad (1)$$

We can use the degrees of possibility to compare with the interval numbers.

2.2. The interval neutrosophic set

Definition 3 [10]. Let X be a universe of discourse, with a generic element in X denoted by x . A interval valued neutrosophic set A in X is

$$A = \{x(T_A(x), I_A(x), F_A(x)) | x \in X\} \quad (2)$$

Where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership, indeterminacy-membership and falsity-membership function, separately. For each point x in X , we have that $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$, and $0 \leq \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \leq 3$.

Definition 4 [10]. If and only if its $\inf T_A(x) = \sup T_A(x) = 0$, $\inf I_A(x) = \sup I_A(x) = 1$, and $\inf F_A(x) = \sup F_A(x) = 0$ for all x in X , we can say the INS A is null.

Definition 5 [10]. A^c denote the complement of an INS A and written as $A^c = \{x(T_A^c(x), I_A^c(x), F_A^c(x)) | x \in X\}$, i.e. $T_A^c(x) = F_A(x)$, $\inf I_A^c(x) = 1 - \sup I_A(x)$, $\sup I_A^c(x) = 1 - \inf I_A(x)$, $F_A^c(x) = T_A(x)$ for all x in X .

Definition 6 [10]. If and only if $\inf T_A(x) \leq \inf T_B(x)$, $\sup T_A(x) \leq \sup T_B(x)$, $\inf I_A(x) \geq \inf I_B(x)$, $\sup I_A(x) \geq \sup I_B(x)$, $\inf F_A(x) \geq \inf F_B(x)$, and $\sup F_A(x) \geq \sup F_B(x)$ for all x in X , we can say an INS A is contained in the INS B and is denoted by $A \subseteq B$.

Definition 7 [10]. If and only if $A \subseteq B$ and $B \subseteq A$, we can say two INSs A and B are identical, and written as $A = B$.

2. 3 Some concepts of HFSs and IVHFSs

Definition 8 [14, 15]. Let X be a **non-empty** finite set, a HFS A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of $[0,1]$, we can express HFSs by:

$$A = \{ \langle x, h_A(x) \rangle | x \in X \} \quad (3)$$

Where $h_A(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to A . we call $h_A(x)$ a hesitant fuzzy element (HFE), denoted by h , which reads $h = \{\gamma | \gamma \in h\}$.

For three hesitant fuzzy elements h, h_1 and h_2 , Torra [15] defined three basic operations shown as follows.

$$(1) h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}, \quad (4)$$

$$(2) h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}, \quad (5)$$

$$(3) h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}. \quad (6)$$

After that, Xia and Xu [26] defined four operations on the HFES h, h_1, h_2 with a positive scale n :

$$(1) h^n = \bigcup_{\gamma \in h} \{\gamma^n\}, \quad (7)$$

$$(2) nh = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^n\}, \quad (8)$$

$$(3) h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}, \quad (9)$$

$$(4) h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}. \quad (10)$$

Definition 9 [16, 25] Let X be a **non-empty** finite set, an interval valued hesitant fuzzy set (IVHFS) on X is represented by:

$$E = \left\{ \left\langle x, \tilde{h}_E(x) \right\rangle \mid x \in X \right\} \quad (11)$$

where $\tilde{h}_E(x)$ is a set of some different interval values in $[0,1]$, which denotes the possible membership degrees of the element $x \in X$ to the set E , $\tilde{h}_E(x)$ can be represented by an IVHFE $\tilde{h}$, which reads $\tilde{h} = \{\tilde{\gamma} \mid \tilde{\gamma} \in \tilde{h}\}$, where $\tilde{\gamma} = [\gamma^L, \gamma^U]$ is an interval number.

Chen et al. [25] gave the operations for three IVHFEs $\tilde{h}, \tilde{h}_1, \tilde{h}_2$ with a positive scale n shown as follows:

$$(1) \tilde{h}^n = \bigcup_{\tilde{\gamma} \in \tilde{h}} \left\{ \left[(\gamma^L)^n, (\gamma^U)^n \right] \right\}; \quad (12)$$

$$(2) n\tilde{h} = \bigcup_{\tilde{\gamma} \in \tilde{h}} \left\{ \left[1 - (1 - \gamma^L)^n, 1 - (1 - \gamma^U)^n \right] \right\}; \quad (13)$$

$$(3) \tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \left\{ \left[\gamma_1^L + \gamma_2^L - \gamma_1^L \gamma_2^L, \gamma_1^U + \gamma_2^U - \gamma_1^U \gamma_2^U \right] \right\}; \quad (14)$$

$$(4) \tilde{h}_1 \otimes \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \left\{ \left[\gamma_1^L \gamma_2^L, \gamma_1^U \gamma_2^U \right] \right\}. \quad (15)$$

2.4 Some aggregation operators

GWA operator is a generalization of the weighted average operator, which is defined as follows:

Definition 10 [23]. Let $GWA: R^n \rightarrow R$, if

$$GWA(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n w_j a_j^\lambda \right)^{1/\lambda} \quad (16)$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector of the $(a_1, a_2, \dots, a_n)$ such that

$w_j \in [0,1]$, $\sum_{j=1}^n w_j = 1$, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$, then GWA is

called the generalized weighted average (GWA) operator.

Definition 11 [23]. Let $GOWA: R^n \rightarrow R$, if

$$GOWA(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n \omega_j b_j^\lambda \right)^{1/\lambda} \quad (17)$$

where b_j is the j th largest of the $(a_1, a_2, \dots, a_n)$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weighting

vector such that $\omega_j \in [0,1]$, $j=1,2,\dots,n$, $\sum_{j=1}^n \omega_j = 1$, and λ is a parameter such that

$\lambda \in (-\infty, 0) \cup (0, +\infty)$, then GOWA is called the generalized ordered weighted average (GOWA)

operator.

Definition 12 [24]. Let $GHWA : R^n \rightarrow R$, if

$$GHWA(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n \omega_j b_j^\lambda \right)^{1/\lambda} \quad (18)$$

where b_j is the j th largest of the weighted arguments $nw_i a_i (i = 1, 2, \dots, n)$, $w = (w_1, w_2, \dots, w_n)^T$ is

the weighting vector of the $a_i (i = 1, 2, \dots, n)$, $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$, λ is a parameter such that

$\lambda \in (-\infty, 0) \cup (0, +\infty)$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the aggregation-associated vector such that

$\omega_j \in [0, 1]$, $j = 1, 2, \dots, n$, $\sum_{j=1}^n \omega_j = 1$. then $GHWA$ is called the generalized hybrid weighted

average (GHWA) operator.

3. The interval neutrosophic hesitant fuzzy set

In this section, we will present the concept of interval neutrosophic hesitant fuzzy set based on the combination of interval neutrosophic set with interval valued hesitant fuzzy set.

Definition 13. Let X be a non-empty finite set, an interval neutrosophic hesitant fuzzy set (INHFS) on X is represented by:

$$N = \left\{ \langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x) \rangle \mid x \in X \right\} \quad (19)$$

Where $\tilde{t}(x) = \{ \tilde{\gamma} \mid \tilde{\gamma} \in \tilde{t}(x) \}$, $\tilde{i}(x) = \{ \tilde{\delta} \mid \tilde{\delta} \in \tilde{i}(x) \}$ and $\tilde{f}(x) = \{ \tilde{\eta} \mid \tilde{\eta} \in \tilde{f}(x) \}$ are three sets of some

interval values in real unit interval $[0, 1]$, which denotes the possible truth-membership hesitant degrees,

indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $x \in X$ to the set N , and satisfies these

limits : $\tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0, 1]$, $\tilde{\delta} = [\delta^L, \delta^U] \subseteq [0, 1]$, $\tilde{\eta} = [\eta^L, \eta^U] \subseteq [0, 1]$ and

$0 \leq \sup \tilde{\gamma}^+ + \sup \tilde{\delta}^+ + \sup \tilde{\eta}^+ \leq 3$, where $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in \tilde{t}(x)} \max \{ \tilde{\gamma} \}$, $\tilde{\delta}^+ = \bigcup_{\tilde{\delta} \in \tilde{i}(x)} \max \{ \tilde{\delta} \}$, and

$\tilde{\eta}^+ = \bigcup_{\tilde{\eta} \in \tilde{f}(x)} \max \{ \tilde{\eta} \}$ for $x \in X$.

