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New logarithmic operational laws and their applications to multiattribute decision making for single-valued neutrosophic numbers

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Abstract

Neutrosophic set, initiated by Smarandache, is a novel tool to deal with vagueness considering the truth, indeterminacy and falsity memberships satisfying the condition that their sum is less than 3. This set can be used to characterize the information more accurately than the intuitionistic fuzzy set. Under this set, the objective of this manuscript is to present some new operational laws called as logarithm operational laws with real number base λ for the single-valued neutrosophic (SVN) numbers. Various desirable properties of the proposed operational laws are contemplated. Further, based on these laws, different weighted averaging and geometric aggregation operators are developed. The properties such as idempotency, monotonicity, boundedness are provided to support the proposed operators. Then, we utilized these operations and operators to present a multiattribute decision making method to solve the decision-making problems. A real numerical example is given to demonstrate the approach under SVN environment. The legitimacy of the proposed strategy is exhibited with a numerical illustration and compared the results with the several existing approaches result.

Keywords: Single valued neutrosophic numbers; Logarithm operations; Aggregation operators; Decision making approaches

1. Introduction

Multiattribute decision making (MADM) methods is one of the cognitive-based human activity to rank a finite set of alternatives based on existing decision making information. Traditionally, researchers are expressed the alternatives preference in terms of crisp numbers; however, these properties have not been observed. Thus, to handle the uncertainty in the data, the theory of fuzzy sets (FSs) (Zadeh, 1965) and its extensions such as intuitionistic fuzzy sets (IFSs) (Atanassov, 1986), interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov & Gargov, 1989) are widely used by the researchers to solve the MADM problems. Over the last decades, several researchers have presented different types of the operational laws under these theories. For instances, Atanassov (1999) defined the basic operations such as ‘union’, ‘intersection’, ‘power’, and so on. Xu and Yager (2006) defined some basic operational laws such as ‘addition’, ‘subtraction’, ‘scalar multiplication’ for different intuitionistic fuzzy numbers (IFNs). Lei and Xu (2015) defined the subtraction and division operations for IFNs. Garg and Ansha (2018), Garg (2018b) developed some basic arithmetic operations on generalized parabolic and sigmoidal fuzzy numbers respectively. Li and Wei (2017) presented some logarithm operational laws of IFSs. Garg (2018c) presented logarithm operations laws to the Pythagorean fuzzy sets. Based on these laws, researchers have developed some aggregation operators (AOs) to solve

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MADM problems. For instance, Xu (2007) developed some weighted averaging AOs under IFSs environment. Garg (2017), Garg (2016) developed some generalized interaction geometric AOs using Einstein norm operations. Kaur and Garg (2018) presented cubic intuitionistic fuzzy aggregation operators. Apart from these, other kinds of AOs are developed by the authors and are summarized in Wang and Triantaphyllou (2008), Kumar and Garg (2018), De, Biswas, and Roy (2000), Garg and Singh (2018), Abdel-Basset, Mohamed, and Smarandache (2018), Abdel-Basset, Mohamed, Zhou, and Hezam (2017), Abdel-Basset, Zhou, Mohamed, and Chang (2018), Garg (2018a), and Rani and Garg (2018) to solve MADM problems. But from these studies, it has been analyzed that they are unable to considered the indeterminate and inconsistent data.

To resolve this, Smarandache (1998) presented a new component named as "indeterminacy-membership function" along with "truth membership function" and "falsity membership function", all which are independent of each other and lying in $]0^-, 1^+[$, and the corresponding set is known as a neutrosophic set (NS). Smarandache (1998), Wang, Smarandache, Zhang, and Sunderraman (2010) presented the concept of a single-valued neutrosophic (SVN) set (SVNS). Wang et al. (2010) introduced some basic operations of SVNSs. Smarandache (2016) defined the subtraction and division operators of SVN numbers (SVNNs). Peng, Wang, Wang, Zhang, and Chen (2016) introduced some basic operational laws such as "addition", "multiplication", "scalar multiplication" and hence presented an AO based on these laws. Nancy and Garg (2016a) presented some improved score function to rank the different SVNSs. Later on, some different kinds of the AOs have been proposed by the authors using algebraic norm (Ye, 2014), Hamacher norm (Liu, Chu, Li, & Chen, 2014), Frank norm (Nancy & Garg, 2016b), hybrid operator (Garg & Nancy, 2018a). Ye (2016) presented an exponential operational law and the aggregation operators. Garg and Nancy (2018b) presented a TOPSIS method under an interval NS environment to solve MADM problems. Garg and Nancy (2018c) presented some prioritized aggregation operators under the linguistic SVNS environment. Aside from these, various authors incorporated the idea of NS theory into the different fields (Biswas, Pramanik, & Giri, 2016; Broumi & Smarandache, 2014; Garg & Nancy, 2016, 2017, 2018d; Jha et al., 2018; Li, Liu, & Chen, 2016; Liu & Wang, 2014; Peng & Liu, 2017; Peng & Dai, 2018a, 2018b; Smarandache, 2018).

It is well known that during the aggregation process, the most important process is to define the operational laws. But from the existing literature, it is observed that most of the existing aggregation operators are based on the assumption that weight is a crisp number within [0,1]. However, Ye (2016) introduced the exponential operational laws as a supplement of operational laws of SVNSs, where the bases are the real numbers and the exponents are SVNSs. With the growing sound of the SVNS both in depth and scope, different kinds of some new operational laws and the aggregation methods are needed. As a kind of important mathematical operation, the logarithmic operational law of SVNSs is necessary to be developed in the field of the aggregation process. By taking the advantages of SVNS and in order to consummate the logarithmic operational laws under the SVNS circumstances, we define the logarithmic operational law (LOL) of SVNSs and SVNNs, in which the logarithm base λ is taken as a positive real number. Also, some properties of LOL are discussed. Furthermore, in the field of the aggregation process, to aggregate the different value into a single one, the weighted averaging and geometric operators are developed with the help of LOLs under SVNS environment. These operators are named as logarithm single-valued neutrosophic (L-SVN) weighted average (L-SVNWA), L-SVN weighted geometric (L-SVNWG), L-SVN ordered weighted average (L-SVNOWA) and L-SVN ordered weighted geometric (L-SVNOWG) which are in the general form and depends on each λ. For instance, the basic averaging and geometric operators are the special case of the proposed operators. Various prominent characteristics of these operators are discussed in details. Then, we utilized these operations and operators to develop multiattribute decision making approach. At last, we influence the selection of the logarithm base and the logarithm operations for SVNNs in practice. Since proposed operators has different forms through choosing different values of λ. Therefore, the decision maker's can obtain different decision results by using proposed aggregation operators, which greatly enhances the flexibility and agility of decision making method.

