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A short note on constrained linear control systems with multiplicative ellipsoidal uncertainty

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***Abstract*—This paper revisits the classical robust control question of how to design linear control laws for uncertain linear dynamic systems. We formulate this robust control design problem from a modern optimal control perspective which allows us to take into account control and state constraints that have to be satisfied by the closed-loop system for all possible uncertainty scenarios. The contribution of this paper is the derivation of a conservative but computationally tractable robust control design problem formulation for the case that multiplicative uncertainties are present in the linear dynamic systems, which render the robust control problem non-convex in general. Here, our approximation technique relies on propagating ellipsoidal bounds on the reachable states of the closed-loop system. The methods developed in this paper can also be used as a building block for tube based model predictive control schemes, where robustly designed linear control laws can help to reduce the conservatism of open-loop predictions. We illustrate the proposed techniques with a numerical case study.**

I. INTRODUCTION

The question of how to design robust control laws has—due to its practical relevance—been addressed by a huge amount of articles. The origins of the classical robust control theory go back to the mid 20th century. Highlights are the contributions by Glover and Schweppe [14], [34] as well as Zames [40], who analyzed linear control systems with set constrained disturbances. The classical robust control theory, including H_∞-control, is covered by many text books, e.g., [10], [35] or [41] and is therefore not reviewed in detail in this introduction.

Many approaches in robust control are influenced by the field of robust convex optimization [3]–[6], [11]. These approaches frequently use the concept of duality in convex optimization to reformulate robust min-max into standard minimization problems. Especially, the work of Scherer [33] and Löfberg [27] have helped to establish robust convex optimization techniques in the context of robust control—including linear fractional representations and linear matrix inequalities.

An important sub-problem of robust optimal control is to analyze the propagation of uncertainty in dynamic systems, also known under the name reachability analysis. We can find mature literature on set theoretic methods including Aubin's viability theory [2], Isaacs' differential games [19], as well as recent set propagation methods based on generalized differential inequalities [38]. In this context, Kurzhanski, Valyi, and Varaiya [23]–[25] contributed significantly with their analysis of ellipsoidal methods for linear dynamic systems with direct applications in robust control. In addition, Kothare, Balakrishnan, and Morari [22] analyzed ellipsoidal invariant sets in order to construct robust linear feedback laws for LPV systems subject to input and state constraints. For an overview over these developments we refer to a text book by Blanchini and Miani [8].

Modern approaches on robust closed-loop control can also be found in the model predictive control theory. We refer to the min-max model predictive control techniques of Kerrigan and Maciejowski [20], [21], the affine disturbance-feedback parameterization approach by Kerrigan, Goulart, and Maciejowski [15], as well as the work of Langson and Chryssochoos [26], Mayne [28], and Rakovic [31], [32] on tube based model predictive control.

In this paper, we design linear control laws for uncertain dynamic systems with state and control constraints. In Section II we formulate this problem mathematically by regarding the reachable set of the closed-loop system as the "state" of a generalized optimal control problem. A time-varying linear feedback control law is regarded as an optimization variable. Problem formulations of this form are not new and similar strategies for optimizing linear feedback laws have been published earlier, e.g., in [1], [16], [18], [28], where stability and robustness performance measures for linear control laws are proposed. Moreover, the design of robust MPC controllers for linear systems based on linear ancillary feedback laws has been proposed by [36] using linear matrix inequality techniques. However, the main contribution of this paper is presented in Sections III and IV, where we develop a parametric set propagation strategy for control systems with multiplicative ellipsoidal uncertainties. Here, a major contribution of this paper compared to existing robust control design methods is that we consider uncertainty sets that are given in the form of general matrix ellipsoids. This is in contrast to the work in [36], [9], and many other articles on LMI techniques for robust control, which are based on a less general class of matrix ellipsoids. Section III explains how the results in this paper relate to existing approaches and that our derivation contains famous robust control achievements, such as the bounded real lemma, as a special case. Section V presents a numerical case study. Section VI concludes the paper.