The $\tilde{n} = \{ \tilde{t}(x), \tilde{i}(x), \tilde{f}(x) \}$ is called an interval neutrosophic hesitant fuzzy element (INHFE) which

is the basic unit of the INHFS and be represented by the symbol $\tilde{n} = \{ \tilde{t}, \tilde{i}, \tilde{f} \}$

Thus, we can regard FSs, IFs, IVIFs, SVNFSs, INFs, HFSs, DHFSs, and IVHFSs as special cases of INHFSs from the definition 13.

Then, we can define the basic operations of INHFEs as follows:

Definition 14. Let $\tilde{n}_1 = \{\tilde{t}_1, \tilde{i}_1, \tilde{f}_1\}$ and $\tilde{n}_2 = \{\tilde{t}_2, \tilde{i}_2, \tilde{f}_2\}$ be two INHFEs in a **non-empty** finite set X, then

$$(1) \tilde{n}_1 \cup \tilde{n}_2 = \{\tilde{t}_1 \cup \tilde{t}_2, \tilde{i}_1 \cap \tilde{i}_2, \tilde{f}_1 \cap \tilde{f}_2\} \quad (20)$$

$$(2) \tilde{n}_1 \cap \tilde{n}_2 = \{\tilde{t}_1 \cap \tilde{t}_2, \tilde{i}_1 \cup \tilde{i}_2, \tilde{f}_1 \cup \tilde{f}_2\} \quad (21)$$

Therefore, for two INHFEs $\tilde{n}_1, \tilde{n}_2$ and a positive scale $k > 0$, these operations can be denoted as follows:

$$(1) \tilde{n}_1 \oplus \tilde{n}_2 = \{\tilde{t}_1 \oplus \tilde{t}_2, \tilde{i}_1 \otimes \tilde{i}_2, \tilde{f}_1 \otimes \tilde{f}_2\} \quad (22)$$

$$= \bigcup_{\substack{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1, \tilde{\gamma}_2 \in \tilde{t}_2, \tilde{\delta}_2 \in \tilde{i}_2, \tilde{\eta}_2 \in \tilde{f}_2}} \{[\gamma_1^L + \gamma_2^L - \gamma_1^L \gamma_2^L, \gamma_1^U + \gamma_2^U - \gamma_1^U \gamma_2^U][\delta_1^L \delta_2^L, \delta_1^U \delta_2^U][\eta_1^L \eta_2^L, \eta_1^U \eta_2^U]\}$$

$$(2) \tilde{n}_1 \otimes \tilde{n}_2 = \{\tilde{t}_1 \otimes \tilde{t}_2, \tilde{i}_1 \oplus \tilde{i}_2, \tilde{f}_1 \oplus \tilde{f}_2\} = \bigcup_{\substack{\tilde{\gamma}_2 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1, \tilde{\gamma}_2 \in \tilde{t}_2, \tilde{\delta}_2 \in \tilde{i}_2, \tilde{\eta}_2 \in \tilde{f}_2}} \{[\gamma_1^L \gamma_2^L, \gamma_1^U \gamma_2^U][\delta_1^L + \delta_2^L - \delta_1^L \delta_2^L, \delta_1^U + \delta_2^U - \delta_1^U \delta_2^U][\eta_1^L + \eta_2^L - \eta_1^L \eta_2^L, \eta_1^U + \eta_2^U - \eta_1^U \eta_2^U]\} \quad (23)$$

$$(3) k\tilde{n}_1 = \bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1} \{[1 - (1 - \gamma_1^L)^k, 1 - (1 - \gamma_1^U)^k], [(\delta_1^L)^k, (\delta_1^U)^k], [(\eta_1^L)^k, (\eta_1^U)^k]\} \quad (24)$$

$$(4) \tilde{n}_1^k = \bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1} \{[(\gamma_1^L)^k, (\gamma_1^U)^k][1 - (1 - \delta_1^L)^k, 1 - (1 - \delta_1^U)^k], [1 - (1 - \eta_1^L)^k, 1 - (1 - \eta_1^U)^k]\} \quad (25)$$

Definition 15. For an INHFE $\tilde{n}$,

$$s(\tilde{n}) = \left[\frac{1}{l} \sum_{i=1}^l \tilde{\gamma}_i + \left(\frac{\sum_{i=1}^p (1 - \tilde{\delta}_i)}{p} \right) + \left(\frac{\sum_{i=1}^q (1 - \tilde{\eta}_i)}{q} \right) \right] / 3 \quad (26)$$

is called the score function of $\tilde{n}$. where l, p, q being the number of the interval values in

$\tilde{\gamma}, \tilde{\delta}, \tilde{\eta}$, respectively. Obviously, $s(\tilde{n})$ is an interval value **included in** $[0,1]$. For two INHFE

$\tilde{n}_1$ and $\tilde{n}_2$, because $s(\tilde{n}_1)$ and $s(\tilde{n}_2)$ are interval numbers, they can be compared by the degrees of

possibility defined in definition 1. If $s(\tilde{n}_1) > s(\tilde{n}_2)$, then $\tilde{n}_1 > \tilde{n}_2$.

4. Some aggregation operators based on interval **neutrosophic** hesitant fuzzy numbers

Since the GWA, GOWA and GHWA operators can only aggregate the crisp numbers, and cannot aggregate the interval **neutrosophic** hesitant fuzzy information. In this section, we will extend the GWA,

GOWA and GHWA operators to aggregate interval **neutrosophic** hesitant fuzzy information, and propose an interval **neutrosophic** hesitant fuzzy generalized weighted average (INHFGWA) operator, an interval **neutrosophic** hesitant fuzzy generalized ordered weighted average (INHFGOWA) operator and an interval **neutrosophic** hesitant fuzzy generalized hybrid weighted average (INHFGHWA) operator described as follows.

Definition 16. Let $\lambda > 0$ and $\tilde{n}_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j) (j=1,2,\dots,n)$ be a collection of interval **neutrosophic** hesitant fuzzy numbers with the weight vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$, then an interval **neutrosophic** hesitant fuzzy generalized weighted average

(INHFGWA) operator of dimension n is a mapping $\text{INHFGWA}: \Omega^n \rightarrow \Omega$, and has

$$\text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \left(\sum_{j=1}^n w_j \tilde{n}_j^\lambda \right)^{1/\lambda} \quad (27)$$

where Ω is the set of all the interval **neutrosophic** hesitant fuzzy numbers.

Based on the operational rules of the interval **neutrosophic** hesitant fuzzy numbers, we have the following theorems.