The remainder of this paper is set out as follows: Section 2 gives some basic knowledge and operations on SVNNs. Section 3 defined the LOL for SVNNs and the aggregation operators based on these laws. Section 4 proposes the decision making approach for solving the multiattribute decision making problems with SVNN information. The applicability of the proposed work has been demonstrated through an illustrated example in Section 5. The paper ends with some conclusions in Section 6.

2. Basic concepts

In this section, some basic definitions related to NS, SVNS on the universal set X are discussed.

Definition 2.1. (Smarandache, 1998) A neutrosophic set (NS) β on X is defined as

$$\beta = \{\langle x, \zeta_\beta(x), \kappa_\beta(x), \varphi_\beta(x) | x \in X\rangle\} \quad (1)$$

where $\zeta_\beta(x), \kappa_\beta(x), \varphi_\beta(x) \in]0^-, 1^+[$ such that $0^- \leqslant \zeta_\beta(x) + \kappa_\beta(x) + \varphi_\beta(x) \leqslant 3^+$.

Definition 2.2. (Wang et al., 2010) A SVNS β in X is stated as

$$\beta = \{\langle x, \zeta_\beta(x), \kappa_\beta(x), \varphi_\beta(x) | x \in X \rangle\} \tag{2}$$

where $\zeta_\beta, \kappa_\beta, \varphi_\beta : X \to [0,1]$ such that $0 \leqslant \zeta_\beta(x) + \kappa_\beta(x) + \varphi_\beta(x) \leqslant 3$. We denote this pair as $\beta = \langle \zeta_\beta, \kappa_\beta, \varphi_\beta \rangle$, throughout this article, and called as SVNN.

Definition 2.3. (Peng et al., 2016; Wang et al., 2010; Ye, 2016) Let $\beta = \langle \zeta, \kappa, \varphi \rangle, \beta_1 = \langle \zeta_1, \kappa_1, \varphi_1 \rangle$ and $\beta_2 = \langle \zeta_2, \kappa_2, \varphi_2 \rangle$ be three SVNNs, then

(i) $\beta^c = \langle \varphi, \kappa, \zeta \rangle$;
(ii) $\beta_1 \leqslant \beta_2$ if $\zeta_1 \leqslant \zeta_2, \kappa_1 \geqslant \kappa_2$ and $\varphi_1 \geqslant \varphi_2$;
(iii) $\beta_1 = \beta_2$ if $\beta_1 \leqslant \beta_2$ and $\beta_2 \leqslant \beta_1$;
(iv) $\beta_1 \cap \beta_2 = \langle \min(\zeta_1, \zeta_2), \max(\kappa_1, \kappa_2), \max(\varphi_1, \varphi_2) \rangle$;
(v) $\beta_1 \cup \beta_2 = \langle \max(\zeta_1, \zeta_2), \min(\kappa_1, \kappa_2), \min(\varphi_1, \varphi_2) \rangle$;
(vi) $\beta_1 \oplus \beta_2 = \langle \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \kappa_1\kappa_2, \varphi_1\varphi_2 \rangle$;
(vii) $\beta_1 \otimes \beta_2 = \langle \zeta_1\zeta_2, \kappa_1 + \kappa_2 - \kappa_1\kappa_2, \varphi_1 + \varphi_2 - \varphi_1\varphi_2 \rangle$;
(viii) $\lambda\beta_1 = \langle 1 - (1-\zeta_1)^\lambda, \kappa_1^\lambda, \varphi_1^\lambda \rangle; \lambda > 0$;
(ix) $\beta_1^\lambda = \langle \zeta_1^\lambda, 1 - (1-\kappa_1)^\lambda, 1 - (1-\varphi_1)^\lambda \rangle; \lambda > 0$;
(x) $\lambda^\beta = \begin{cases} \langle \lambda^{1-\zeta}, 1 - \lambda^\kappa, 1 - \lambda^\varphi \rangle & \text{if } \lambda \in (0,1) \\ \langle (1/\lambda)^{1-\zeta}, 1 - (1/\lambda)^\kappa, 1 - (1/\lambda)^\varphi \rangle & \text{if } \lambda \geqslant 1 \end{cases}$

Definition 2.4. (Wang et al., 2010) An order relation, based on score function (S) and accuracy function (H), between two SVNNs β and γ is stated as, if $S(\beta) > S(\gamma)$ then $\beta > \gamma$ and if $S(\beta) = S(\gamma)$ and $H(\beta) > H(\gamma)$ then $\beta > \gamma$, if $H(\beta) = H(\gamma)$ then $\beta = \gamma$, where $S(\beta) = \zeta_\beta - \kappa_\beta - \varphi_\beta$ and $H(\beta) = \zeta_\beta + \kappa_\beta + \varphi_\beta$.