Notation and Preliminaries: We use the symbols $\mathbb{K}^n$ and $\mathbb{K}^n_C$ to denote the set of compact and compact convex subsets

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of $\mathbb{R}^n$ respectively. The support function $\sigma_Z(c) : \mathbb{R}^n \to \mathbb{R}$ of a set $Z \in \mathbb{K}^n_C$ is defined as

$$\forall c \in \mathbb{R}^n, \quad \sigma_Z(c) = \max_z \{c^\top z | z \in Z\}\,.$$

The Kronecker product $\otimes : \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{nm\times nm}$ of two matrices M and N is given by

$$M \otimes N = \begin{pmatrix} M_{1,1}N & \dots & M_{1,n}N \\ \vdots & \ddots & \vdots \\ M_{n,1}N & \dots & M_{n,n}N \end{pmatrix}$$

For symmetric matrices $M \in \mathbb{R}^{n\times n}$ we write $M \succeq 0$ if the matrix M is positive semi-definite. The set of positive semi-definite and positive definite matrices is denoted by $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$ respectively. For the derivations in Section III, we suppress the index k of a discrete-time system, e.g., we write

$$x^+ = Ax \quad \text{instead of} \quad x_{k+1} = A_k x_k\ .$$

Lemma 1.1: Let $A \in \mathbb{R}^{n\times m}$ be any given matrix and let $b \in \mathbb{R}^m$ be any given vector. The optimal value of the optimization problem

$$V_1 = \max_{x,y}\ \{x^\mathsf{T} Ay + b^\mathsf{T} y\} \quad \text{s.t.} \quad \left\{ \begin{array}{c} x^\mathsf{T} x \le 1 \\ y^\mathsf{T} y \le 1 \end{array} \right\}$$

coincides with the optimal value of its associated dual optimization problem

$$V_2 = \inf_{r>0,s>0}\ \max_{x,y}\ \{x^\mathsf{T} Ay + b^\mathsf{T} y - r x^\mathsf{T} x - s y^\mathsf{T} y + r + s\}\,,$$

i.e., there is no duality gap, $V_1 = V_2$.

A proof of Lemma 1.1 can be found in Appendix A.

II. Problem Formulation

We consider uncertain linear control systems of the form

$$x_{k+1} = (A_k + D_k)x_k + B_k u_k\ , \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the current state and $x_{k+1} \in \mathbb{R}^n$ is the successive state. The matrices $A_k \in \mathbb{R}^{n\times n}$ and $B_k \in \mathbb{R}^{n\times l}$ are assumed to be given, but the matrices $D_k \in \mathbb{R}^{n\times n}$ are uncertain and only known to be in a given set. We assume that the uncertainty set is an ellipsoid of the form

$$\mathcal{P} = \left\{ \sum_{i=1}^{p} P_i\, \delta_i \ \middle|\ \delta \in \mathbb{R}^p,\ \delta^\top \delta \le 1 \right\} \subseteq \mathbb{R}^{n\times n}\,.$$

Here, $P_1, \dots, P_p \in \mathbb{R}^{n\times n}$ are given scaling matrices. The analysis below can be extended easily for the case that the matrices B_k are affected by uncertainty, too, by using similar arguments as in [36]. However, this paper assumes that the matrices B_k are given in order to keep the notation light.

Remark 2.1: Notice that existing robust control design methods [9], [36] often assume that the matrix uncertainty set has the form

$$\bar{\mathcal{P}} = \left\{ D \in \mathbb{R}^{n_x\times n_x} \mid \|\bar{P}_1 D \bar{P}_2\|_* \le 1 \right\}\ ,$$

where $\bar{P}_1$ and $\bar{P}_2$ are scaling matrices and $\|\cdot\|_*$ denotes either the Frobenius- or the spectral norm. However, in general it is not possible to find matrices $\bar{P}_1$ and $\bar{P}_2$ such that $\mathcal{P} = \bar{\mathcal{P}}$. This aspect is relevant in practical applications, where the scaling matrices $P_1, \dots, P_p$ are often sparse and contain important structural information about how the uncertainty affects the model equations. ◇

Remark 2.2: A recent trend in tube based robust model predictive control is to model the multiplicative uncertainty set by a matrix polytope, which is given in the form of a convex hull of its vertices [12], [29]. This has the advantage that the uncertainty set can be modelled very accurately. A disadvantage of polytopes, however, is that the number of vertices that are needed to model high-dimensional uncertainty sets can be very large. Thus, if the matrix uncertainty set is structured but not low dimensional, the ellipsoidal modelling approach, which is analyzed in this paper, has advantages in terms of computational complexity. ◇

