International J.Math. Combin. Vol.1(2019), 53-61

Unique Metro Domination Number of Circulant Graphs

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Abstract: A dominating set D of G which is also a resolving set of G is called a *metro dominating set*. A metro dominating set D of a graph $G(V, E)$ as a *unique metro dominating set* (in short an UMD-set) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$ and the minimum cardinality of an UMD-set of G is the *unique metro domination number* of G. In this paper, we determine unique metro domination number of circulant graphs.

Key Words: Domination, Smarandachely k-dominating set, metric dimension, metro domination, uni-metro domination.

AMS(2010): 05C20, 05C26.

§1. Introduction

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d(u, v)$. For a vertex v of a graph, $N(v)$ denotes the set of all vertices adjacent to v and is called open neighborhood of v. Similarly, the closed neighborhood of v is defined as $N[v] = N(v) \cup \{v\}$.

Let $G(V, E)$ be a graph. For each ordered subset $S = \{v_1, v_2, \cdots, v_k\}$ of V, each vertex $v \in V$ can be associated by a vector of distances denoted by $\Gamma(v/S) = (d(s_1, v), d(s_2, v), \cdots, d(s_k, v))$. The set S is said to be a *resolving set* of G, if $\Gamma(v/S) \neq \Gamma(u/S)$, for every $u, v \in V - S$. A resolving set of minimum cardinality is the *metric basis* and cardinality of a metric basis is the *metric dimension* of G. The k-tuple, $\Gamma(v/S)$ associated to the vertex $v \in V$ with respect to a Metric basis S, is referred as a *code generated by* S for that vertex v. If $\Gamma(v/S) = \{v_1, v_2, \cdots, v_k\}$, then $v_1, v_2, \cdots, v_k$ are called components of the code of v generated by S and in particular v_i, $1 \leq i \leq k$, is called i^{th}-component of the code of v generated by S.

A *dominating set* D of a graph $G(V, E)$ is the subset of V having the property that for each vertex $v \in V - D$ there exists a vertex u in D such that $uv \in E$. Generally, a set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S with $k \geq 1$. Clearly, a dominating set is a Smarandachely 1-dominating

[1]Received April 22, 2018, Accepted February 24, 2019.

set. A dominating set D of G which is also a resolving set of G is called a *metro dominating set* or in short an $MD-set$. A metro dominating set D of a graph $G(V,E)$ as a *unique metro dominating set* (in short an UMD-set) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$ and the minimum cardinality of an UMD-set of G is the *unique metro domination number* of G, denoted by $\gamma_{u\beta}(G)$.

Metric dimensions and locating dominating sets of certain classes of graphs were studied in [1-14].

§2. Resolvability of Circulant Graphs

A graph whose vertex set is $\{v_i | i \in Z^+\}$ and two vertices v_i and v_j are adjacent if and only if $i - j \pmod{n} \in C$, for a given $C \subseteq Z_n$ with $0 \notin C$, is called a *circulant diagraph*. If the set C has the property that $C = -C$, then the underlying graph is called *circulant graph*, and we denote it by $X_{n,\Delta}$, where $|C| = \Delta$. The set C is referred to as a connected set. The circulant graph $X_{n,\Delta}$ is a Δ-regular graph. In this paper, we consider a family of circulant graph $X_{n,3}$ with connection set $C = \{1, \frac{n}{2}, n-1\}$, where n is even.

We state the following lemma whose proof follows directly by the definition of domination, and is most helpful to find UMD-sets.

Lemma 2.1 *In the circulant graph $X_{n,3}$, n is even, with connection set $C = \{1, \frac{n}{2}, n-1\}$, a vertex v_i dominates v_{i-1}, v_{i+1} and $v_{i+\frac{n}{2}}$, where $i + \frac{n}{2}$ is under modulo n.*

Now we consider $G = X_{n,3}$, n is even, where the connection set $C = \{1, \frac{n}{2}, n-1\}$. Let S be a dominating set of G. Then by Lemma 2.1, a vertex $v_i \in S$ can dominate at most 3 vertices in $V - S$. Hence $|V - S| \leqslant 3|S|$. Therefore,

$$|V| - |S| \leqslant 3|S| \Rightarrow 4|S| \geqslant |V| \Rightarrow |S| \geqslant \frac{n}{4}.$$

Thus we have the following lemma.