Definition 2.5. (Peng et al., 2016) If $\beta_j = \langle \zeta_j, \kappa_j, \varphi_j \rangle$ $(j = 1, 2, \ldots, n)$ be n SVNNs having weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$, then the weighted averaging and geometric aggregation operators which are defined as

(a) SVNWA and SVNOWA operators

$$\text{SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) = \left\langle 1 - \prod_{j=1}^n (1-\zeta_j)^{\omega_j}, \prod_{j=1}^n (\kappa_j)^{\omega_j}, \prod_{j=1}^n (\varphi_j)^{\omega_j} \right\rangle \tag{3}$$

$$\text{SVNOWA}(\beta_1, \beta_2, \ldots, \beta_n) = \left\langle 1 - \prod_{j=1}^n (1-\zeta_{\sigma(j)})^{\omega_j}, \prod_{j=1}^n (\kappa_{\sigma(j)})^{\omega_j}, \prod_{j=1}^n (\varphi_{\sigma(j)})^{\omega_j} \right\rangle \tag{4}$$

(b) SVNWG and SVNOWG operators

$$\text{SVNWG}(\beta_1, \beta_2, \ldots, \beta_n) = \left\langle \prod_{j=1}^n (\zeta_j)^{\omega_j}, 1 - \prod_{j=1}^n (1-\kappa_j)^{\omega_j}, 1 - \prod_{j=1}^n (1-\varphi_j)^{\omega_j} \right\rangle \tag{5}$$

$$\text{SVNOWG}(\beta_1, \beta_2, \ldots, \beta_n) = \left\langle \prod_{j=1}^n (\zeta_{\sigma(j)})^{\omega_j}, 1 - \prod_{j=1}^n (1-\kappa_{\sigma(j)})^{\omega_j}, 1 - \prod_{j=1}^n (1-\varphi_{\sigma(j)})^{\omega_j} \right\rangle \tag{6}$$

where σ is a permutation of $(1, 2, \ldots, n)$ such that $\beta_{\sigma(j-1)} \geqslant \beta_{\sigma(j)}$ for $j = 2, \ldots, n$.

3. Logarithmic operational laws and its based aggregation operators of SVNSs

In this section, we have introduced some new logarithmic operational laws (LOL) for the SVNNs.

3.1. Logarithmic Operational laws

Let β be SVNNs and $\lambda > 0$ be a real number. Since $\log_\lambda 0$ and $\log_1 x$ is not defined in real numbers, so we assume that $\beta \neq \mathbf{0}$ where $\mathbf{0}$ is the zero SVNN, $\beta \neq \langle 0, 1, 1 \rangle$ and $\lambda \neq 1$ throughout the study.

Definition 3.1. Let X be the non-empty fixed set and $A = \{\langle x, \zeta_A(x), \kappa_A(x), \varphi_A(x) \rangle \mid x \in X\}$ be SVNS, then we can define a logarithm operational laws of SVNS A as follows:

$$\log_\lambda A = \{\langle x, 1 - \log_\lambda \zeta_A(x), \log_\lambda(1 - \kappa_A(x)), \log_\lambda(1 - \varphi_A(x)) \rangle \mid x \in X\} \tag{7}$$

where $0 < \lambda \leqslant \min\{\zeta_A, 1 - \kappa_A, 1 - \varphi_A\} \leqslant 1, \lambda \neq 1$. It is clearly seen that the $\log_\lambda A$ is also SVNS. As it is clear from the definition of SVNS, for all $x \in X$, the functions ζ_A, κ_A and φ_A satisfy:

$$\zeta_A : X \to (0,1], \kappa_A : X \to [0,1), \varphi_A : X \to [0,1)$$

and $0 \leqslant \zeta_A(x) + \kappa_A(x) + \varphi_A(x) \leqslant 3$. If $0 < \lambda \leqslant \min\{\zeta_A, 1 - \kappa_A, 1 - \varphi_A\} \leqslant 1$ and $\lambda \neq 1$, then the membership function:

$$1 - \log_\lambda \zeta_A : X \to [0,1],\ \forall\, x \in X \to 1 - \log_\lambda \zeta_A(x) \in [0,1],$$

the indeterminacy function

$$\log_\lambda(1 - \kappa_A) : X \to [0,1], \forall x \in X \to \log_\lambda(1 - \kappa_A(x)) \in [0,1],$$

and the non-membership function:

$$\log_\lambda(1 - \varphi_A) : X \to [0,1], \forall x \in X \to \log_\lambda(1 - \varphi_A(x)) \in [0,1],$$

Therefore,

$$\log_\lambda A = \{\langle x, 1 - \log_\lambda \zeta_A(x), \log_\lambda(1 - \kappa_A(x)), \log_\lambda(1 - \varphi_A(x)) \rangle \mid x \in X\},$$

where $0 < \lambda \leqslant \min\{\zeta_A, 1 - \kappa_A, 1 - \varphi_A\} \leqslant 1, \lambda \neq 1$ is SVNS.

Definition 3.2. Let $\beta = \langle \zeta, \kappa, \varphi \rangle$ be SVNN. If

$$\log_\lambda \beta = \begin{cases} \langle 1 - \log_\lambda \zeta, \log_\lambda(1-\kappa), \log_\lambda(1-\varphi) \rangle; & 0 < \lambda \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} < 1 \\ \left\langle 1 - \log_{\frac{1}{\lambda}} \zeta, \log_{\frac{1}{\lambda}}(1-\kappa), \log_{\frac{1}{\lambda}}(1-\varphi) \right\rangle; & 0 < \frac{1}{\lambda} \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} < 1, \lambda \neq 1 \end{cases} \tag{8}$$

then the function $\log_\lambda \beta$ is called a logarithmic operator, and the value $\log_\lambda \beta$ is called Logarithmic SVNN (L-SVNN). Here, we take $\log_\lambda 0 = 0, \lambda > 0, \lambda \neq 1$.

Theorem 3.1. *For SVNN β, the value of operator $\log_\lambda \beta$ is SVNN.*

Proof. Let SVNN $\beta = \langle \zeta, \kappa, \varphi \rangle$ satisfies $0 < \zeta \leqslant 1, 0 \leqslant \kappa < 1, 0 \leqslant \varphi < 1$ and $\zeta + \kappa + \varphi \leqslant 3$. The, following two cases happens.

Case 1: When $0 < \lambda \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} < 1, \lambda \neq 1$. Thus, $0 \leqslant \log_\lambda \zeta, \log_\lambda(1-\kappa), \log_\lambda(1-\varphi) \leqslant 1$ and hence $0 \leqslant 1 - \log_\lambda \zeta \leqslant 1, 0 \leqslant \log_\lambda(1-\kappa) \leqslant 1, 0 \leqslant \log_\lambda(1-\varphi) \leqslant 1$ and $0 \leqslant 1 - \log_\lambda \zeta + \log_\lambda(1-\kappa) + \log_\lambda(1-\varphi) \leqslant 3$. Therefore, $\log_\lambda \beta$ is SVNN.