Theorem 1. Let $\lambda > 0$ and $\tilde{n}_i = ([\gamma_i^L, \gamma_i^U], [\delta_i^L, \delta_i^U], [\eta_i^L, \eta_i^U]) (i=1,2,\dots,n)$ be the collection of INHFs, then the result aggregated from Definition 16 is still an INHFs, and even

$$\begin{aligned} \text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = & \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} \left(1 - (\gamma_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} \left(1 - (\gamma_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right], \right. \\ & \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\delta_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\delta_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right], \\ & \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} \left(1 - (\eta_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} \left(1 - (\eta_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right] \right\} \quad (28) \end{aligned}$$

Proof.

(1) We firstly prove that

$$\begin{aligned} \sum_{j=1}^n w_j \tilde{n}_j^\lambda = & \left\{ \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} \left(1 - (\gamma_j^L)^\lambda \right)^{w_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} \left(1 - (\gamma_j^U)^\lambda \right)^{w_j} \right], \right. \\ & \left[\prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\delta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\delta_j^U)^\lambda \right)^{w_j} \right], \quad (29) \end{aligned}$$

$$\left[\prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta_j^U)^\lambda \right)^{w_j} \right]$$

(29) can be proved by Mathematical induction on n as follows

(i) When $n = 1$,

According to the operational rules of INHFs, we have

$$\begin{aligned} \tilde{n}_1^\lambda &= \bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{t}_1, \tilde{\eta}_1 \in \tilde{f}_1} \left\{ \left[(\gamma_1^L)^\lambda, (\gamma_1^U)^\lambda \right], \left[1 - (1 - \delta_1^L)^\lambda, 1 - (1 - \delta_1^U)^\lambda \right], \left[1 - (1 - \eta_1^L)^\lambda, 1 - (1 - \eta_1^U)^\lambda \right] \right\} \\ w_1 \tilde{n}_1^\lambda &= \bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{t}_1, \tilde{\eta}_1 \in \tilde{f}_1} \left\{ \left[1 - \left(1 - (\gamma_1^L)^\lambda \right)^{w_1}, 1 - \left(1 - (\gamma_1^U)^\lambda \right)^{w_1} \right], \left[\left(1 - (1 - \delta_1^L)^\lambda \right)^{w_1}, \left(1 - (1 - \delta_1^U)^\lambda \right)^{w_1} \right], \right. \\ &\quad \left. \left[\left(1 - (1 - \eta_1^L)^\lambda \right)^{w_1}, \left(1 - (1 - \eta_1^U)^\lambda \right)^{w_1} \right] \right\} \end{aligned}$$

So, when $n=1$, (29) is right.

(ii) Suppose when $n = k$, (29) is right, i.e.,

$$\begin{aligned} \sum_{j=1}^k w_j \tilde{n}_j^\lambda &= \left\{ \left[1 - \prod_{j=1}^k \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^L)^\lambda \right)^{w_j}, 1 - \prod_{j=1}^k \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^U)^\lambda \right)^{w_j} \right], \right. \\ &\quad \left[\prod_{j=1}^k \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^k \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta_j^U)^\lambda \right)^{w_j} \right], \left[\prod_{j=1}^k \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^k \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta_j^U)^\lambda \right)^{w_j} \right] \left. \right\} \end{aligned}$$

Then when $n = k + 1$, we have

$$\begin{aligned} w_{k+1} \tilde{n}_{k+1}^\lambda &= \bigcup_{\tilde{\gamma}_{k+1} \in \tilde{t}_{k+1}, \tilde{\delta}_{k+1} \in \tilde{t}_{k+1}, \tilde{\eta}_{k+1} \in \tilde{f}_{k+1}} \left\{ \left[1 - \left(1 - (\gamma_{k+1}^L)^\lambda \right)^{w_{k+1}}, 1 - \left(1 - (\gamma_{k+1}^U)^\lambda \right)^{w_{k+1}} \right], \left[\left(1 - (1 - \delta_{k+1}^L)^\lambda \right)^{w_{k+1}}, \left(1 - (1 - \delta_{k+1}^U)^\lambda \right)^{w_{k+1}} \right], \right. \\ &\quad \left. \left[\left(1 - (1 - \eta_{k+1}^L)^\lambda \right)^{w_{k+1}}, \left(1 - (1 - \eta_{k+1}^U)^\lambda \right)^{w_{k+1}} \right] \right\} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{k+1} w_j \tilde{n}_j^\lambda &= \sum_{j=1}^k w_j \tilde{n}_j^\lambda \oplus w_{k+1} \tilde{n}_{k+1}^\lambda \\ &= \left\{ \left[1 - \prod_{j=1}^k \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^L)^\lambda \right)^{w_j}, 1 - \prod_{j=1}^k \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^U)^\lambda \right)^{w_j} \right], \right. \\ &\quad \left[\prod_{j=1}^k \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^k \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta_j^U)^\lambda \right)^{w_j} \right], \left[\prod_{j=1}^k \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^k \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta_j^U)^\lambda \right)^{w_j} \right] \left. \right\} \\ &\quad \oplus \end{aligned}$$

$$\begin{aligned}
& \bigcup_{\tilde{\gamma}_{k+1} \in \tilde{t}_{k+1}, \tilde{\delta}_{k+1} \in \tilde{i}_{k+1}, \tilde{\eta}_{k+1} \in \tilde{f}_{k+1}} \left\{ \left[1 - \left(1 - (\gamma_{k+1}^L)^\lambda \right)^{w_{k+1}}, 1 - \left(1 - (\gamma_{k+1}^U)^\lambda \right)^{w_{k+1}} \right], \left[\left(1 - (\delta_{k+1}^L)^\lambda \right)^{w_{k+1}}, \left(1 - (\delta_{k+1}^U)^\lambda \right)^{w_{k+1}} \right], \right. \\
& \left. \left[\left(1 - (\eta_{k+1}^L)^\lambda \right)^{w_{k+1}}, \left(1 - (\eta_{k+1}^U)^\lambda \right)^{w_{k+1}} \right] \right\} \\
& = \left\{ \left[1 - \prod_{j=1}^{k+1} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^L)^\lambda \right)^{w_j}, 1 - \prod_{j=1}^{k+1} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^U)^\lambda \right)^{w_j} \right], \right. \\
& \left. \left[\prod_{j=1}^{k+1} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - (\delta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^{k+1} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - (\delta_j^U)^\lambda \right)^{w_j} \right], \left[\prod_{j=1}^{k+1} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (\eta_j^L)^\lambda \right)^{w_j}, \prod_{j=1}^{k+1} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (\eta_j^U)^\lambda \right)^{w_j} \right] \right\}
\end{aligned}$$

So, when $n = k + 1$, (29) is also right.

According to (i) and (ii), we can get when (29) is right for all n .

(2) According step (1), we have

$$\begin{aligned}
& \text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \left(\sum_{j=1}^n w_j \tilde{n}_j^\lambda \right)^{1/\lambda} \\
& = \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right], \right. \\
& \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - (\delta_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - (\delta_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right], \\
& \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (\eta_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (\eta_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right] \right\}.
\end{aligned}$$

which completes the proof Theorem 1..

Moreover, the INHFGWA operator has the following properties.

(1) Theorem 2 (Idempotency).