Lemma 2.2 *For any positive even integer n,*

$$\gamma(X_{n,3}) \geqslant \left\lceil \frac{n}{4} \right\rceil.$$

Let R be a set of two or more vertices of the principal cycle. Consider two distinct vertices u and v of R. Let P, P' be two distinct uv-path on the principal cycle. The vertices u and v are said to be neighboring vertices if u and v are the only vertices of S contained in one of the paths P, P'. If P (or P') is the path containing only u, v from S, then the set of all vertices of $P - \{u, v\}$ is called a gap of S determined by u and v and is denoted by γ. The number of vertices in the gap is called order of the gap and is denoted by $o(\gamma)$.

Notice that it is shown that $X_{26,3}$ with a unique metro dominating set $S = \{v_i, v_{i+4}, v_{i+8}, v_{i+15}, v_{i+19}, v_{i+23}, v_{i+24}, v_{i+25}\}$ in Figure 1. We observe that v_i, v_{i+4} are neighboring vertices of S and the gap determined by v_i and v_{i+4} is of order 3. Similarly, the gap determined by v_{i+8} and v_{i+15} is of order 6. The gap between v_{i+24} and v_{i+25} is of order 0 or empty gap.

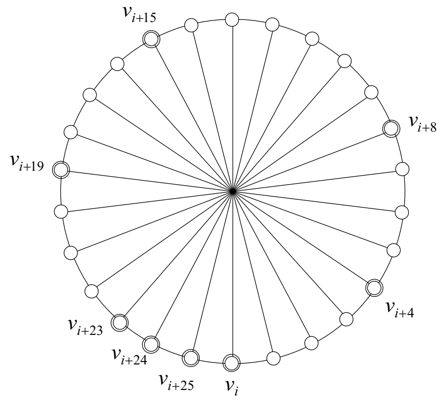

Figure 1 Circulant graph $X_{26,3}$ with $C = \{1, 13, 25\}$.

Consider $G = X_{n,3}$, n is even. Let S be a UMD-set of G. Suppose there is a gap of S of order 1. Let u, v be two neighboring vertices and w be the only vertex of the gap. Then w is dominated by both u and v and hence w is not uniquely dominated. Thus, we have

Lemma 2.1 *If γ is a gap of a UMD-set S of the graph $G = X_{n,3}$, then $0(\gamma) \neq 1$.*

In the discussion to follow, we want to find suitable gaps of a dominating set S, so that S becomes a UMD-set. Gaps of order 4 or more will introduce gaps of order 0 and thereby increase $|S|$; for, consider a gap γ of order 4 between the neighboring vertices v_i and v_{i+5}. Vertices in the gaps are $v_{i+1}, v_{i+2}, v_{i+3}$ and v_{i+4}. Vertices v_{i+1} and v_{i+4} are dominated by v_i and v_{i+5} respectively. It is therefore obvious that v_{i+2} and v_{i+3} should be dominated by $v_{i+2+\frac{n}{2}}$ and $v_{i+3+\frac{n}{2}}$. Thus, $v_{i+2+\frac{n}{2}}$ and $v_{i+3+\frac{n}{2}}$ belongs to S. The gap between them is empty. Hence $|S|$ is increased.

If all the gaps of S are of order 3, then $|S|$ is the least; for, if $v_i, v_{i+4} \in S$ and are neighboring vertices, then the gap determined by them is of order 3. As v_{i+1} and v_{i+3} are dominated by v_i and v_{i+4} respectively, the vertex v_{i+2} has to be dominated by $v_{i+2+\frac{n}{2}}$. Thus, $v_{i+2+\frac{n}{2}} \in S$.