Case 2: When $\lambda > 1$ and $0 < \frac{1}{\lambda} < 1$ and $\frac{1}{\lambda} \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\}$, so it is easy to obtain that $\log_\lambda \beta$ is SVNN.

Hence, the operator $\log_\lambda \beta$ is SVNN. □

Example 3.1. Let $\beta = \langle 0.6, 0.4, 0.5 \rangle$ be SVNN, $\lambda = 0.3$, then

$$\log_\lambda \beta = \langle 1 - \log_{0.3}(0.6), \log_{0.3}(0.6), \log_{0.3}(0.5) \rangle$$
$$= \langle 0.5757, 0.4243, 0.5757 \rangle$$

If $\lambda = 3$ then,

$$\log_{\frac{1}{\lambda}} \beta = \log_{\frac{1}{3}} \langle 0.6, 0.4, 0.5 \rangle$$
$$= \langle 1 - \log_{\frac{1}{3}}(0.6), \log_{\frac{1}{3}}(0.6), \log_{\frac{1}{3}}(0.5) \rangle$$
$$= \langle 0.5350, 0.4650, 0.6309 \rangle$$

Next, we discuss some basic properties of L-SVNN $\log_\lambda \beta$ based on LOL by taking $\lambda \in (0,1)$, while for $\lambda > 1$ it can be obtained analogously.

Theorem 3.2. *Let $\beta = \langle \zeta, \kappa, \varphi \rangle$ be SVNN. If $0 < \lambda \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} \leqslant 1, \lambda \neq 1$ then*

(i) $\lambda^{\log_\lambda \beta} = \beta$;
(ii) $\log_\lambda \lambda^\beta = \beta$.

Proof.

(i) According to the Definitions 2.3 and 3.2, we have

$$\lambda^{\log_\lambda \beta} = \langle \lambda^{1-(1-\log_\lambda \zeta)}, 1 - \lambda^{\log_\lambda(1-\kappa)}, 1 - \lambda^{\log_\lambda(1-\varphi)} \rangle$$
$$= \langle \lambda^{\log_\lambda \zeta}, 1 - (1-\kappa), 1 - (1-\varphi) \rangle = \langle \zeta, \kappa, \varphi \rangle = \beta$$

(ii) From Definition 3.2, we have

$$\log_\lambda \lambda^\beta = \log_\lambda \left(\lambda^{1-\zeta}, 1 - \lambda^\kappa, 1 - \lambda^\varphi \right)$$
$$= \langle 1 - \log_\lambda \lambda^{1-\zeta}, \log_\lambda(1-(1-\lambda^\kappa)), \log_\lambda(1-(1-\lambda^\varphi)) \rangle$$
$$= \langle \zeta, \kappa, \varphi \rangle = \beta \quad \square$$

Theorem 3.3. *Let $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle (i = 1, 2)$, be two SVNNs, $0 < \lambda \leqslant \min_i\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1$ and $\lambda \neq 1$. Then,*

(i) $\log_\lambda \beta_1 \oplus \log_\lambda \beta_2 = \log_\lambda \beta_2 \oplus \log_\lambda \beta_1$,
(ii) $\log_\lambda \beta_1 \otimes \log_\lambda \beta_2 = \log_\lambda \beta_2 \otimes \log_\lambda \beta_1$.

Theorem 3.4. *Let $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle (i = 1, 2, 3)$ be three SVNNs, $0 < \lambda \leqslant \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1$ and $\lambda \neq 1$. Then,*

(i) $(\log_\lambda \beta_1 \oplus \log_\lambda \beta_2) \oplus \log_\lambda \beta_3 = \log_\lambda \beta_1 \oplus (\log_\lambda \beta_2 \oplus \log_\lambda \beta_3)$,
(ii) $(\log_\lambda \beta_1 \otimes \log_\lambda \beta_2) \otimes \log_\lambda \beta_3 = \log_\lambda \beta_1 \otimes (\log_\lambda \beta_2 \otimes \log_\lambda \beta_3)$.

Proof. The proof is trial. □

Theorem 3.5. *Let $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle (i = 1, 2)$ be two SVNNs, $0 < \lambda \leqslant \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda \neq 1$ and $k, k_1, k_2 > 0$ be three real numbers. Then,*

(i) $k(\log_\lambda \beta_1 \oplus \log_\lambda \beta_2) = k\log_\lambda \beta_1 \oplus k\log_\lambda \beta_2$,
(ii) $(\log_\lambda \beta_1 \otimes \log_\lambda \beta_2)^k = (\log_\lambda \beta_1)^k \otimes (\log_\lambda \beta_2)^k$,
(iii) $k_1\log_\lambda \beta_1 \oplus k_2\log_\lambda \beta_1 = (k_1 + k_2)\log_\lambda \beta_1$,
(iv) $(\log_\lambda \beta_1)^{k_1} \otimes (\log_\lambda \beta_1)^{k_2} = (\log_\lambda \beta_1)^{k_1+k_2}$,
(v) $\left((\log_\lambda \beta_1)^{k_1} \right)^{k_2} = (\log_\lambda \beta_1)^{k_1 k_2}$.

Proof. For SVNNs β_1, β_2 and by Definition 3.2, we get

$$\log_\lambda \beta_1 = \langle 1 - \log_\lambda \zeta_1, \log_\lambda(1-\kappa_1), \log_\lambda(1-\varphi_1) \rangle$$

and

$$\log_\lambda \beta_2 = \langle 1 - \log_\lambda \zeta_2, \log_\lambda(1-\kappa_2), \log_\lambda(1-\varphi_2) \rangle$$

and hence by using the operations laws between two SVNNs, we have

$$\log_\lambda \beta_1 \oplus \log_\lambda \beta_2 = \langle 1 - (\log_\lambda \zeta_1)(\log_\lambda \zeta_2), (\log_\lambda(1-\kappa_1))(\log_\lambda(1-\kappa_2)), (\log_\lambda(1-\varphi_1))(\log_\lambda(1-\varphi_2))$$

and

$$\log_\lambda \beta_1 \otimes \log_\lambda \beta_2 = \langle (1 - \log_\lambda \zeta_1)(1 - \log_\lambda \zeta_2), 1 - (1 - \log_\lambda(1-\kappa_1))(1 - \log_\lambda(1-\kappa_2)), 1 - (1 - \log_\lambda(1-\varphi_1))(1 - \log_\lambda(1-\varphi_2)) \rangle$$