Let $\tilde{n}_j = \tilde{n} = \{\tilde{t}, \tilde{i}, \tilde{f}\}$, where $\tilde{t} = [\gamma^L, \gamma^U]$, $\tilde{i} = [\delta^L, \delta^U]$ and $\tilde{f} = [\eta^L, \eta^U]$,

then

$$\text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \tilde{n}. \quad (30)$$

Proof. Since $\tilde{n}_j = \tilde{n} = \{\gamma^L, \gamma^U\}[\delta^L, \delta^U][\eta^L, \eta^U]$, for all j , we have

$$\text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \left(\sum_{j=1}^n w_j \tilde{n}_j^\lambda \right)^{1/\lambda} = \left(\sum_{j=1}^n w_j \tilde{n}^\lambda \right)^{1/\lambda}$$

$$\begin{aligned}
&= \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right] \right. \\
&\quad \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right], \\
&\quad \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta^L)^\lambda \right)^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \left[\left(1 - \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma^L)^\lambda \right)^{\sum_{j=1}^n w_j} \right)^{1/\lambda}, \left(1 - \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma^U)^\lambda \right)^{\sum_{j=1}^n w_j} \right)^{1/\lambda} \right], \right. \\
&\quad \left[1 - \left(1 - \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta^L)^\lambda \right)^{\sum_{j=1}^n w_j} \right)^{1/\lambda}, 1 - \left(1 - \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta^U)^\lambda \right)^{\sum_{j=1}^n w_j} \right)^{1/\lambda} \right], \\
&\quad \left. \left[1 - \left(1 - \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta^L)^\lambda \right)^{\sum_{j=1}^n w_j} \right)^{1/\lambda}, 1 - \left(1 - \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta^U)^\lambda \right)^{\sum_{j=1}^n w_j} \right)^{1/\lambda} \right] \right\} \\
&= \left\{ \left[\left(1 - \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma^L)^\lambda \right) \right)^{1/\lambda}, \left(1 - \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\gamma^U)^\lambda \right) \right)^{1/\lambda} \right], \right. \\
&\quad \left[1 - \left(1 - \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta^L)^\lambda \right) \right)^{1/\lambda}, 1 - \left(1 - \bigcup_{\tilde{\delta}_j \in \tilde{t}_j} \left(1 - (1 - \delta^U)^\lambda \right) \right)^{1/\lambda} \right], \\
&\quad \left. \left[1 - \left(1 - \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta^L)^\lambda \right) \right)^{1/\lambda}, 1 - \left(1 - \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \eta^U)^\lambda \right) \right)^{1/\lambda} \right] \right\} \\
&= \tilde{n}.
\end{aligned}$$

(2) Theorem 3 (monotonicity)

Let $\tilde{n}_j (j=1,2,\dots,n)$ and $\tilde{n}'_j (j=1,2,\dots,n)$ be two sets of INHFS, and $w = (w_1, w_2, \dots, w_n)^T$ be the weighting vector of $\tilde{n}_j, \tilde{n}'_j (j=1,2,\dots,n)$ which satisfying $w_j \in [0,1], \sum_{j=1}^n w_j = 1$, if $\tilde{n}_j \geq \tilde{n}'_j$ for all j , then

$$\text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \geq \text{INHFGWA}(\tilde{n}'_1, \tilde{n}'_2, \dots, \tilde{n}'_n) \quad (31)$$

Proof.

Because $\tilde{n}_j \geq \tilde{n}'_j$ for all j , we may suppose $\gamma_j^L \geq \gamma_j'^L, \gamma_j^U \geq \gamma_j'^U, \delta_j^L \leq \delta_j'^L, \delta_j^U \leq \delta_j'^U$,
 $\eta_j^L \leq \eta_j'^L$ and $\eta_j^U \leq \eta_j'^U$ for all j .

As the aggregated result of INHFGWA $(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n)$ operator is still an INHEs, and it has three parts, i.e., truth-membership, indeterminacy-membership and falsity-membership. We can prove them separately. The proof is shown as follows.

(i) We firstly prove the truth-membership part.

As $\gamma_j^L \geq \gamma_j'^L$ and $\lambda > 0, w_j \geq 0$ for all j , then

$$\left(1 - (\gamma_j^L)^\lambda\right)^{w_j} \leq \left(1 - (\gamma_j'^L)^\lambda\right)^{w_j}$$

And

$$\prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} \left(1 - (\gamma_j^L)^\lambda\right)^{w_j} \leq \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} \left(1 - (\gamma_j'^L)^\lambda\right)^{w_j}$$

So we have

$$\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} \left(1 - (\gamma_j^L)^\lambda\right)^{w_j}\right)^{1/\lambda} \geq \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} \left(1 - (\gamma_j'^L)^\lambda\right)^{w_j}\right)^{1/\lambda}$$

Similarly, we have

$$\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} \left(1 - (\gamma_j^U)^\lambda\right)^{w_j}\right)^{1/\lambda} \geq \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} \left(1 - (\gamma_j'^U)^\lambda\right)^{w_j}\right)^{1/\lambda}$$

(ii) We prove the indeterminacy-membership part.

As $\delta_j^L \leq \delta_j'^L$ and $\lambda > 0, w_j \geq 0$ for all j , then

$$(1 - \delta_j^L)^\lambda \geq (1 - \delta_j'^L)^\lambda$$

$$\left(1 - (1 - \delta_j^L)^\lambda\right)^{w_j} \leq \left(1 - (1 - \delta_j'^L)^\lambda\right)^{w_j}$$

$$\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} \left(1 - (1 - \delta_j^L)^\lambda\right)^{w_j}\right)^{1/\lambda} \geq \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} \left(1 - (1 - \delta_j'^L)^\lambda\right)^{w_j}\right)^{1/\lambda}$$

So,

$$1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} \left(1 - (1 - \delta_j^L)^\lambda\right)^{w_j}\right)^{1/\lambda} \leq 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} \left(1 - (1 - \delta_j'^L)^\lambda\right)^{w_j}\right)^{1/\lambda}$$

Similarly, we have

$$1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} \left(1 - (1 - \delta_j^U)^\lambda\right)^{w_j}\right)^{1/\lambda} \leq 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} \left(1 - (1 - \delta_j'^U)^\lambda\right)^{w_j}\right)^{1/\lambda}$$

(iii) We prove the falsity-membership part.

Similar to the indeterminacy-membership part, we have

$$1 - \left(1 - \prod_{j=1}^n \bigcup_{\delta_j \in \tilde{t}_j} \left(1 - (1 - \eta_j^L)^\lambda \right)^{w_j} \right)^{1/\lambda} \leq 1 - \left(1 - \prod_{j=1}^n \bigcup_{\delta_j \in \tilde{t}_j} \left(1 - (1 - \eta_j'^L)^\lambda \right)^{w_j} \right)^{1/\lambda}$$

Similarly, we have

$$1 - \left(1 - \prod_{j=1}^n \bigcup_{\delta_j \in \tilde{t}_j} \left(1 - (1 - \eta_j^U)^\lambda \right)^{w_j} \right)^{1/\lambda} \leq 1 - \left(1 - \prod_{j=1}^n \bigcup_{\delta_j \in \tilde{t}_j} \left(1 - (1 - \eta_j'^U)^\lambda \right)^{w_j} \right)^{1/\lambda}$$

According to (i) and (ii), we can get

$$\text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \geq \text{INHFGWA}(\tilde{n}'_1, \tilde{n}'_2, \dots, \tilde{n}'_n)$$

(3) Theorem 4 (Boundedness)

The INHFGWA operator lies between the max and min operators:

$$\min(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \leq \text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \leq \max(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \quad (32)$$

Proof.

Let $m = \min(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n)$, $M = \max(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n)$,

Since $m \leq \tilde{n}_j \leq M$, we can get the result as follows according to theorem 3.

$$\left(\sum_{j=1}^n w_j m^\lambda \right)^{1/\lambda} \leq \left(\sum_{j=1}^n w_j \tilde{n}_j^\lambda \right)^{1/\lambda} \leq \left(\sum_{j=1}^n w_j M^\lambda \right)^{1/\lambda}$$

that is

$$m \leq \left(\sum_{j=1}^n w_j \tilde{n}_j^\lambda \right)^{1/\lambda} \leq M$$

$$\text{i.e. } \min(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \leq \text{INHFGWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \leq \max(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n).$$

Now we can consider some special cases of INHFGWA operator.

