

## Hypergeometric DE Applied to the Time-Dependent Schrodinger Equation

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In a series of notes (1), the wavefunction of the time-independent Schrodinger equation  $W(x)$  is written as  $W_0(x)W_1(x)$ , where  $W_0(x)$  is the ground state solution, and:

$$-1/2m \frac{d}{dx} \frac{d}{dx} W_1(x) - 1/m \left( \frac{d}{dx} W_1 \right) \left( \frac{d}{dx} W_0 / W_0 \right) = (E - E_0) W_1(x) \quad ((1))$$

Equation ((1)) is then compared to the hypergeometric form from (2):

$$p(x) \frac{d}{dx} \frac{d}{dx} W(x) + q(x) \frac{d}{dx} W(x) + \text{constant } W(x) = 0 \quad ((2))$$

Where  $p(x)$  is at most quadratic in  $x$  and  $q(x)$  linear. Equation ((2)) has solutions in the form of Rodrigues polynomials given in (2). In this note, we wish to examine the time dependent Schrodinger equation:

$$i \frac{d}{dt} W(x,t) = -1/2m \frac{d}{dx} \frac{d}{dx} W(x,t) + V(x,t) W(x,t) \quad ((3))$$

We would like to again use a form similar to ((1)), but this means the time-dependent Schrodinger equation should be converted, through the transformation  $g(y)=x+b(t)$ , into an equation in  $y$  which has the appearance of the time independent Schrodinger equation. This approach has been presented in (3) and (4) for a specific potential  $V(x,t) = .5k x^2 + f(t) x$ . Here, we wish to find a general method which may be linked to ((1)) and the hypergeometric equation. From this, one may find potentials  $V(x,t)$  which yield Rodrigues polynomial solutions.

### Transforming the Time-Dependent Schrodinger

We consider using independent variables  $y(x,t)$  and  $t$  as in (3), but instead of only considering a linear transformation:  $y=x-b(t)$ , we use:

$$g(y) = x - b(t) \quad ((4))$$

The goal is to write  $W(x,t)$  as  $W(y)$  multiplied by  $\exp(i \text{ quantity})$  factors with all time dependent coefficients disappearing. From there, one may easily obtain an equation similar to ((1)) by writing  $W(y)=W_0(y)W_1(y)$ .

Let us write a factor:  $\exp[i q(y) \frac{d}{dt} b(t)]$ . Then (as argued in (4)), we wish to have time flux  $\frac{d}{dt} W$  cancel with the spatial flux  $\frac{d}{dx} W$  because we assume  $W(y)$  is a real function.

$$i \frac{d}{dt} W(y) \frac{dy}{dt} = -i \left( \frac{d}{dt} b \right) / g' \frac{d}{dy} W \quad \text{where } g'(y) = \frac{d}{dy} g(y) \quad ((5))$$

There will also be a term from  $i \frac{d}{dt} \exp[i q(y) \frac{d}{dt} b(t)] = - \frac{d}{dt} \frac{d}{dt} b(t) q(y)$  ((6)), but this is real.

From  $-1/2m \frac{d}{dx} \frac{d}{dx}$  in ((3)) an imaginary term occurs:

$$\frac{d}{dx} F = (\frac{d}{dy} F) \frac{1}{g'} \quad \text{and} \quad \frac{d}{dx} \frac{d}{dx} F = -g'' / (g'g') \frac{d}{dy} F + (1/g') (1/g') \frac{d}{dy} \frac{d}{dy} F \quad ((6))$$

where  $F(x)$  is some function.

Thus:

$$(\exp(-i q \frac{d}{dt} b) \frac{d}{dx} \frac{d}{dx} [W \exp(i q \frac{d}{dt} b)]) = 2i (\frac{d}{dt} b) q' (\frac{d}{dy} W) / [g'g'] + 1/(g'g') \frac{d}{dy} \frac{d}{dy} W - g'' (1/g'g') \frac{d}{dy} W (1/g') + i W (\frac{d}{dt} b) g'' q' / (g'g') - W (\frac{d}{dt} b)^2 (q')^2 / g'^2 \quad ((7))$$

The complex terms (assuming  $W(y)$  is real) are:

$$(\frac{d}{dt} b) (\frac{d}{dy} W) (1/g') = 2 (\frac{d}{dt} b) q' (\frac{d}{dy} W) / (g'g') + W (\frac{d}{dt} b) q'' / (g'g') - W (\frac{d}{dt} b) g'' q' / (g'g') \quad ((8))$$

If one sets  $g' = q'/2$ , there is a solution.