A. Reformulating constant offsets

Discrete-time system of the more general form

$$y_{k+1} = (\tilde{A}_k + \tilde{D}_k)y_k + \tilde{B}_k u_k + \tilde{c}_k + \tilde{d}_k\ , \tag{2}$$

where $\tilde{c}_k$ is an offset and $\tilde{d}_k$ an additional vector-valued uncertainty, can be reformulated in the form (1) by introducing the stacked state vector $x_k = (y_k^\top, 1)^\top$ for all k and defining

$$A_k = \begin{pmatrix} \tilde{A}_k & \tilde{c}_k \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B_k = \begin{pmatrix} \tilde{B}_k \\ 0 \end{pmatrix}.$$

The scaling matrices P_i have to be defined accordingly in order to take into account that the last row of the uncertainty matrix

$$D_k = \begin{pmatrix} \tilde{D}_k & \tilde{d}_k \\ 0 & 0 \end{pmatrix}$$

is equal to zero and known explicitly.

B. Affine Feedback Control Parameterizations

In this paper we are interested in designing control gains $K_k \in \mathbb{R}^{l\times n}$. The corresponding affine feedback control law[1] is given by

$$u_k = K_k x_k \tag{3}$$

such that the associated closed-loop system becomes

$$x_{k+1} = (F_k + D_k)\, x_k \tag{4}$$

Here, $F_k = A_k + B_k K_k$ denotes the nominal closed-loop system matrix.

[1]We call our feedback law affine, since if we have $x_k^\top = (y_k, 1)^\top$ the—on the first view linear—control law $u_k = K_k x_k = K_k^1 y_k + K_k^2$ with the composed matrix $K_k = (K_k^1, K_k^2)$ allows us to model offsets, too.

C. Performance Specifications

Our goal is to design the potentially time-varying feedback gains K_k in equation (3) such that mixed state-control constraints of the form $Hx_k + Ju_k + h \geq 0$ are satisfied for all possible realizations of the uncertainty matrices D_k. Notice that this constraint can be written in the equivalent form

$$\forall k \in \mathbb{N}, \qquad (H + JK_k)X_k + h \subseteq \mathbb{R}^n_+$$

where X_k denotes the reachable set at time k. We are interested in minimizing the cost function $\sum_{k=0}^{T} L(X_k)$, where L is a monotone function of the reachable set i.e.

$$L(X) \leq L(Y) \qquad \text{for all } \; X, Y \in \mathbb{K}^n \; \text{ with } \; X \subseteq Y \; .$$

For example, if we are interested in minimizing the generalized inertia of the set X with respect to a given reference point x_{ref}, we define

$$L(X) \;=\; \frac{\int_X \; (x - x_{\mathrm{ref}})^\top R \, (x - x_{\mathrm{ref}}) \, \mathrm{d}x}{\int_X 1 \, \mathrm{d}x} \; ,$$

where $R \in \mathbb{R}^{n \times n}$ is a given positive definite weighting matrix. The generalized inertia can be interpreted as one way to generalize least-squares tracking objectives for set-valued trajectories, which leads to a computationally tractable objective function when working with ellipsoidal sets. However, other objectives are possible, too. For example, in [29] an objective based on the volume of the set X is used as an objective. The feedback gain matrices $K_1, \ldots, K_N$ are then computed from

$$\begin{aligned} \min_{\substack{X_1,\ldots,X_N,\\ K_1,\ldots,K_N}} \quad & \sum_{k=1}^{N} L(X_k) \qquad\qquad (5) \\ \text{s.t.} \quad & X_k = \left\{ (A_k + B_k K_k + D)\, x \;\middle|\; \begin{array}{l} x \in X_{k-1} \\ D \in \mathcal{P} \end{array} \right\} \\ & (H + JK_k)X_k + h \;\subseteq\; \mathbb{R}^n_+, \quad k = 1, \ldots, N \; , \end{aligned}$$

where $N \in \mathbb{N}$ denotes the horizon length. Here, we assume that the set X_0 of possible initial states is given.