(1) If $\lambda = 1$, then INHFGWA operator becomes the interval neutrosophic hesitant fuzzy weighted averaging (INHFWA) operator

$$\text{INHFWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \sum_{j=1}^n w_j \tilde{n}_j$$

$$= \left[\begin{aligned} & \left[1 - \prod_{j=1}^n \bigcup_{\gamma_j \in \tilde{t}_j} (1 - \gamma_j^L)^{w_j}, 1 - \prod_{j=1}^n \bigcup_{\gamma_j \in \tilde{t}_j} (1 - \gamma_j^U)^{w_j} \right], \left[\prod_{j=1}^n \bigcup_{\delta_j \in \tilde{t}_j} (\delta_j^L)^{w_j}, \prod_{j=1}^n \bigcup_{\delta_j \in \tilde{t}_j} (\delta_j^U)^{w_j} \right], \\ & \left[\prod_{j=1}^n \bigcup_{\eta_j \in \tilde{f}_j} (\eta_j^L)^{w_j}, \prod_{j=1}^n \bigcup_{\eta_j \in \tilde{f}_j} (\eta_j^U)^{w_j} \right] \end{aligned} \right] \quad (33)$$

(2) If $\lambda \rightarrow 0$, then INHFGWA operator becomes the interval neutrosophic hesitant fuzzy weighted geometric (INHFWG) operator:

$$\begin{aligned} \text{INHFWG}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \prod_{j=1}^n (\tilde{n}_j)^{w_j} \\ &= \left\{ \left[\prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{r}_j} (\gamma_j^L)^{w_j}, \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{r}_j} (\gamma_j^U)^{w_j} \right], \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{r}_j} (1 - \delta_j^L)^{w_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{r}_j} (1 - \delta_j^U)^{w_j} \right], \right. \\ &\quad \left. \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{r}_j} (1 - \eta_j^L)^{w_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{r}_j} (1 - \eta_j^U)^{w_j} \right] \right\} \end{aligned} \quad (34)$$

(3) If $\lambda = 2$, then INHFGWA operator becomes the interval neutrosophic hesitant fuzzy weighted quadratic averaging (INHFWQA) operator:

$$\begin{aligned} \text{INHFWQA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \left(\sum_{j=1}^n w_j \tilde{n}_j^2 \right)^{1/2} \\ &= \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{r}_j} (1 - (\gamma_j^L)^2)^{w_j} \right)^{1/2}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{r}_j} (1 - (\gamma_j^U)^2)^{w_j} \right)^{1/2} \right], \right. \\ &\quad \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{r}_j} (1 - (1 - \delta_j^L)^2)^{w_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{r}_j} (1 - (1 - \delta_j^U)^2)^{w_j} \right)^{1/2} \right], \\ &\quad \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{r}_j} (1 - (1 - \eta_j^L)^2)^{w_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{r}_j} (1 - (1 - \eta_j^U)^2)^{w_j} \right)^{1/2} \right] \right\} \end{aligned} \quad (35)$$

In definition 16, we considered the weight vector of attributes. However, in some cases, we may need to consider the position of aggregated data. For example, in the diving contest of Olympic Games, in general, after removing the most high and low scores, we can take the average value of the remaining. i.e., we can assign the weights of the most high and low scores are 0. So, the positional weights are very important in some real decision making problems. thus, we further define a new aggregation operator to process this case.

Definition 17. An interval neutrosophic hesitant fuzzy generalized ordered weighted averaging (INHFGOWA) operator of dimension n is a mapping $I : \Omega^n \rightarrow \Omega$, defined by an associated weighting

vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$ and parameter $\lambda \in (0, +\infty)$, so,

$$\text{INHFGOWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \left(\sum_{j=1}^n \omega_j \tilde{n}_{\sigma(j)}^\lambda \right)^{1/\lambda} \quad (36)$$

where, the $\tilde{n}_{\sigma(j)}$ is the j th largest of the $\tilde{n}_i$ ($i=1,2,\dots, n$), $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1,2,\dots, n)$, such that $\tilde{n}_{\sigma(j-1)} \geq \tilde{n}_{\sigma(j)}$ for all $j=2,\dots,n$, and $\tilde{n}_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j)$ ($j=1,2,\dots,n$) are interval neutrosophic hesitant fuzzy elements (INHFE).

Theorem 5. Let $\tilde{n}_j = ([\gamma_j^L, \gamma_j^U], [\delta_j^L, \delta_j^U], [\eta_j^L, \eta_j^U])$ ($j=1,2,\dots,n$) be the collection of INHES, $\tilde{n}_{\sigma(j)}$ is the j th largest of the $\tilde{n}_i$ ($i=1,2,\dots, n$) then the result aggregated from Definition 17 is still an INHES, and even

$$\begin{aligned} \left(\sum_{j=1}^n \omega_j \tilde{n}_{\sigma(j)}^\lambda \right)^{1/\lambda} = & \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} \left(1 - (\gamma_{\sigma(j)}^L)^\lambda \right)^{\omega_j} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} \left(1 - (\gamma_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda} \right] \right. \\ & \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\delta_{\sigma(j)}^L)^\lambda \right)^{\omega_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\delta_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda} \right], \\ & \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} \left(1 - (\eta_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} \left(1 - (\eta_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda} \right] \right\} \end{aligned} \quad (37)$$

The proof of this theorem is similar with theorem 1, it's omitted here.

Similar to Theorems 2-4, it is easy to prove the INHFGOWA operator has the following properties.

(1) Theorem 6 (Idempotency).

If $\tilde{n}_j = \tilde{n} = \{\tilde{t}, \tilde{i}, \tilde{f}\}$ for all j , then $\text{INHFGOWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \tilde{n}$.

(2) Theorem 7 (monotonicity)

Let $\tilde{n}_j$ ($j=1,2,\dots,n$) and $\tilde{n}'_j$ ($j=1,2,\dots,n$) be two sets of INHFS, if $\tilde{n}_j \geq \tilde{n}'_j$, then

$$\text{INHFGOWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \geq \text{INHFGOWA}(\tilde{n}'_1, \tilde{n}'_2, \dots, \tilde{n}'_n)$$

(3) Theorem 8 (Boundedness)

The INHFGOWA operator lies between the max and min operators:

$$\min(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \leq \text{INHFGOWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \leq \max(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n).$$

(4) Theorem 9 (Commutativity)

Let $(\tilde{n}'_1, \tilde{n}'_2, \dots, \tilde{n}'_n)$ be any permutation of $(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n)$, then

$$\text{INHFGOWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \text{INHFGOWA}(\tilde{n}'_1, \tilde{n}'_2, \dots, \tilde{n}'_n)$$

Now we can consider some special cases of INHFGOWA operator:

(1) If $\lambda = 1$, then INHFGOWA operator becomes the interval neutrosophic hesitant fuzzy ordered weighted averaging operator (INHFWA) operator :

$$\begin{aligned} \text{INHFWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \sum_{j=1}^n \omega_j \tilde{n}_{\sigma(j)} \\ &= \left[\left[1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} (1 - \gamma_{\sigma(j)}^L)^{\omega_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} (1 - \gamma_{\sigma(j)}^L)^{\omega_j} \right], \left[\prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} (\delta_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} (\delta_{\sigma(j)}^U)^{\omega_j} \right], \right. \\ &\quad \left. \left[\prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{I}_j} (\eta_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{I}_j} (\eta_{\sigma(j)}^U)^{\omega_j} \right] \right] \end{aligned} \quad (38)$$