Observe that $v_{i+\frac{n}{2}+4}$ is dominated by v_{i+4}. Therefore, we take $v_{i+\frac{n}{2}+6} \in S$, so that $v_{i+\frac{n}{2}+6}$ and $v_{i+\frac{n}{2}+2}$ are neighboring vertices of a gap of order 3. As $v_{i+\frac{n}{2}+6}$ dominates v_{i+6}, we include v_{i+8} in S so that v_{i+4} and v_{i+8} are neighboring vertices of gap of order 3. Thus, $S = \{v_i, v_{i+4}, v_{i+2+\frac{n}{2}}, v_{i+8}, \cdots\}$.

If the above set of vertices has $v_i + \frac{n}{2} - 2$ as the last vertex, then the above sequence of vertices in S terminates at v_{i-4}. In this case each vertex in S is uniquely dominating exactly 3 vertices in $V - S$. Thus by Lemma 2.1, $|S|$ is the least. This leads to the lemma.

Lemma 2.4 *A dominating set S of $X_{n,3}$ has a least $|S|$, when each gap of S is of order 3.*

Lemma 2.5 *If G is a graph of order n having a dominating set S such that every gap of S is of order 3, then $n \equiv 4(mod\ 8)$.*

Proof If every gap is of order 3, then $v_{i+\frac{n}{2}+6}$, $v_{i+\frac{n}{2}+10}$, $\cdots$, v_{i-4} are in S. Hence $\frac{n}{2}+4k+2 \equiv 0 \pmod{n} \Rightarrow n \equiv 4 \pmod{8}$. □

Note that when $n \equiv 4 \pmod 8$, there are exactly $\frac{n}{4}$ gaps of order 3 on the principal cycle. Also observe that $\frac{n}{4}$ is an odd integer. From these we conclude the following result.

Lemma 2.6 *When $n \equiv 0 (mod\ 8)$, $n > 8$, there is at least one gap of order less than 3.*

In a circulant graph $G = X_{n,3}$, let $v_i, v_{i+1}, \cdots, v_{i+(n-1)}, v_{i+n} = v_i$ (subscripts increase in anti-clockwise direction) form the principal cycle. Each vertex v_i, on the principal cycle, also lies on two other cycles:

(1) $v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}-1}, \ldots, v_{i+1}, v_i$ (Clockwise) and

(2) $v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}, \ldots, v_{i-1}, v_i$ (anti-clockwise).

Length of these cycles is $\frac{n}{2} + 1$. Hence maximum distance between any two vertices on these cycles is $\frac{1}{2}\left(\frac{n}{2}+1\right)$, if $\frac{n}{2}+1$ is even and is $\frac{1}{2}\left(\frac{n}{2}\right)$ if $\frac{n}{2}+1$ is odd. Thus we have

Lemma 2.7 *If x and y are any two vertices of $X_{n,3}$ then $d(x, y) = k \leqslant \lceil \frac{n}{4} \rceil$.*

Note that the subscripts of the vertex names are in Z_n, i.e. congruent modulo n.

Lemma 2.8 *If $n > 8$ and $n \equiv 0$ (mod 4), then for a fixed i, $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}\}$ is a resolving set.*

Proof For the cases, $n = 12, 16$ and 20, it is easy to see that the set S is a resolving set. We prove the case when $n > 20$. Suppose that x and y are vertices on the principal cycle such that $d(v_i, x) = d(v_i, y)$. Then there are four cases, in each case $k \leqslant \lceil \frac{n}{4} \rceil$.

Case 1. $x = v_{i-k}$ and $y = v_{i+k}$.

In this case we see that $d(v_{i+4}, y) = |k-4|$. If $k \leqslant \frac{n}{4}-4$, then $d(v_{i+4}, x) = k+4 \neq d(v_{i+4}, y)$; If $k > \frac{n}{4} - 4$, then $d(v_{i+4}, x) = \frac{n}{2} + 1 - (k+4) = \frac{n}{2} - k - 3 \neq d(v_{i+4}, y)$ for if $\frac{n}{2} - k - 3 = k - 4$, then $2k = \frac{n}{2} + 1 \Rightarrow k = \frac{n}{4} + \frac{1}{2}$, which is not possible as $n \equiv 0$ modulo 4. Thus v_{i+4} resolves x and y.