III. Ellipsoidal Calculus for Discrete-Time Systems with Multiplicative Uncertainty

Let us assume that the current state $x \in \mathbb{R}^n$ is known to be in the ellipsoid

$$\mathcal{E}(Q) = \left\{ Q^{\frac{1}{2}} v \mid v \in \mathbb{R}^n, \; v^\top v \leq 1 \right\} \subseteq \mathbb{R}^n \; ,$$

where $Q \in \mathbb{S}^n_+$ denotes a positive semi-definite shape matrix and $Q^{\frac{1}{2}}$ its symmetric positive-definite square root. Now, the set Y^+ of reachable next iterates is defined by

$$Y^+ = \{ (F + D)x \mid x \in \mathcal{E}(Q), D \in \mathcal{P} \} \; .$$

Unfortunately, Y^+ is in general not an ellipsoid. However, it is possible to compute an ellipsoid $\mathcal{E}(Q^+)$ that contains Y^+ and touches this set in a given direction c.

Theorem 3.1: Let $S \succ 0$ be any positive definite scaling matrix with $I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \succ 0$. The ellipsoid

$$\mathcal{E}(Q^+) = \left\{ \left(Q^+\right)^{\frac{1}{2}} v \mid v \in \mathbb{R}^n, \; v^\top v \leq 1 \right\} \; ,$$

with shape matrix

$$Q^+ = FQ^{\frac{1}{2}} \left(I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \right)^{-1} Q^{\frac{1}{2}} F^\top + P \left(I \otimes S^{-1} \right) P^\top$$

contains the set Y^+. Moreover, there exist for any direction $c \in \mathbb{R}^n$ a matrix $S \succ 0$ and $I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \succ 0$ such that the associated ellipsoid $\mathcal{E}(Q^+)$ touches the set Y^+ in the direction c, $\sigma_{Y^+}(c) = \sigma_{\mathcal{E}(Q^+)}(c)$.

Proof.
The support function S of the set Y^+ is given by

$$\begin{aligned} \sigma_{Y^+}(c) \;&=\; \max_{x \in \mathcal{E}(Q), D \in \mathcal{P}} \; c^\top (F + D) x \\ &=\; \left\{ \begin{array}{ll} \max\limits_{v,\delta} & c^\top (F + \sum_{i=1}^{p} P_i \, \delta_i) Q^{\frac{1}{2}} v \\ \text{s.t.} & \left\{ \begin{array}{lcl} v^\top v & \leq & 1 \\ \delta^\top \delta & \leq & 1 \; . \end{array} \right. \end{array} \right. \end{aligned}$$

The maximization problem on the right-hand of this equation has no duality gap, as it satisfies the requirements from Lemma 1.1. We replace the constraint $\delta^\top \delta \leq 1$ by an equivalent constraint $\delta \delta^\top \preceq I$. It can be checked easily that this replacement does not change anything about the above no-duality-gap statement, as we can only get more degrees of freedom in the dual problem when switching from scalar to matrix-valued multipliers. We will explain below why this introduction of redundant dual variables is needed. We have

$$\begin{aligned} \sigma_{Y^+}(c) = \inf_{\lambda > 0, M \succ 0} \max_{v,\delta} \; & c^\top (F + \textstyle\sum_{i=1}^{p} P_i \, \delta_i) Q^{\frac{1}{2}} v \\ & -\lambda v^\top v + \lambda - \delta^\top M \delta + \mathrm{Tr}(M). \end{aligned}$$

We introduce the shorthand

$$W = \left(P_1^\top c, \ldots, P_p^\top c \right)^\top \; .$$

If W has full-rank, the maximizer δ^* for the optimization variable δ can be written in the form $\delta^* = \frac{1}{2} M^{-1} W Q^{\frac{1}{2}} v$. Substituting this expression yields

$$\begin{aligned} & \sigma_{Y^+}(c) = \\ & \inf_{M \succ 0, \lambda > 0} \max_{v} \; c^\top F Q^{\frac{1}{2}} v + \frac{v^\top Q^{\frac{1}{2}} W^\top M^{-1} W Q^{\frac{1}{2}} v}{4} \\ & \qquad -\lambda v^\top v + \mathrm{Tr}(M) + \lambda \; . \end{aligned}$$