(2) If $\lambda \rightarrow 0$, then INHFGOWA operator becomes the interval neutrosophic hesitant fuzzy ordered weighted geometric (INHFWG) operator:

$$\begin{aligned} \text{INHFWG}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \prod_{j=1}^n (\tilde{n}_{\sigma(j)})^{\omega_j} \\ &= \left[\left[\prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} (\gamma_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} (\gamma_{\sigma(j)}^U)^{\omega_j} \right], \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} (1 - \delta_{\sigma(j)}^L)^{\omega_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} (1 - \delta_{\sigma(j)}^U)^{\omega_j} \right], \right. \\ &\quad \left. \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{I}_j} (1 - \eta_{\sigma(j)}^L)^{\omega_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{I}_j} (1 - \eta_{\sigma(j)}^U)^{\omega_j} \right] \right] \end{aligned} \quad (39)$$

(4) If $\lambda = 2$, then INHFGOWA operator becomes the interval neutrosophic hesitant fuzzy ordered weighted quadratic averaging (INHFWQA) operator:

$$\begin{aligned} \text{INHFWQA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \left(\sum_{j=1}^n \omega_j \tilde{n}_{\sigma(j)}^2 \right)^{1/2} \\ &= \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} (1 - (\gamma_{\sigma(j)}^L)^2)^{\omega_j} \right)^{1/2}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{I}_j} (1 - (\gamma_{\sigma(j)}^U)^2)^{\omega_j} \right)^{1/2} \right], \right. \\ &\quad \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} (1 - (1 - \delta_{\sigma(j)}^L)^2)^{\omega_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{I}_j} (1 - (1 - \delta_{\sigma(j)}^U)^2)^{\omega_j} \right)^{1/2} \right], \\ &\quad \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{I}_j} (1 - (1 - \eta_{\sigma(j)}^L)^2)^{\omega_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{I}_j} (1 - (1 - \eta_{\sigma(j)}^U)^2)^{\omega_j} \right)^{1/2} \right] \right\} \end{aligned} \quad (40)$$

About the positional weight vector, we can get it according to the real application. In some cases, we can obtain it by mathematical method. O'Hagan[27, 28] developed a method to get the OWA weights which have maximize the entropy and a predefined degree of orness. They can be gotten by the following models.

Maximize: $-\sum_{i=1}^n \omega_i \ln \omega_i$

Subject to: $\frac{1}{n-1} \sum_{i=1}^n (n-i) \omega_i = \alpha, 0 \leq \alpha \leq 1$

$\sum_{i=1}^n \omega_i = 1, 0 \leq \omega_i \leq 1, i = 1, 2, \dots, n$ where, α is the predefined degree of orness.

In definitions 16 and 17, the attribute weight vector and the positional weight vector can be considered separately. However, in many decision making problems, we need take two kinds of weight into account. We can define the interval neutrosophic hesitant fuzzy generalized hybrid weighted averaging (INHFGHWA) operator to process the problems.

Definition 18. An interval neutrosophic hesitant fuzzy generalized hybrid weighted averaging (INHFGHWA) operator of dimension n is a mapping $I : \Omega^n \rightarrow \Omega$, defined by an associated weighting

vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$ and parameter $\lambda \in (0, +\infty)$, so,

$$\text{INHFGHWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \left(\sum_{j=1}^n \omega_j \tilde{b}_{\sigma(j)}^\lambda \right)^{1/\lambda}. \quad (41)$$

where, the $\tilde{b}_{\sigma(j)}$ is the j th largest of the $nw_j \tilde{n}_j$ ($j = 1, 2, \dots, n$), and $w = (w_1, w_2, \dots, w_n)^T$ is the

weighting vector of the $\tilde{n}_j$ ($j = 1, 2, \dots, n$), $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, and $\tilde{n}_i$ ($i = 1, 2, \dots, n$) are

interval neutrosophic hesitant fuzzy elements (INHFE), and $\tilde{n}_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j)$ ($j = 1, 2, \dots, n$).

Theorem 10. Let $\tilde{n}_j = ([\gamma_j^L, \gamma_j^U], [\delta_j^L, \delta_j^U], [\eta_j^L, \eta_j^U])$ ($j = 1, 2, \dots, n$) be the collection of INHFEs,

$\tilde{b}_{\sigma(j)} = ([\alpha_{\sigma(j)}^L, \alpha_{\sigma(j)}^U], [\beta_{\sigma(j)}^L, \beta_{\sigma(j)}^U], [\chi_{\sigma(j)}^L, \chi_{\sigma(j)}^U])$ is the j th largest of

the $nw_j \tilde{n}_j$ ($j = 1, 2, \dots, n$) then the result aggregated from Definition 18 is still an INHFEs, and even

$$\begin{aligned} \left(\sum_{j=1}^n \omega_j \tilde{b}_{\sigma(j)}^\lambda \right)^{1/\lambda} = & \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{t}_j \in \tilde{t}_j} \left(1 - (\alpha_{\sigma(j)}^L)^\lambda \right)^{\omega_j} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{t}_j \in \tilde{t}_j} \left(1 - (\alpha_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda} \right] \right. \\ & \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\beta_{\sigma(j)}^L)^\lambda \right)^{\omega_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} \left(1 - (\beta_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda} \right] \\ & \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} \left(1 - (\chi_{\sigma(j)}^L)^\lambda \right)^{\omega_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} \left(1 - (\chi_{\sigma(j)}^U)^\lambda \right)^{\omega_j} \right)^{1/\lambda} \right] \right\} \end{aligned} \quad (42)$$

Where

$$\begin{aligned} \tilde{b}_j &= \{[\alpha_j^L, \alpha_j^U], [\beta_j^L, \beta_j^U], [\chi_j^L, \chi_j^U]\} \\ &= \left\{ \left[1 - \left(1 - \left(1 - (\gamma_j^L)^{w_j} \right) \right)^n, 1 - \left(1 - \left(1 - (\gamma_j^U)^{w_j} \right) \right)^n \right], \left[\left((\delta_j^L)^{w_j} \right)^n, \left((\delta_j^U)^{w_j} \right)^n \right], \left[\left((\eta_j^L)^{w_j} \right)^n, \left((\eta_j^U)^{w_j} \right)^n \right] \right\} \end{aligned} \quad (43)$$

The proof of this theorem is similar with theorem 1, it's omitted here.

Now we can consider some special cases of INHFGHWA operator:

(1) If $\lambda = 1$, then INHFGHWA operator becomes the interval neutrosophic hesitant fuzzy hybrid weighted averaging (INHFWA) operator :

$$\begin{aligned} \text{INHFWA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \sum_{j=1}^n \omega_j \tilde{b}_{\sigma(j)} \\ &= \left\{ \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} (1 - \alpha_{\sigma(j)}^L)^{\omega_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} (1 - \alpha_{\sigma(j)}^U)^{\omega_j} \right], \left[\prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} (\beta_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} (\beta_{\sigma(j)}^U)^{\omega_j} \right], \right. \\ &\quad \left. \left[\prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} (\chi_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} (\chi_{\sigma(j)}^U)^{\omega_j} \right] \right\} \end{aligned} \quad (44)$$

Where, $\tilde{b}_j$ can be obtained by (43).