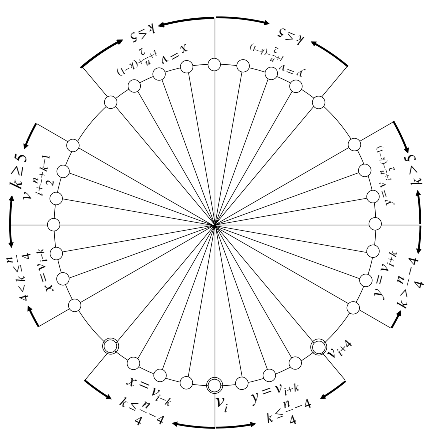

Figure 2. Circulant graph $X_{32,3}$.

Case 2. $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

If $k \leqslant 5$, then $d(v_{i+4}, x) = 6-k$ and $d(v_{i+4}, y) = 4+k$. Now $6-k = 4+k \Rightarrow 2k = 2 \Rightarrow k = 1$ which is not possible. Hence $d(v_{i+4}, x) \neq d(v_{i+4}, y)$; If $k > 5$, then $d(v_{i+4}, x) = k-4$, which is not equal to $d(v_{i+4}, y)$. Thus v_{i+4} resolves x and y.

Case 3. $x = v_{i-k}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

If $k \leqslant \lceil \frac{n}{4} \rceil - 4$, then we have $d(v_{i+4}, x) = 4+k$ and $d(v_{i+4}, y) = 4+k$. But $d(v_{i+\frac{n}{2}+2}, x) = 3+k \neq d(v_{i+\frac{n}{2}+2}, y) = k+1$; If $\frac{n}{4} - 4 < k \leqslant \frac{n}{4}$ then $d(v_{i+4}, x) = \frac{n}{2} - k - 3 = d(v_{i+4}, y)$ and $d(v_{i+\frac{n}{2}+2}, x) \neq d(v_{i+\frac{n}{2}+2}, y)$. Hence in all the cases, $v_{i+\frac{n}{2}+2}$ resolves x and y.

Case 4. If $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+k}$ then $d(v_{i+4}, y) = |k-4|$.

If $k < 5$, then $d(v_{i+4}, x) = 6-k \neq d(v_{i+4}, y)$; If $k \geqslant 5$, then $d(v_{i+4}, x) = k-4 = d(v_{i+4}, y)$. However, $d(v_{i+\frac{n}{2}+2}, x) = k-3 \neq d(v_{i+\frac{n}{2}+2}, y) = k-1$. Hence $v_{i+\frac{n}{2}+2}$ resolves x and y.

Note that the Theorem 1 of Muhammad Salman et al in [6], states that for all $n \geqslant 4$ and $n \equiv 0 \pmod 4$, the metric dimension of $X_{n,3} > 2$. Hence $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}\}$ is a resolving set. Hence the lemma. □

Lemma 2.9 *For any integer $n > 8$ and $n \equiv 2 (mod\ 4)$, the set $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}, v_{i+\frac{n}{2}+6}\}$ is a resolving set.*

Proof The cases where $n \leqslant 22$ follows easily. We now suppose $n > 22$ and, x and y are two vertices of G such that $d(v_i, x) = d(v_i, y)$. Then there are four cases in each case $k \leqslant \frac{n+2}{4}$.

Case 1. $x = v_{i-k}$ and $y = v_{i+k}$.

In this case $d(v_{i+4}, y) = |k-4|$. If $k \leqslant \frac{n+2}{4} - 4$, then $d(v_{i+4}, x) = k+4 \neq d(v_{i+4}, y)$; If $k > \frac{n+2}{4} - 4$, then $d(v_{i+4}, x) = \frac{n}{2} - k - 3$. Now $d(v_{i+4}, y) = d(v_{i+4}, x) \Rightarrow k-4 = \frac{n}{2} - k - 3 \Rightarrow k = \frac{n+2}{4}$. If $k = \frac{n+2}{4}$, $d(v_{i+\frac{n}{2}+2}, x) = k-3 \neq d(v_{i+\frac{n}{2}+2}, y) = k-1$.