Next, the maximization problem for the variable v can be solved explicitly finding

$$\sigma_{Y^+}(c) \;=\; \left\{ \begin{array}{ll} \inf\limits_{\lambda > 0, M \succ 0} & \frac{1}{4} c^\top F Q^{\frac{1}{2}} \left(\lambda I - \frac{Q^{\frac{1}{2}} W^\top M^{-1} W Q^{\frac{1}{2}}}{4} \right)^{-1} Q^{\frac{1}{2}} F^\top c \\ & + \mathrm{Tr}\,(M) + \lambda \\ \text{s.t.} & \lambda I - \dfrac{Q^{\frac{1}{2}} W^\top M^{-1} W Q^{\frac{1}{2}}}{4} \succ 0 \; . \end{array} \right.$$

An optimizing sequence of matrices $M \succ 0$ can be constructed by setting

$$M_\epsilon = \epsilon I - \frac{1}{4\lambda} W S^{-1} W^\top \; ,$$

where $S \in \mathbb{S}^{n_x}_{++}$ is a new optimization variable. This construction is such that we have

$$\lim_{\epsilon \to 0} \; \mathrm{Tr}(M_\epsilon) = -\frac{1}{4\lambda} c^\top P \left(I \otimes S^{-1} \right) P^\top c$$

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with $P = (P_1, \ldots, P_p) \in \mathbb{R}^{n \times pn}$. Notice that this construction is only possible due to the introduction of matrix valued multipliers for the quadratic constraint on the primal variable δ. If we assume for a moment that the matrix W has full-rank, we can employ the Sherman-Morrison-Woodbury formula in order to find the limit

$$\lim_{\epsilon \to 0} \frac{Q^{\frac{1}{2}} W^\top M_\epsilon^{-1} W Q^{\frac{1}{2}}}{4} = \lambda Q^{\frac{1}{2}} S Q^{\frac{1}{2}} .$$

As this substitution is lossless by construction, we find

$$\sigma_{Y^+}(c) = \begin{cases} \inf\limits_{\lambda > 0, S \succ 0} & \frac{1}{4\lambda} c^\top F Q^{\frac{1}{2}} \left(I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \right)^{-1} Q^{\frac{1}{2}} F^\top c \\ & + \frac{1}{4\lambda} c^\top P \left(I \otimes S^{-1} \right) P^\top c + \lambda \\ \text{s.t.} & I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \succ 0 . \end{cases}$$

Rescaling λ and S gives

$$\sigma_{Y^+}(c) = \inf_{S \succ 0} \sqrt{c^\top Q^+ c} \ \text{ s.t. } \ I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \succ 0 ,$$

where we have introduced the shorthand notation

$$Q^+ = F Q^{\frac{1}{2}} \left(I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \right)^{-1} Q^{\frac{1}{2}} F^\top + P \left(I \otimes S^{-1} \right) P^\top .$$

In this form it becomes clear that the ellipsoid

$$\mathcal{E}(Q^+) = \left\{ (Q^+)^{\frac{1}{2}} v \mid v \in \mathbb{R}^n, \ v^\top v \leq 1 \right\} ,$$

with shape matrix Q^+ contains the set Y^+, where $S \succ 0$ can be any positive definite scaling matrix with $I - Q^{\frac{1}{2}} S Q^{\frac{1}{2}} \succ 0$. This follows from the fact that the support function of the ellipsoid $\mathcal{E}(Q^+)$ is for all c larger than or equal to the support function of the set Y^+,

$$\sigma_{\mathcal{E}(Q^+)}(c) = \sqrt{c^\top Q^+ c} \geq \sigma_{Y^+}(c) .$$

The above derivation admits an even stronger statement: there exists for any direction c and any $\varepsilon > 0$ a feasible $S \succ 0$ such that

$$\sigma_{\mathcal{E}(Q^+)}(c) - \sigma_{Y^+}(c) < \varepsilon .$$

Finally, if P is a degenerate matrix, we can set $P_i \to P_i + \epsilon I$ for a small $\epsilon > 0$ such that our proof is applicable. Since our final expression for Q^+ can be regarded as a continuous function of P, we can, afterwards, take the limit for $\epsilon \to 0$ and prove that our overestimation works without any assumptions on P. $\diamond$