(2) If $\lambda \rightarrow 0$, then INHFGHWA operator becomes the interval neutrosophic hesitant fuzzy hybrid weighted geometric (INHFWG) operator.

$$\begin{aligned} \text{INHFWG}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) &= \prod_{j=1}^n (\tilde{b}_{\sigma(j)})^{\omega_j} \\ &= \left\{ \left[\prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} (\alpha_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{\gamma}_j} (\alpha_{\sigma(j)}^U)^{\omega_j} \right], \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} (1 - \beta_{\sigma(j)}^L)^{\omega_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{\delta}_j} (1 - \beta_{\sigma(j)}^U)^{\omega_j} \right], \right. \\ &\quad \left. \left[1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} (1 - \chi_{\sigma(j)}^L)^{\omega_j}, 1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{\eta}_j} (1 - \chi_{\sigma(j)}^U)^{\omega_j} \right] \right\} \end{aligned} \quad (45)$$

Where, $\tilde{b}_j$ can be obtained by (43).

(3) If $\lambda = 2$, then INHFGHWA operator becomes the interval neutrosophic hesitant fuzzy hybrid quadratic weighted averaging (INHFWQA) operator:

$$\text{INHFWQA}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) = \left(\sum_{j=1}^n \omega_j \tilde{b}_{\sigma(j)}^2 \right)^{1/2}$$

$$\begin{aligned}
&= \left\{ \left[\left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\alpha_{\sigma(j)}^L)^2 \right)^{\omega_j} \right)^{1/2}, \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - (\alpha_{\sigma(j)}^U)^2 \right)^{\omega_j} \right)^{1/2} \right] \right. \\
&\quad \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - (1 - \beta_{\sigma(j)}^L)^2 \right)^{\omega_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - (1 - \beta_{\sigma(j)}^U)^2 \right)^{\omega_j} \right)^{1/2} \right], \\
&\quad \left. \left[1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \chi_{\sigma(j)}^U)^2 \right)^{\omega_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^n \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - (1 - \chi_{\sigma(j)}^U)^2 \right)^{\omega_j} \right)^{1/2} \right] \right\}
\end{aligned} \tag{46}$$

Where, $\tilde{b}_j$ can be obtained by (43).

5. A decision-making method based on the INHFGHWA operator

In this section, we will use the **developed** operators **for** the interval neutrosophic hesitant fuzzy numbers to the **MADM** problems **where** attribute values take the form of the interval neutrosophic hesitant fuzzy information.

For a multiple attribute decision making problem, let $X = \{x_1, x_2, \dots, x_m\}$ be a set of

alternatives, $C = \{c_1, c_2, \dots, c_n\}$ be a set of attributes, and $w = (w_1, w_2, \dots, w_n)^T$ be the weighting

vector of attributes such that $w_j \in [0,1]$. Suppose that $\tilde{n}_{ij} = (\tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ is the evaluation

information of the criteria C_j on the alternative A_i which is represented by the form of the interval

neutrosophic hesitant fuzzy information. Where $\tilde{t}_{ij} = \{\tilde{\gamma}_{ij} | \tilde{\gamma}_{ij} \in \tilde{t}_{ij}\}$, $\tilde{i}_{ij} = \{\tilde{\delta}_{ij} | \tilde{\delta}_{ij} \in \tilde{i}_{ij}\}$, and

$\tilde{f}_{ij} = \{\tilde{\eta}_{ij} | \tilde{\eta}_{ij} \in \tilde{f}_{ij}\}$ are three sets of some interval values in real unit interval $[0,1]$, which denotes the

possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees, and satisfies these

limits : $\tilde{\gamma}_{ij} = [\gamma_{ij}^L, \gamma_{ij}^U] \subseteq [0,1]$, $\tilde{\delta}_{ij} = [\delta_{ij}^L, \delta_{ij}^U] \subseteq [0,1]$, $\tilde{\eta}_{ij} = [\eta_{ij}^L, \eta_{ij}^U] \subseteq [0,1]$ and

$0 \leq \sup \tilde{\gamma}^+ + \sup \tilde{\delta}^+ + \sup \tilde{\eta}^+ \leq 3$, where $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma}_{ij} \in \tilde{t}_{ij}} \max \{\gamma_{ij}^U\}$, $\tilde{\delta}^+ = \bigcup_{\tilde{\delta}_{ij} \in \tilde{i}_{ij}} \max \{\delta_{ij}^U\}$, and

$\tilde{\eta}^+ = \bigcup_{\tilde{\eta}_{ij} \in \tilde{f}_{ij}} \max \{\eta_{ij}^U\}$. Then we can rank the order of the alternatives. The steps are shown as

follows.

Step 1. Utilize the INHFGHWA operator

$$\tilde{I}_i = \text{INHFGHWA}(\tilde{n}_{i1}, \tilde{n}_{i2}, \dots, \tilde{n}_{in}) = \left(\sum_{j=1}^n \omega_j \tilde{b}_{i\sigma(j)}^\lambda \right)^{1/\lambda} \quad (i = 1, 2, \dots, m). \tag{47}$$

to derive the collective overall preference values $\tilde{I}(i = 1, 2, \dots, m)$. Where, $\tilde{b}_{ij} = nw_j \tilde{n}_{ij}$.

Step 2. Utilize the score function expressed by (26) to calculate the ranking values. Because the score function is an interval value, it can be compared by possibility degrees of the interval numbers defined in (1). We can use a simple average method to improve the score function as follows.

$$s_i = s(\tilde{I}_i) = \left[\frac{1}{l_i} \sum_{i=1}^{l_i} \left(\frac{\gamma_i^L + \gamma_i^U}{2} \right) + \left(\frac{\sum_{i=1}^{p_i} \left(1 - \frac{\delta_i^L + \delta_i^U}{2} \right)}{p_i} \right) + \left(\frac{\sum_{i=1}^{q_i} \left(1 - \frac{\eta_i^L + \eta_i^U}{2} \right)}{q_i} \right) \right] / 3 \quad (48)$$

Where l_i , p_i , and q_i are the numbers of interval values in $\tilde{t}_i, \tilde{f}_i, \tilde{f}_i$.

Step 3. Rank all the alternatives x_i ($i = 1, 2, \dots, m$) and select the best one(s).

Step 4. End.

6. An numerical example

In this section, we will provide an example to illustrate the application of INHFGHWA operator.

Suppose that an investment company wants to select an enterprise from four possible alternatives to invest. The four candidates are marked by A_i ($i=1,2,3,4$), and they are evaluated by three attributes: (1) C1 (the risk index); (2) C2 (the growth index); (3) C3 (environmental impact index), and the evaluation values are expressed by INHFNs. Suppose the weight vector of the attributes is $w = (0.35, 0.25, 0.4)^T$.

We can construct the following matrix R shown in the Table 1.