Case 2. $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

If $k \leqslant 5$, $d(v_{i+4}, x) = 6-k \neq d(v_{i+4}, y) = k+4$; except when $k = 1$. But when $k = 1$, x and y coincide. Now if $5 < k < \frac{n+2}{4}$, then $d(v_{i+4}, y) = k-4 \neq d(v_{i+4}, x) = k+4$.

Case 3. $x = v_{i-k}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

When $k \leqslant \frac{n+2}{4} - 4$, $d(v_{i+4}, x) = 4+k$ and $d(v_{i+4}, y) = 4+k$. But $d(v_{i+\frac{n}{2}+2}, x) = 3+k \neq d(v_{i+\frac{n}{2}+2}, y) = k+1$, and when $\frac{n+2}{4} - 4 < k \leqslant \frac{n+2}{4}$, $d(v_{i+4}, x) = \frac{n}{2} - k - 3$ and $d(v_{i+4}, y) = \frac{n}{2} - k - 3$. However $d(v_{i+\frac{n}{2}+2}, x) = \frac{n}{2} - k - 2$ and $d(v_{i+\frac{n}{2}+2}, y) = k+1$.

Now $\frac{n}{2} - k - 2 = k+1 \Rightarrow 2k = \frac{n}{2} - 3 \Rightarrow k = \frac{n-6}{4}$.

If $k \neq \frac{n-6}{4}$, then $v_{i+\frac{n}{2}+2}$ resolves x and y; If $k = \frac{n-6}{4}$, then $d(v_{i+\frac{n}{2}+6}, x) = \frac{n}{2} - k - 6$ and $d(v_{i+\frac{n}{2}+6}, y) = \frac{n}{2} - k - 4$, which are not equal. Hence $v_{i+\frac{n}{2}+6}$ resolves x and y.

Case 4. $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+k}$.

If $k \leqslant 5$, then $d(v_{i+4}, x) = 6-k \neq d(v_{i-4}, y) = |k-4|$; If $5 < k < \frac{n+2}{4}$, then $d(v_{i+4}, x) = k-4 = d(v_{i+4}, y)$. But $d(v_{i+\frac{n}{2}+2}, x) = k-3 \neq d(v_{i+\frac{n}{2}+2}, y) = k-1$. Thus $v_{i+\frac{n}{2}+2}$ resolves x and y.

Now by the Theorem 2 of Muhammad Salman et al [6], we have for all $n \geqslant 6$ and $n \equiv 2 \pmod 6$, the metric dimension of the graph $X_{n,3} > 3$.

Hence the set $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}, v_{i+\frac{n}{2}+6}\}$ as per the above four cases becomes a resolving set. Hence the lemma. □

§4. Algorithm to Extend Circulant Graphs and Resolving Sets

We give an algorithm, which constructs new circulant graph from the old one by increasing its order and extending the its resolving set to suit for the newly constructed one (as in Figure 3).

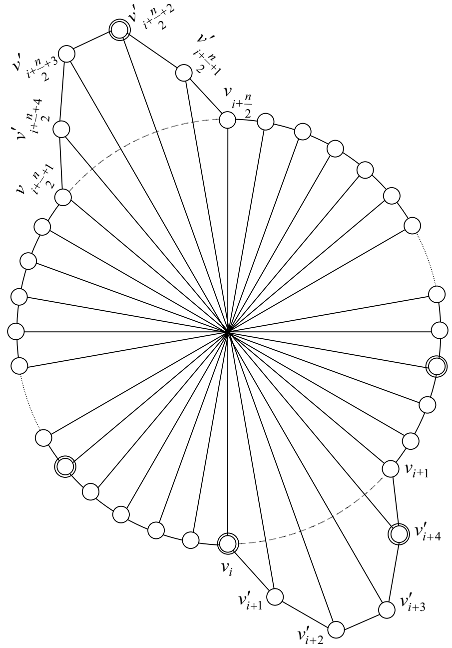

Figure 3. Algorithmic construction of $X_{n+8,3}$ from $X_{n,3}$.