(i) For a real number $k > 0$, we have

$$k(\log_\lambda \beta_1 \oplus \log_\lambda \beta_2)$$
$$= \left\langle 1 - (\log_\lambda \zeta_1 \log_\lambda \zeta_2)^k, (\log_\lambda(1-\kappa_1)\log_\lambda(1-\kappa_2))^k, (\log_\lambda(1-\varphi_1)\log_\lambda(1-\varphi_2))^k \right\rangle$$
$$= \left\langle 1 - (\log_\lambda \zeta_1)^k(\log_\lambda \zeta_2)^k, (\log_\lambda(1-\kappa_1))^k(\log_\lambda(1-\kappa_2))^k, (\log_\lambda(1-\varphi_1))^k(\log_\lambda(1-\varphi_2))^k \right\rangle$$
$$= \left\langle 1 - (\log_\lambda \zeta_1)^k, (\log_\lambda(1-\kappa_1))^k, (\log_\lambda(1-\varphi_1))^k \right\rangle \oplus \left\langle 1 - (\log_\lambda \zeta_2)^k, (\log_\lambda(1-\kappa_2))^k, (\log_\lambda(1-\varphi_2))^k \right\rangle$$
$$= k\log_\lambda \beta_1 \oplus k\log_\lambda \beta_2$$

(ii) For a real number $k > 0$, we have

$$(\log_\lambda\beta_1 \otimes \log_\lambda\beta_2)^k = \Big\langle ((1-\log_\lambda\zeta_1)(1-\log_\lambda\zeta_2))^k, 1-((1-\log_\lambda(1-\kappa_1))(1-\log_\lambda(1-\kappa_2)))^k, 1-((1-\log_\lambda(1-\varphi_1))(1-\log_\lambda(1-\varphi_2)))^k \Big\rangle$$
$$= \Big\langle (1-\log_\lambda\zeta_1)^k(1-\log_\lambda\zeta_2)^k, 1-(1-\log_\lambda(1-\kappa_1))^k(1-\log_\lambda(1-\kappa_2))^k, 1-(1-\log_\lambda(1-\varphi_1))^k(1-\log_\lambda(1-\varphi_2))^k \Big\rangle$$
$$= \Big\langle (1-\log_\lambda\zeta_1)^k, 1-(1-\log_\lambda(1-\kappa_1))^k, 1-(1-\log_\lambda(1-\varphi_1))^k \Big\rangle \otimes \Big\langle (1-\log_\lambda\zeta_2)^k, 1-(1-\log_\lambda(1-\kappa_2))^k, 1-(1-\log_\lambda(1-\varphi_2))^k \Big\rangle = (\log_\lambda\beta_1)^k \otimes (\log_\lambda\beta_2)^k$$

(iii) For real positive number k_1 and k_2, we have

$$k_1\log_\lambda\beta_1 \oplus k_2\log_\lambda\beta_1 = \Big\langle 1-(\log_\lambda\zeta_1)^{k_1}, (\log_\lambda(1-\kappa_1))^{k_1}, (\log_\lambda(1-\varphi_1))^{k_1} \Big\rangle \oplus \Big\langle 1-(\log_\lambda\zeta_1)^{k_2}, (\log_\lambda(1-\kappa_1))^{k_2}, (\log_\lambda(1-\varphi_1))^{k_2} \Big\rangle$$
$$= \Big\langle 1-(\log_\lambda\zeta_1)^{k_1+k_2}, (\log_\lambda(1-\kappa_1))^{k_1+k_2}, (\log_\lambda(1-\varphi_1))^{k_1+k_2} \Big\rangle = (k_1+k_2)\log_\lambda\beta_1$$

(iv) For real positive number k_1 and k_2. Since

$$(\log_\lambda\beta_1)^{k_1} = \Big\langle (1-\log_\lambda\zeta_1)^{k_1}, 1-(1-\log_\lambda(1-\kappa_1))^{k_1}, 1-(1-\log_\lambda(1-\varphi_1))^{k_1} \Big\rangle$$

and

$$(\log_\lambda\beta_1)^{k_2} = \Big\langle (1-\log_\lambda\zeta_1)^{k_2}, 1-(1-\log_\lambda(1-\kappa_1))^{k_2}, 1-(1-\log_\lambda(1-\varphi_1))^{k_2} \Big\rangle$$

and hence

$$(\log_\lambda\beta_1)^{k_1} \otimes (\log_\lambda\beta_1)^{k_2} = \Big\langle (1-\log_\lambda\zeta_1)^{k_1}, 1-(1-\log_\lambda(1-\kappa_1))^{k_1}, 1-(1-\log_\lambda(1-\varphi_1))^{k_1} \Big\rangle \otimes \Big\langle (1-\log_\lambda\zeta_1)^{k_2}, 1-(1-\log_\lambda(1-\kappa_1))^{k_2}, 1-(1-\log_\lambda(1-\varphi_1))^{k_2} \Big\rangle$$
$$= \Big\langle (1-\log_\lambda\zeta_1)^{k_1}(1-\log_\lambda\zeta_1)^{k_2}, 1-(1-\log_\lambda(1-\kappa_1))^{k_1}(1-\log_\lambda(1-\kappa_1))^{k_2}, 1-(1-\log_\lambda(1-\varphi_1))^{k_1}(1-\log_\lambda(1-\varphi_1))^{k_2} \Big\rangle$$
$$= \Big\langle (1-\log_\lambda\zeta_1)^{k_1+k_2}, 1-(1-\log_\lambda(1-\kappa_1))^{k_1+k_2}, 1-(1-\log_\lambda(1-\varphi_1))^{k_1+k_2} \Big\rangle = (\log_\lambda\beta_1)^{k_1+k_2}$$