Thus, one has an equation in terms of the real function  $W(y)$  of the form of the time-independent Schrodinger equation. One may have pieces of the potential of the form  $q(y) d(t)$  where  $d(t)$  is some function of  $t$ . Then:

$$\frac{d}{dt} \frac{d}{dt} b(t) = d(t) \quad ((10))$$

This DE may be solved for  $b(t)$ . (See (4) for an example.)

### Obtaining an Equation for $W_1(y)$ where $W(y)=W_0(y)W_1(y)$

One may now attempt to find an equation similar to ((1)) for  $W_1(y)$ , where  $W_0(y)$  is a ground state solution. One obtains:

$$-1/2m [1/(g'g')] \frac{d}{dy} \frac{d}{dy} W_1 - 1/2m g''/(g'g') \frac{d}{dy} W_1 - 1/m (\frac{d}{dy} W_0 / W_0) (1/g'g') \frac{d}{dy} W_1 \quad ((11))$$

This may then be compared with the hypergeometric equation ((3)). A further transformation  $z(y)$  may be necessary to ensure the coefficient of  $d/dz \frac{d}{dz} W(z)$  is at most quadratic in  $z$  and that of  $d/dz W(z)$  linear.

It may be noted that ((11)) contains  $d/dy W_0 / W_0$  which is associated with the coefficient of  $d/dz W(z)$ . Thus, when one equates the coefficient of  $d/dz W(z)$  with  $az+c$ , where  $a$  and  $c$  are constants, one obtains an equation for  $d/dy W_0 / W_0$ .

Now:

$$1/(g'g') \frac{d}{dy} \left( \frac{d}{dy} W_0 / W_0 \right) - 1/(g'g') \left( \frac{d}{dy} W_0 / W_0 \right)^2 = (E_0 - V(y)) \quad ((12))$$

Thus, knowing  $d/dy W_0 / W_0$  as a function of  $y$  yields  $d/dy (d/dy W_0 / W_0)$  and one may solve for  $V(y)$ . This yields a potential compatible with a Rodrigues polynomial solution. In general, if  $b(t)$  is not  $vt$  ( $v=\text{constant}$ ), one might add a term  $g(y)d(t)$  to the potential where  $d/dt d/dt b(t)=d(t)$ .

### Example of $g(y)=y = x - b(t)$

This is the example of (3) and (4) to which we apply ((12)). In this case:

$g(y)=y$  and  $g'=1$  so:

$$-1/2m \frac{d}{dy} \frac{d}{dy} W_1 - 1/m \frac{d}{dy} W_1 \left( \frac{d}{dy} W_0 / W_0 \right) = (E - E_0) W_1 \quad ((13))$$

If one leaves the coefficient of  $d/dy d/dy W_1$  as a constant, then:

$$\left( \frac{d}{dy} W_0 / W_0 \right) = ay+c \quad ((14))$$

One may see that this leads to a harmonic oscillator potential.

One may, however, argue that the coefficient of  $d/dy d/dy W_1$  should be a quadratic (or even linear). Consider  $z=s(y)$ , then:

$$\frac{d}{dy} W = \frac{d}{dz} W (z') \quad \text{and} \quad \frac{d}{dy} \frac{d}{dy} W = \frac{d}{dz} \frac{d}{dz} W (z'z') + \left( \frac{d}{dz} W \right) (z'') \quad ((14))$$

Thus, as an example, one may take the coefficient of  $d/dz d/dz W$  as  $c_1 c_1 z^* z$ . Then:

$$z^* z' = z^* z \quad \text{or} \quad z' = c_1 z \quad \text{thus} \quad z = \exp(-c_1 y)$$

One might even obtain a more general result. The exponential, however, leads to a Hulthen type of potential.