Input: The graph $X_{n,3}$ and a metric basis S with $|S| = k$.

Step 1: Select two neighboring vertices on the principal cycle with a gap of 3. Let v_i and v_{i+4} be the vertices. Then v_i, v_{i+4} and $v_{i+\frac{n}{2}+2} \in S$.

Step 2: Delete the edge v_iv_{i+1}. Add four vertices v'_{i+1}, v'_{i+2}, v'_{i+3} and v'_{i+4} between the vertices v_i and v_{i+1}. Join the vertices to get the edges $v_iv'_{i+1}$, $v'_{i+1}v'_{i+2}$, $v'_{i+2}v'_{i+3}$, $v'_{i+3}v'_{i+4}$ and $v'_{i+4}v_{i+1}$.

Step 3: Delete the edge $v_{i+\frac{n}{2}}v_{i+\frac{n}{2}+1}$. Add four vertices vertices $v'_{i+\frac{n}{2}+1}$, $v'_{i+\frac{n}{2}+2}$, $v'_{i+\frac{n}{2}+3}$ and $v'_{i+\frac{n}{2}+4}$ between the vertices $v_{i\frac{n}{2}}$ and $v_{i+\frac{n}{2}+1}$. Join these vertices to get the edges $v_{i\frac{n}{2}}v'_{i+\frac{n}{2}+1}$, $v'_{i+\frac{n}{2}+1}v'_{i+\frac{n}{2}+2}$, $v'_{i+\frac{n}{2}+2}v'_{i+\frac{n}{2}+3}$, $v'_{i+\frac{n}{2}+3}v'_{i+\frac{n}{2}+4}$ and $v'_{i+\frac{n}{2}+4}v_{i+\frac{n}{2}+1}$.

Step 4: Join these 8 vertices to get the edges, $v_{i+1}v'_{i+\frac{n}{2}+1}$, $v_{i+2}v'_{i+\frac{n}{2}+2}$, $v_{i+3}v'_{i+\frac{n}{2}+3}$ and $v_{i+4}v'_{i+\frac{n}{2}+4}$.

Step 5: Add v'_{i+4} and $v_{i+\frac{n}{2}+2}$ into S.

Output: The graph $X_{n+8,3}$ and a metric basis S with $|S| = k+2$

§4. Unique Metro Domination Number of $X_{n,3}$

In this section we completely determine unique metro domination number of 3-regular circulant graphs in the form of following sequence of theorems.

Theorem 4.1 *If $n > 8$ is an even integer and $n \equiv 6$ or 4 (mod 8), then $\gamma_{\mu\beta}(X_{n,3}) = \lceil \frac{n}{4} \rceil$.*

Proof We consider two cases separately following.

Case 1. $n \equiv 4 \pmod 8$ and $n > 8$.

We first take the smallest possible n, that is $n = 12$. Define $S = \{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}\}$ for any i, $1 \leqslant i \leqslant 12$. In view of Lemma 2.8, the set S so defined is a resolving set. Further, each gap is of order 3. Hence S is uniquely dominates $V - S$. So, by Lemma 2.2, $\gamma_{\mu\beta}(X_{12,3}) \geqslant \lceil \frac{n}{4} \rceil$ and here $|S| = 3$. Thus, S is a UMD-set. Therefore, $\gamma_{\mu\beta}(X_{12,3}) = \lceil \frac{n}{4} \rceil$, when $n = 12$.

Now we apply the algorithm to construct the next cases for n. At each time algorithm increases the order by 8 and $|S|$ by 2. Hence $\gamma_{\mu\beta}(X_{n+8,3}) = \lceil \frac{n}{4} \rceil + 2 = \lceil \frac{n+8}{4} \rceil$.

Case 2. $n \equiv 6 \pmod 8$ and $n > 8$.