(v) For positive real numbers k_1 and k_2, we have

$$\Big((\log_\lambda\beta_1)^{k_1}\Big)^{k_2} = \Big(\Big\langle (1-\log_\lambda\zeta_1)^{k_1}, 1-(1-\log_\lambda(1-\kappa_1))^{k_1}, 1-(1-\log_\lambda(1-\varphi_1))^{k_1} \Big\rangle\Big)^{k_2}$$
$$= \Big\langle (1-\log_\lambda\zeta_1)^{k_1k_2}, 1-(1-\log_\lambda(1-\kappa_1))^{k_1k_2}, 1-(1-\log_\lambda(1-\varphi_1))^{k_1k_2} \Big\rangle = (\log_\lambda\beta_1)^{k_1k_2}$$

□

Theorem 3.6. *Let $\beta = \langle\zeta,\kappa,\varphi\rangle$ be SVNN. If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} \leqslant 1, \lambda_1,\lambda_2 \neq 1$, then $\log_{\lambda_1}\beta \geqslant \log_{\lambda_2}\beta$ and $\log_{\lambda_1}\beta \leqslant \log_{\lambda_2}\beta$ for $0 < \frac{1}{\lambda_2} \leqslant \frac{1}{\lambda_1} \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} \leqslant 1, \lambda_1,\lambda_2 \neq 1$.*

Proof. By Definition 3.2, we have

$$\log_{\lambda_1}\beta = \langle 1-\log_{\lambda_1}\zeta, \log_{\lambda_1}(1-\kappa), \log_{\lambda_1}(1-\varphi)\rangle$$

and

$$\log_{\lambda_2}\beta = \langle 1-\log_{\lambda_2}\zeta, \log_{\lambda_2}(1-\kappa), \log_{\lambda_2}(1-\varphi)\rangle$$

If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} \leqslant 1$ and $\lambda_1,\lambda_2 \neq 1$, then $1-\log_{\lambda_1}\zeta \geqslant 1-\log_{\lambda_2}\zeta, \log_{\lambda_1}(1-\kappa) \leqslant \log_{\lambda_2}(1-\kappa)$ and $\log_{\lambda_1}(1-\varphi) \leqslant \log_{\lambda_2}(1-\varphi)$ which implies that $\log_{\lambda_1}\beta \geqslant \log_{\lambda_2}\beta$.

On the other hand, when $\lambda_1,\lambda_2 > 1$ and $\lambda_1 \leqslant \lambda_2$, we get $0 < \frac{1}{\lambda_2} \leqslant \frac{1}{\lambda_1} \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} \leqslant 1, \lambda_1,\lambda_2 \neq 1$. Therefore, as discussed above, we can also obtain $\log_{\lambda_1}\beta \leqslant \log_{\lambda_2}\beta$. □

Theorem 3.7. *Let $\beta_1 = \langle\zeta_1,\kappa_1,\varphi_1\rangle$ and $\beta_2 = \langle\zeta_2,\kappa_2,\varphi_2\rangle$ be two SVNNs. If $\zeta_1 \leqslant \zeta_2, \varphi_1 \geqslant \varphi_2$ and $\kappa_1 \geqslant \kappa_2$, i.e., $\beta_1 \leqslant \beta_2, 0 < \lambda \leqslant \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda \neq 1$, then $\log_\lambda\beta_1 \leqslant \log_\lambda\beta_2$.*

Proof. Similar with Theorem 3.6. □

3.2. Aggregation operators

Based on the LOL of SVNNs, we define some weighted aggregation operators as follows.

Definition 3.3. Let $\beta_i = \langle\zeta_i,\kappa_i,\varphi_i\rangle (i = 1,2,\ldots,n)$ be a collection of SVNNs, $0 < \lambda_i \leqslant \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda_i \neq 1$ and let L-SVNWA : $\Theta^n \to \Theta$. If

$$\text{L-SVNWA}(\beta_1,\beta_2,\ldots,\beta_n) = \omega_1\log_{\lambda_1}\beta_1 \oplus \omega_2\log_{\lambda_2}\beta_2 \oplus \ldots \oplus \omega_n\log_{\lambda_n}\beta_n \quad (9)$$

then the function L-SVNWA is called logarithmic SVN weighted averaging operator, where $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\log_{\lambda_i}\beta_i$ with $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$.

Theorem 3.8. *Let $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle (i = 1, 2, \ldots, n)$ be a collection of SVNNs. Then, the aggregated value by using L-SVNWA operator is also SVNN and is given by*

$$\text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) = \begin{cases} \left\langle 1 - \prod_{i=1}^{n} (\log_{\lambda_i} \zeta_i)^{\omega_i}, \prod_{i=1}^{n} (\log_{\lambda_i}(1-\kappa_i))^{\omega_i}, \prod_{i=1}^{n} (\log_{\lambda_i}(1-\varphi_i))^{\omega_i} \right\rangle; & 0 < \lambda_i \leqslant \min\left\{ \begin{array}{c} \zeta_i, \\ 1-\kappa_i, \\ 1-\varphi_i \end{array} \right\} \leqslant 1, \lambda_i \neq 1 \\ \left\langle 1 - \prod_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}} \zeta_i\right)^{\omega_i}, \prod_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}(1-\kappa_i)\right)^{\omega_i}, \prod_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}(1-\varphi_i)\right)^{\omega_i} \right\rangle; & 0 < \frac{1}{\lambda_i} \leqslant \min\left\{ \begin{array}{c} \zeta_i, \\ 1-\kappa_i, \\ 1-\varphi_i \end{array} \right\} \leqslant 1, \lambda_i \neq 1 \end{cases} \quad (10)$$

Proof. We prove the result given in Eq. (10) by employing mathematical induction on n for $0 < \lambda_i \leqslant \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda_i \neq 1$. Since for each $i, \beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle$ is SVNN which implies that $\zeta_i, \kappa_i, \varphi_i \in [0, 1]$ and $\zeta_i + \kappa_i + \varphi_i \leqslant 3$. Then the following steps of the mathematical induction are executed.