If we have $Q \succ 0$, the equation for Q^+ becomes

$$Q^+ = F \left(Q^{-1} - S \right)^{-1} F^\top + P \left(I \otimes S^{-1} \right) P^\top .$$

If we want to find robust control invariant ellipsoids, we can substitute $Q = Q^+$. If the corresponding equation has a positive semi-definite solution $Q \succ 0$ for a suitable feedback law K, then the discrete-time system is robustly stabilizable. Notice that this result is known in less general versions, e.g., Souza and Xie [37] have analyzed a variant of the bounded real lemma for discrete-time systems under similar assumptions but less general matrix ellipsoidal uncertainties.

A. Generalization to Continuous-Time Systems

The above result can be generalized to continuous-time systems of the form $\dot{x} = (\tilde{F} + \tilde{D})x$ by replacing

$$F \to I + h\tilde{F},\ D \to h\tilde{D},\ S \to h\tilde{S}^{-1},\ P \to h\tilde{P},\ Q^+ \to \tilde{Q} + h\dot{\tilde{Q}}$$

and taking the limit for $h \to 0$. This yields

$$\dot{\tilde{Q}} = \tilde{F}\tilde{Q} + \tilde{Q}\tilde{F}^\top + \tilde{Q}\tilde{S}^{-1}\tilde{Q} + \tilde{P} \left(I \otimes \tilde{S} \right) \tilde{P}^\top ,$$

for any $\tilde{S} \succ 0$. For the case that $\tilde{Q}$ is invertible and $\tilde{F} = \tilde{A} + \tilde{B}\tilde{K}$ depends on a feedback gain $\tilde{K}$, it is advisable to redefine variables and set $\bar{K} = \tilde{K}\tilde{Q}$ such that

$$\dot{\tilde{Q}} \succeq \tilde{A}\tilde{Q} + \tilde{B}\bar{K} + \tilde{Q}\tilde{A}^\top + \bar{K}^\top \tilde{B}^\top + \tilde{Q}\tilde{S}^{-1}\tilde{Q} + \tilde{P} \left(I \otimes \tilde{S} \right) \tilde{P}^\top$$

is jointly matrix-convex in $(\dot{\tilde{Q}}, \tilde{Q}, \bar{K}, \tilde{S})$. If we are interested in computing robust control invariant ellipsoids, we can substitute $\dot{\tilde{Q}} = 0$ and write the above inequality as

$$\begin{pmatrix} \tilde{A}\tilde{Q} + \tilde{B}\bar{K} + \tilde{Q}\tilde{A}^\top + \bar{K}^\top \tilde{B}^\top + \tilde{P} \left(I \otimes \tilde{S} \right) \tilde{P}^\top & \tilde{Q} \\ \tilde{Q} & -\tilde{S} \end{pmatrix} \preceq 0 .$$

This is an LMI condition for the existence of robustly stabilizing linear controllers, which is in a less general version known under the name "Bounded Real Lemma" [8], [33].

IV. Robust Optimization of Feedback Controllers

The original control law design problem (5) can be solved conservatively by first bounding the initial set $X_0 \subseteq \mathcal{E}(Q_0)$ by an ellipsoid with shape matrix $Q_0 \in \mathbb{S}^n_{++}$ and then solving

$$\begin{aligned} \inf_{Q,K,S} \quad & \textstyle\sum_{k=1}^N L\left(\mathcal{E}(Q_k)\right) \\ \text{s.t.} \quad & \begin{cases} Q_{k+1} = (A_k + B_k K_k) \left(Q_k^{-1} - S_k \right)^{-1} (A_k + B_k K_k)^\top \\ \qquad + P \left[I \otimes S_k^{-1} \right] P^\top \\ (H + JK_k)\mathcal{E}(Q_k) + h \subseteq \mathbb{R}^n_+ \ , \ Q_k^{-1} - S_k \succ 0 . \end{cases} \end{aligned}$$

If L is monotone, the objective value of this optimization problem is an upper bound on the objective value of the original control design problem (5). The constraints of the form $(H + JK_k)\mathcal{E}(Q_k) + h \subseteq \mathbb{R}^n_+$ ensure robust feasibility.