For the least possible $n = 14$, by Lemma 2.8, it follows that the set $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}, v_{i+\frac{n}{2}+6}\}$ resolves $V - D$. Now from Lemma 2.2, $\gamma_{\mu\beta}(X_{n,3}) \geqslant \lceil \frac{n}{4} \rceil = 4$. There is a gap of order 0 in S. The middle vertex v_k of a gap of order 3 is dominated by $v_{k+\frac{n}{2}} \in S$. Hence S is a UMD-set.

We apply the algorithm on $X_{14,3}$. It increases $n = 14$ to $n = 22$. It also increases $|S|$ by 2. Hence $\gamma_{\mu\beta}(X_{22,3}) = \lceil \frac{n}{4} \rceil$. Repeated application of the algorithm gives the theorem. □

Theorem 4.2 *If n is any even integer, $n > 8$ and $n \equiv 0$ or 2 (mod 8), then $\gamma_{\mu\beta}(X_{n,3}) = \lceil \frac{n}{4} \rceil + 1$.*

Proof We prove the theorem in two different cases as follows:

Case 1. $n \equiv 0 \pmod 8$.

The graph $X_{16,3}$ is the graph of least possible order in this case. Invoking Lemma 2.4, consider a gap of order 3, having v_i and v_{i+4} as the neighboring vertices. Then v_i, v_{i+4}, and $v_{i+\frac{n}{2}+2} \in S$.

This leads to a gap of order 5 between v_{i+4} and $v_{i+\frac{n}{2}+2}$ in which v_{i+6}, v_{i+7} are not dominated and a gap of order 5 between $v_{i+\frac{n}{2}+2} = v_{i+10}$ and v_i in which v_{i-2} and v_{i-3} are not dominated. As $\lceil \frac{n}{4} \rceil = 4$, S contains a minimum of 4 vertices by Lemma 2.2. If $v_{i+6} \in S$, then v_{i+5} is not uniquely dominated. If $v_{i+7} \in S$, then v_{i-1} is not uniquely dominated and if $v_{i-3} \in S$, then v_{i+5} is not uniquely dominated. Hence none of $v_{i+6}, v_{i+7}, v_{i-2}, v_{i-3}$ can be included in S. If $v_{i-1} \in S$, then v_{i-2} and $v_{i+7} \in V - S$ are uniquely dominated. Similarly if $v_{i+5} \in S$, then v_{i+6} and v_{i-3} in $V - S$ are uniquely dominated. However $|S| = 5$ and is not possible to reduce it to 4. (Because each gap of order 5 can be converted to two gaps of order 2, But it will have $|S| = 5$).

Hence $\gamma_{\mu\beta}(X_{n,3}) = 5$. Application of algorithm now will increase the order by 8 and $|S|$ increases by 2. Therefore $\gamma_{\mu\beta}(X_{n,3}) = \lceil \frac{n}{4} \rceil + 1$.

Case 2. $n \equiv 2 \pmod 8$.

As in Case 1, we take v_i, v_{i+4} and $v_{i+\frac{n}{2}+2} \in S$. If $v_{i+\frac{n}{2}+6} \in S$, then it leaves a gap of order 2 between v_i and v_{i-3} and a gap of order 6 between v_{i+4} and $v_{i+\frac{n}{2}+2}$. In this gap of order 6, the vertices $v_{i+5}, v_{i+6}, v_{i+9}$ and v_{i+10} are uniquely dominated. If $v_{i+7} \in S$, then v_{i-2} and v_{i+6} are not uniquely dominated. If $v_{i+8} \in S$, then v_{i-1} and v_{i+9} are not uniquely dominated. If we include v_{i-2} and v_{i-1} in S, then the domination of all vertices in $V - S$ is unique. However $|S| = 6$. It can not be reduced to $5 = \lceil \frac{n}{4} \rceil$. If the gap of order 6 between v_{i+4} and v_{i+11} is converted into 2 gaps by including $v_{i+8} \in S$, then v_{i-1} is not uniquely dominated. Similarly including v_{i+4} fails. Hence $\gamma_{\mu\beta}(X_{n,3}) = 6$. As before we apply algorithm to conclude the rest of the theorem. $\square$

Acknowledgment The authors are thankful to the learned referees for their valuable suggestions for the improvement of the paper.

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