### Potential Associated with $g(y)=y*y$

Apart from the linear function  $g(y)=y$ , consider, as an example,  $g(y)=y^2$ . This does not necessarily lead to a physically important potential, but demonstrates the method.

In such a case,  $g'=2y$  and equation ((1)) becomes:

$$(E-E_0) W_1 = -1/2m \left[ \frac{1}{(4y^2)} \frac{d}{dy} \frac{d}{dy} W_1 - .5/y^2 \frac{d}{dy} W_1 \right] - 1/m \left( \frac{d}{dy} W_0 / W_0 \right) \frac{1}{(4y^2)} \quad ((15))$$

Then:  $(z'^2 z') \frac{1}{(4y^2)} = 1$  or  $\frac{d}{dy} z = 2y$  and  $z = y^2$ . Then:

$$(-1/2m) \left( -\frac{1}{(2y^2)} (2y) \right) - (1/m) \left( \frac{d}{dy} W_0 / W_0 \right) (2y / (4y^2)) = az + c = ay^2 + c \text{ or}$$

$$\frac{d}{dy} W_0 / W_0 = -ay^2 y - 2cy + 2$$

$$\frac{d}{dy} \left( \frac{d}{dy} W_0 / W_0 \right) = -3ay^2 y - 2c \text{ and:}$$

$$-3ay^2 y - 2c - 4y^2 y (E_0 - V) = (-ay^2 y - 2cy + 2)^2$$

This leads to a polynomial potential with  $y$  to the powers of -2, -1, 0, 1, 2, 4.

Alternatively, one may set the coefficient of  $\frac{d}{dz} \frac{d}{dz} W_1$  to  $z$  i.e.

$$(z'^2 z') \frac{1}{(4y^2)} = z \text{ or } \frac{dz}{z^{.5}} = 4y^2 dy$$

This seems to lead again to polynomials, although the orders will be different

Finally, we consider the coefficient of  $\frac{d}{dz} \frac{d}{dz} W_1$  as  $z^2 z$  ( $c_1 z^2 z + c_2$  would be more general). Then:

$$(z'^2 z') \frac{1}{(4y^2)} = A z^2 z \text{ or } \ln(z) = Ay^2 y \text{ or } z = \exp(Ay^2 y) \text{ and } z' = 2Ay \exp(Ay^2 y)$$

$$\text{Then } \frac{d}{dy} W_0 / W_0 = (2 + 2ya/A + (2cy/A) \exp(-Ay^2 y)) \text{ and}$$

$$\frac{d}{dy} \left( \frac{d}{dy} W_0 / W_0 \right) - 4y^2 y (E_0 - V) = \left( \frac{d}{dy} W_0 / W_0 \right)^2 \quad ((16))$$

$$V(y) \text{ is of the form: } C_1 + C_2 \frac{1}{(y^2 y)} \exp(-Ay^2 y) + C_2 / (y^2 y) + C_3 \exp(-Ay^2 y) + C_5 / y + C_6 (1/y) \exp(-Ay^2 y) + C_7 \exp(-2Ay^2 y) \quad ((17))$$

It should be noted that for all of these potentials above:

$y^*y = x - b(t)$ . Furthermore, if  $b(t)$  is not  $vt$  ( $v = \text{constant}$ ), then one may add  $y^*y d(t)$  where  $d(t)$  could be a periodic function and  $d/dt d/dt b(t) = d(t)$ . Thus, the wavefunctions and potentials which look time independent actually contain a time component. This, however, is consistent with (3).

## Conclusion

In conclusion, we try to present an approach for converting a time-independent Schrodinger equation into one which has the appearance of a time independent one with the variable  $x$  replaced with  $y$  where  $g(y) = x - b(t)$ . We then use  $W(y) = W_0(y)W_1(y)$ , where  $W_0(y)$  is the “ground state” solution, to obtain a DE for  $W_1(y)$ . We compare this to the general hypergeometric equation ((3)) and try to find potentials  $V(y)$  which are compatible, thus ensuring a Rodrigues polynomial solution

## References

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