A. Implementation of the objective function

If L denotes the generalized inertia of the reachable set with respect to $x_{\text{ref}} = 0$, we have

$$L(\mathcal{E}(Q_k)) = \frac{\int_{\mathcal{E}(Q_k)} x^\top R x \, \mathrm{d}x}{\int_{\mathcal{E}(Q_k)} 1 \, \mathrm{d}x} = \frac{1}{n+2} \mathrm{Tr}(R Q_k) .$$

This expression is easy to implement as the objective is linear in the optimization variables $Q_1, \ldots, Q_k$.

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B. Extension to continuous-time systems

The above formulation strategies can be applied to continuous-time systems by using the results from Section III-A. If our aim is to minimize the integral over the generalized inertia of the reachable set, the associated continuous-time control design problem becomes

$$\begin{aligned}&\inf_{\tilde{Q},\tilde{K},\tilde{S}} \quad \int_0^T \mathrm{Tr}\left(R\tilde{Q}(\tau)\right)\,\mathrm{d}\tau \\ &\text{s.t.} \left\{\begin{array}{rcl} \dot{\tilde{Q}}(\tau) & = & \left(\tilde{A}(\tau)+\tilde{B}(\tau)\tilde{K}(\tau)\right)\tilde{Q}(\tau) \\ & & +\tilde{Q}(\tau)\left(\tilde{A}(\tau)+\tilde{B}(\tau)\tilde{K}(\tau)\right)^\top \\ & & +\tilde{Q}(\tau)\tilde{S}(\tau)^{-1}\tilde{Q}(\tau)+\tilde{P}\left[I\otimes\tilde{S}(\tau)\right]\tilde{P}^\top \\ h_i^2 & \geq & (H+J\tilde{K}(\tau))_i\tilde{Q}(\tau)(H+J\tilde{K}(\tau))_i^\top \\ \tilde{S}(\tau) & \succ & 0\,,\ \tilde{Q}(0)=Q_0\,, \end{array}\right.\end{aligned} \tag{6}$$

where the constraints are enforced for all $\tau \in [0,T]$.

V. Numerical Case Study

We consider a second order mass-spring-damper system of the form

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m}\end{bmatrix}}_{\tilde{A}} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m}\end{bmatrix}}_{\tilde{B}} u \quad \text{with} \quad x(0)=\begin{bmatrix}1\\0\end{bmatrix}\,, \tag{7}$$

where $x_1(t)$ is the displacement of the mass block from the equilibrium and $x_2(t)$ the velocity. The variable $u = F$ denotes the force acting on the system. Here, $m = 3$ denotes the mass, $k = 2$ the spring constant, and $c = 1$ the damping constant. The uncertainty ellipsoid is assumed to be given by

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$$

We set $T = 15$, $H = 0$, $J = 1$, $R = I$, $h_1 = 0.3$, and $Q_0 = \mathrm{diag}(0.1, 0.1, 0)$. We are using the ACADO Toolkit [17] in combination with a SQP solver as well as direct multiple shooting discretization and piece-wise control discretization in order to solve the problem (6). Figure 1 shows the ellipsoidal

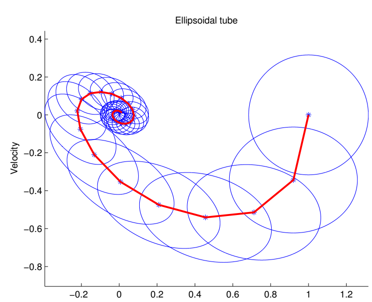

Fig. 1. The ellipsoidal tube $\mathcal{E}(Q(t_i))$ at the time points $t_i = \frac{15}{30} i$.

tube $x_{\text{nominal}}(t) + \mathcal{E}(Q(t))$. Figure 2 depicts the result for the function μ as well as the function $K(t)Q(t)K(t)^\top$, which can be interpreted as an upper bound on the maximum possible value of $u(t)^2$. Also notice that if the auxiliary function $S(t)$