Table 1 interval neutrosophic hesitant generalized aggregation operator decision matrix

	C1	C2	C3
A1	$\{ \{ [0.3, 0.4], [0.4, 0.4], [0.4, 0.5] \}, \{ [0.1, 0.2] \}, \{ [0.3, 0.4] \} \}$	$\{ \{ [0.4, 0.5], [0.5, 0.6] \}, \{ [0.2, 0.3] \}, \{ [0.3, 0.3] \}, \{ [0.3, 0.4] \} \}$	$\{ \{ [0.2, 0.3] \}, \{ [0.1, 0.2] \}, \{ [0.4, 0.5], [0.5, 0.6] \} \}$
A2	$\{ \{ [0.6, 0.7] \}, \{ [0.1, 0.2] \}, \{ [0.1, 0.2], [0.2, 0.3] \} \}$	$\{ \{ [0.6, 0.7] \}, \{ [0.1, 0.1] \}, \{ [0.2, 0.3] \} \}$	$\{ \{ [0.6, 0.7] \}, \{ [0.1, 0.2] \}, \{ [0.1, 0.2] \} \}$
A3	$\{ \{ [0.3, 0.4], [0.5, 0.6] \}, \{ [0.2, 0.4] \}, \{ [0.2, 0.3] \} \}$	$\{ \{ [0.5, 0.6] \}, \{ [0.2, 0.3] \}, \{ [0.3, 0.4] \} \}$	$\{ \{ [0.5, 0.6] \}, \{ [0.1, 0.2], [0.2, 0.3] \}, \{ [0.2, 0.3] \} \}$
A4	$\{ \{ [0.7, 0.8] \}, \{ [0, 0.1] \}, \{ [0.1, 0.2] \} \}$	$\{ \{ [0.6, 0.7] \}, \{ [0, 0.1] \}, \{ [0.2, 0.2] \} \}$	$\{ \{ [0.3, 0.5] \}, \{ [0.2, 0.3] \}, \{ [0.1, 0.2], [0.3, 0.3] \} \}$

6.1 The evaluation steps by the proposed method

Step 1. We can get the INHFGHWA operators based on Eq.(47) (Suppose the positional weight

$$(\omega = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } \lambda = 1):$$

$$\tilde{I}_1 = \left\{ \left\{ (0.2895, 0.3903) \right\}, \left\{ (0.3568, 0.4234) \right\}, \left\{ (0.3568, 0.4590) \right\}, \right. \\ \left. \left\{ (0.1189, 0.2213) \right\}, \left\{ (0.3041, 0.4070) \right\}, \left\{ (0.3680, 0.4704) \right\} \right\};$$

$$\tilde{I}_2 = \left\{ \left\{ (0.6, 0.7) \right\}, \left\{ (0.1, 0.1682) \right\}, \left\{ (0.1189, 0.2213) \right\}, \left\{ (0.1516, 0.2551) \right\} \right\};$$

$$\tilde{I}_3 = \left\{ \left\{ (0.4375, 0.5390) \right\}, \left\{ (0.5, 0.6) \right\}, \left\{ (0.1516, 0.2821) \right\}, \left\{ (0.2, 0.3318) \right\}, \right. \\ \left. \left\{ (0.2213, 0.3224) \right\} \right\};$$

$$\tilde{I}_4 = \left\{ \left\{ (0.5476, 0.6807) \right\}, \left\{ (0, 0.1552) \right\}, \left\{ (0.1189, 0.2) \right\}, \left\{ (0.1845, 0.2352) \right\} \right\}.$$

Step 2. calculate the score function according to Eq.(48):

$$s_1 = 1.0273; s_2 = 1.402; s_3 = 1.1339; s_4 = 1.3969.$$

$$\text{So, } s_2 > s_4 > s_3 > s_1.$$

Step 3. Rank all the alternatives $A_j (j = 1, 2, 3, 4)$ in descending order and select the best one(s) in

accordance with the values of $s_j (j = 1, 2, 3, 4)$, we can get $A_2 > A_4 > A_3 > A_1$ and A_2 is the best option.

Step 4. End.

6.2 The influence of the parameter λ on decision making of this example

In order to show the influence of the parameter λ , we can adopt the different parameter value λ in step 1 to rank the alternatives. The ranking results are shown in Table 2.

Table 2 Ranking of the alternatives by utilizing the different λ in INHFGHWA operator

λ	Score function values $s_j (j = 1, 2, 3, 4)$	Ranking
$\lambda = 0.001$	$s_1 = 0.66646, s_2 = 0.66661$ $s_3 = 0.66662, s_4 = 0.66651$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\lambda = 0.1$	$s_1 = 0.64800, s_2 = 0.66180$ $s_3 = 0.66318, s_4 = 0.65267$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\lambda = 0.7$	$s_1 = 0.61207, s_2 = 0.64704$	$A_3 \succ A_2 \succ A_1 \succ A_4$

	$s_3 = 0.67125, s_4 = 0.59764$	
$\lambda = 0.8$	$s_1 = 0.61471, s_2 = 0.64662$ $s_3 = 0.67620, s_4 = 0.59233$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\lambda = 1.0$	$s_1 = 0.6245, s_2 = 0.6742$ $s_3 = 0.6882, s_4 = 0.5843$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda = 2.0$	$s_1 = 0.71978, s_2 = 0.67197$ $s_3 = 0.77333, s_4 = 0.57952$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda = 5.9$	$s_1 = 1.05053, s_2 = 0.88072$ $s_3 = 1.08634, s_4 = 0.71573$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda = 15$	$s_1 = 1.26617, s_2 = 1.18636$ $s_3 = 1.29611, s_4 = 0.90952$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda = 39$	$s_1 = 1.32926, s_2 = 1.32323$ $s_3 = 1.33213, s_4 = 0.99365$	$A_3 \succ A_1 \succ A_2 \succ A_4$

As we can see from Table 2, the ranking results may be different for the different parameter value λ in INHFGHWA operator.

(1) When $0.001 \leq \lambda \leq 39$, the best alternative is A_3 .

(2) When $0.001 < \lambda < 39$, the ranking of the alternatives is different with respect to different λ .

When $0.001 \leq \lambda \leq 0.1$, the ranking of the alternatives is $A_3 \succ A_2 \succ A_4 \succ A_1$. When $0.7 \leq \lambda \leq 0.8$, the ranking of the alternatives is $A_3 \succ A_2 \succ A_1 \succ A_4$. When $1.0 \leq \lambda \leq 39$, the ranking of the alternatives is $A_3 \succ A_1 \succ A_2 \succ A_4$.

Comparing with the method proposed by Ye [22], this paper proposed the interval neutrosophic hesitant fuzzy set, and Ye [22] proposed a neutrosophic hesitant fuzzy set. Obviously, the neutrosophic hesitant fuzzy set is a special of the interval neutrosophic hesitant fuzzy set. In addition, in information

aggregation operators, we proposed a generalized hybrid weighted average operator based on interval neutrosophic hesitant set, and Ye [22] proposed a weighted averaging operator and weighted geometric operator for SVNHFS. Obviously, they are the special cases of the generalized aggregation operators. So, the method proposed in this paper is more general than Ye's method.

7. Conclusion

The HFS can allow the membership function of an element to a set represented by several possible values, however, it cannot handle indeterminate and inconsistent information, while the NS can easily represent uncertainty, incomplete, and inconsistent information. In this paper, combined the IVHFS and INS, we further propose the concept of interval neutrosophic hesitant fuzzy sets (INHFSs), which extends truth-membership degree, indeterminacy-membership degree, and falsity-membership degree of an element to a given set to IVHFS. The INHFS is a generalization of fuzzy set, IFS, IVIFS, NS, INS, HS, IVHFS, and so on. Then, we proposed the score function and comparison method of NHFSs, and developed some new aggregation operators for the interval neutrosophic hesitant fuzzy information, including interval neutrosophic hesitant fuzzy generalized weighted (INHFGWA) operator, interval neutrosophic hesitant fuzzy generalized ordered weighted (INHFGOWA) operator, and interval neutrosophic hesitant fuzzy generalized hybrid weighted (INHFGHWA) operator, and discuss some properties. Furthermore, we propose the decision making method for multiple attribute group decision making (MAGDM) problems with interval neutrosophic hesitant fuzzy information, and give the detail decision steps. A significant characteristic of the proposed method is that it can process many kinds of fuzzy information. In the future, we can apply the proposed operators to expand the scope of application, such as selection of supplier, science-technology assessment, the performance evaluation.

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