Step 1: For $n = 2$, we get $\text{L-SVNWA}(\beta_1, \beta_2) = \omega_1(\log_{\lambda_1}\beta_1) \oplus \omega_2(\log_{\lambda_2}\beta_2)$. Since by Definition 3.2, we can see that $\log_{\lambda_1}\beta_1$ and $\log_{\lambda_2}\beta_2$ are SVNNs and hence $\omega_1 \log_{\lambda_1}\beta_1 \oplus \omega_2 \log_{\lambda_2}\beta_2$ is also SVNN. Further, for β_1 and β_2, we have

$$\begin{aligned} \text{L-SVNWA}(\beta_1, \beta_2) &= \omega_1 \log_{\lambda_1}\beta_1 \oplus \omega_2 \log_{\lambda_2}\beta_2 \\ &= \langle 1 - (\log_{\lambda_1}\zeta_1)^{\omega_1}, (\log_{\lambda_1}(1-\kappa_1))^{\omega_1}, (\log_{\lambda_1}(1-\varphi_1))^{\omega_1} \rangle \\ &\oplus \langle 1 - (\log_{\lambda_2}\zeta_2)^{\omega_2}, (\log_{\lambda_2}(1-\kappa_2))^{\omega_2}, (\log_{\lambda_2}(1-\varphi_2))^{\omega_2} \rangle \\ &= \left\langle 1 - \prod_{i=1}^{2} (\log_{\lambda_i}\zeta_i)^{\omega_i}, \prod_{i=1}^{2} (\log_{\lambda_i}(1-\kappa_i))^{\omega_i}, \prod_{i=1}^{2} (\log_{\lambda_i}(1-\varphi_i))^{\omega_i} \right\rangle \end{aligned}$$

Thus, result holds for $n = 2$.

Step 2: Assume Eq. (10) holds for $n = k$. Now, for $n = k + 1$, we have

$$\begin{aligned} &\text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_{k+1}) = \text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_k) \\ &\oplus \omega_{k+1}\log_{\lambda_{k+1}}\beta_{k+1} = \left\langle 1 - \prod_{i=1}^{k} (\log_{\lambda_i}\zeta_i)^{\omega_i}, \prod_{i=1}^{k} (\log_{\lambda_i}(1-\kappa_i))^{\omega_i}, \right. \\ &\left. \prod_{i=1}^{k} (\log_{\lambda_i}(1-\varphi_i))^{\omega_i} \right\rangle \oplus \left\langle 1 - \left(\log_{\lambda_{k+1}}\zeta_{k+1}\right)^{\omega_{k+1}}, \right. \\ &\left. \left(\log_{\lambda_{k+1}}(1-\kappa_{k+1})\right)^{\omega_{k+1}}, \left(\log_{\lambda_{k+1}}(1-\varphi_{k+1})\right)^{\omega_{k+1}} \right\rangle \\ &= \left\langle 1 - \prod_{i=1}^{k+1} (\log_{\lambda_i}\zeta_i)^{\omega_i}, \prod_{i=1}^{k+1} (\log_{\lambda_i}(1-\kappa_i))^{\omega_i}, \right. \\ &\left. \prod_{i=1}^{k+1} (\log_{\lambda_i}(1-\varphi_i))^{\omega_i} \right\rangle \end{aligned}$$

and the aggregated value is also SVNN. Therefore, Eq. (10) holds for $n = k + 1$ also. Hence, result is true for all positive integer n by the means of principle of mathematical induction.

On the other hand, if $\lambda_i \geqslant 1$ and $0 < \frac{1}{\lambda_i} \leqslant \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda_i \neq 1$, we can also get

$$\begin{aligned} &\text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) \\ &= \left\langle 1 - \prod_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}\zeta_i\right)^{\omega_i}, \prod_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}(1-\kappa_i)\right)^{\omega_i}, \right. \\ &\left. \prod_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}(1-\varphi_i)\right)^{\omega_i} \right\rangle \end{aligned}$$

and aggregated value is SVNN. □

Remark 3.1. If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda, 0 < \lambda \leqslant \min_i\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda \neq 1$, then L-SVNWA operator reduces to the following

$$\begin{aligned} &\text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) \\ &= \left\langle 1 - \prod_{i=1}^{n} (\log_{\lambda}\zeta_i)^{\omega_i}, \prod_{i=1}^{n} (\log_{\lambda}(1-\kappa_i))^{\omega_i}, \prod_{i=1}^{n} (\log_{\lambda}(1-\varphi_i))^{\omega_i} \right\rangle \end{aligned}$$

Example 3.2. Let $\beta_1 = \langle 0.5, 0.3, 0.2 \rangle, \beta_2 = \langle 0.2, 0.4, 0.3 \rangle$ and $\beta_3 = \langle 0.6, 0.3, 0.5 \rangle$ be three SVNNs and $\omega = (0.2, 0.5, 0.3)^T$ is the weight vector of them. Considering $\lambda_1 = \lambda_2 = \lambda_3 = 0.1$ and then

$$\begin{aligned} &\text{L-SVNWA}(\beta_1, \beta_2, \beta_3) \\ &= \left\langle 1 - \prod_{i=1}^{3} (\log_{\lambda_i}\zeta_i)^{\omega_i}, \prod_{i=1}^{3} (\log_{\lambda_i}(1-\kappa_i))^{\omega_i}, \right. \\ &\left. \prod_{i=1}^{3} (\log_{\lambda_i}(1-\varphi_i))^{\omega_i} \right\rangle \\ &= \left\langle 1 - (\log_{0.1}(0.5))^{0.2} \times (\log_{0.1}(0.2))^{0.5} \times (\log_{0.1}(0.6))^{0.3}, \right. \\ &(\log_{0.1}(1-0.3))^{0.2} \times (\log_{0.1}(1-0.4))^{0.5} \times (\log_{0.1}(1-0.3))^{0.3}, \\ &\left. (\log_{0.1}(1-0.2))^{0.2} \times (\log_{0.1}(1-0.3))^{0.5} \times (\log_{0.1}(1-0.5))^{0.3} \right\rangle \\ &= \langle 0.5814, 0.1854, 0.1721 \rangle \end{aligned}$$

Next, we give some properties of the proposed L-SVNWA operator for $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$, and $0 < \lambda \leqslant \min_i\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1, \lambda_i \neq 1$ and ω_i be the weight vector of SVNN β_i such that $\omega_i > 0$ and $\sum_{i=1}^{n}\omega_i = 1$.