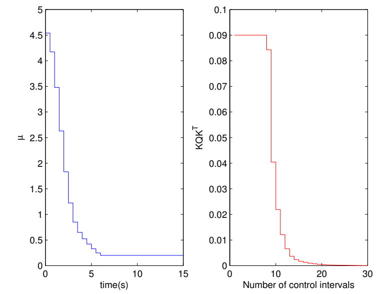

Fig. 2. The result $\mu(t)$ and the function $K(t)Q(t)K(t)^\top$.

is enforced to be a multiple of the unit matrix, $\hat{S}(t) = \hat{s}(t)I$ with $\tilde{s}(t)$ being a time-varying scalar optimization variable, the optimal objective value increases by 5.3%. This indicates that matrix valued multipliers can indeed help to reduce the conservatism of the proposed method when compared to existing robust control analysis methods, e.g., based on the bounded real lemma, which uses scalar mutipliers.

VI. Conclusions

In this paper, we have proposed a framework for the design of robust control laws for linear time-varying systems with multiplicative uncertainty. As our technical derivation is based on existing conceptual ideas from the field of ellipsoidal calculus and robust control design, it is not surprising that the proposed ellipsoidal propagation laws contain existing results such as the bounded real lemma as a special case. However, our framework generalizes these classical concepts in such a way that they can deal with general matrix ellipsoids and such that they become accessible for the use in modern optimal control based feedback design algorithms.

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APPENDIX

Notice that there exists a mature body of literature about the so-called S-procedure [13], [39]. In particular, it is well-known that non-convex QCQPs with one constraint do not possess a duality gap, i.e., the S-procedure is lossless. For real-valued non-convex QCQPs with two constraints, there may be a duality gap in general, although in some special cases strong duality can be established [7]. As the statement of Lemma 1.1 is not among the special cases mentioned in [7], [30], this appendix provides a concise and self-contained proof of this statement. The primal optimization problem is given by

$$\begin{aligned} V_1 &= \max_{x,y} \left\{ x^\intercal A y + b^\intercal y \right\} \quad \text{s.t.} \quad \left\{ \begin{array}{l} x^\intercal x \leq 1 \\ y^\intercal y \leq 1 \end{array} \right\} \\ &= \max_{y} \; b^\intercal y + \|Ay\|_2 \quad \text{s.t.} \quad y^\intercal y \leq 1 \; . \end{aligned} \tag{8}$$

Let us first show that the latter non-convex optimization problem can be solved via its dual. For this aim, we use a change of variables, $y = \alpha z$, in order to establish the equation

$$\begin{aligned} &\inf_{s>0} \max_{y} \; b^\intercal y + \|Ay\|_2 - s\left(y^\intercal y - 1\right) \\ &= \inf_{s>0} \max_{\alpha, z} \; \alpha \left(b^\intercal z + \|Az\|_2\right) - s\left(\alpha^2 - 1\right) \quad \text{s.t.} \;\; z^\intercal z \leq 1 \\ &= \inf_{s>0} \max_{\alpha} \; \alpha V_1 - s\left(\alpha^2 - 1\right) \\ &= \inf_{s>0} \frac{(V_1)^2}{4s} + s \; = \; V_1 \; . \end{aligned}$$

Fortunately, this equation allows us to write V_1 in the form

$$\begin{aligned} V_1 &= \inf_{s>0} \max_{y} \; b^\intercal y + \|Ay\|_2 - s\left(y^\intercal y - 1\right) \\ &= \inf_{s>0} \max_{x,y} \; b^\intercal y + x^\intercal A y - s\left(y^\intercal y - 1\right) \quad \text{s.t.} \;\; x^\intercal x \leq 1 \\ &= \inf_{s>0} \max_{x} \; \tfrac{1}{s} \left\|b + Ax\right\|_2^2 + s \quad \text{s.t.} \;\; x^\intercal x \leq 1 \; . \end{aligned}$$

The tight version of the S-procedure for one constraint yields

$$\begin{aligned} V_1 &= \inf_{s>0, r>0} \max_{x} \; \tfrac{1}{s} \left\|b + Ax\right\|_2^2 + s - r\left(x^\intercal x - 1\right) \\ &= \inf_{s>0, r>0} \max_{x,y} \; y^\intercal \left(b + Ax\right) - s(y^\intercal y - 1) - r\left(x^\intercal x - 1\right) \\ &= V_2 \; . \qquad \diamond \end{aligned}$$