Property 3.1. *If all SVNNs $\beta_i = \beta (i = 1, 2, \ldots, n)$, then* L-SVNWA$(\beta_1, \beta_2, \ldots, \beta_n) = \log_\lambda \beta$

Proof. Let $\beta = \langle \zeta, \kappa, \varphi \rangle$ is SVNN such that $\beta_i = \beta$ for all i. Then, by Theorem 3.8, we get

$$\begin{aligned}
&\text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n)\\
&= \left\langle 1 - \prod_{i=1}^{n}(\log_\lambda \zeta_i)^{\omega_i}, \prod_{i=1}^{n}(\log_\lambda (1-\kappa_i))^{\omega_i}, \prod_{i=1}^{n}(\log_\lambda (1-\varphi_i))^{\omega_i} \right\rangle\\
&= \left\langle 1 - \prod_{i=1}^{n}(\log_\lambda \zeta)^{\omega_i}, \prod_{i=1}^{n}(\log_\lambda (1-\kappa))^{\omega_i}, \prod_{i=1}^{n}(\log_\lambda (1-\varphi))^{\omega_i} \right\rangle\\
&= \left\langle 1 - (\log_\lambda \zeta)^{\sum_{i=1}^{n}\omega_i}, (\log_\lambda (1-\kappa))^{\sum_{i=1}^{n}\omega_i}, (\log_\lambda (1-\varphi))^{\sum_{i=1}^{n}\omega_i} \right\rangle\\
&= \langle 1 - \log_\lambda \zeta, \log_\lambda (1-\kappa), \log_\lambda (1-\varphi) \rangle = \log_\lambda \beta \quad \square
\end{aligned}$$

Property 3.2. *If $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle (i = 1, 2, \ldots, n)$ with $\beta^- = \langle \min_i\{\zeta_i\}, \max_i\{\kappa_i\}, \max_i\{\varphi_i\} \rangle$ and $\beta^+ = \langle \max_i\{\zeta_i\}, \min_i\{\kappa_i\}, \min_i\{\varphi_i\} \rangle$, then*

$$\log_\lambda \beta^- \leqslant \text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \log_\lambda \beta^+$$

Proof. Since, for any $i, \min_i\{\zeta_i\} \leqslant \zeta_i \leqslant \max_i\{\zeta_i\}$, $\min_i\{\kappa_i\} \leqslant \kappa_i \leqslant \max_i\{\kappa_i\}$, and $\min_i\{\varphi_i\} \leqslant \varphi_i \leqslant \max_i\{\varphi_i\}$. This implies that $\beta^- \leqslant \beta_i \leqslant \beta^+$. Assume that L-SVNWA$(\beta_1, \ldots, \beta_n) = \log_\lambda \beta = \langle \zeta_\beta, \kappa_\beta, \varphi_\beta \rangle, \log_\lambda \beta^- = \langle \zeta_{\beta^-}, \kappa_{\beta^+}, \varphi_{\beta^+} \rangle, \log_\lambda \beta^+ = \langle \zeta_{\beta^+}, \kappa_{\beta^-}, \varphi_{\beta^-} \rangle$. Then, based on the monotonicity of logarithm function, we have

$$\begin{aligned}
\zeta_\beta &= 1 - \prod_{i=1}^{n}(\log_\lambda \zeta_i)^{\omega_i} \geqslant 1 - \prod_{i=1}^{n}\left(\log_\lambda \min_i\{\zeta_i\}\right)^{\omega_i}\\
&= 1 - \log_\lambda(\min\{\zeta_i\}) = \zeta_{\beta^-},\\
\kappa_\beta &= \prod_{i=1}^{n}(\log_\lambda (1-\kappa_i))^{\omega_i} \geqslant \prod_{i=1}^{n}\left(\log_\lambda (1-\min_i\{\kappa_i\})\right)^{\omega_i}\\
&= \log_\lambda(1-\min\{\kappa_i\}) = \kappa_{\beta^-}
\end{aligned}$$

and $$\varphi_\beta = \prod_{i=1}^{n}(\log_\lambda (1-\varphi_i))^{\omega_i} \geqslant \prod_{i=1}^{n}\left(\log_\lambda (1-\min_i\{\varphi_i\})\right)^{\omega_i} = \log_\lambda(1-\min\{\varphi_i\}) = \varphi_{\beta^-}$$

Also

$$\begin{aligned}
\zeta_\beta &= 1 - \prod_{i=1}^{n}(\log_\lambda \zeta_i)^{\omega_i} \leqslant 1 - \prod_{i=1}^{n}\left(\log_\lambda \max_i\{\zeta_i\}\right)^{\omega_i}\\
&= 1 - \log_\lambda(\max\{\zeta_i\}) = \zeta_{\beta^+},\\
\kappa_\beta &= \prod_{i=1}^{n}(\log_\lambda (1-\kappa_i))^{\omega_i} \leqslant \prod_{i=1}^{n}\left(\log_\lambda (1-\max_i\{\kappa_i\})\right)^{\omega_i}\\
&= \log_\lambda(1-\max\{\kappa_i\}) = \kappa_{\beta^+}
\end{aligned}$$

and $$\varphi_\beta = \prod_{i=1}^{n}(\log_\lambda (1-\varphi_i))^{\omega_i} \leqslant \prod_{i=1}^{n}\left(\log_\lambda (1-\max_i\{\varphi_i\})\right)^{\omega_i} = \log_\lambda(1-\max\{\varphi_i\}) = \varphi_{\beta^+}$$

Based on score function, we get

$$S(\log_\lambda \beta) = \zeta_\beta - \kappa_\beta - \varphi_\beta \leqslant \zeta_{\beta^+} - \kappa_{\beta^-} - \varphi_{\beta^-} = S(\log_\lambda \beta^+)$$

and

$$S(\log_\lambda \beta) = \zeta_\beta - \kappa_\beta - \varphi_\beta \geqslant \zeta_{\beta^-} - \kappa_{\beta^+} - \varphi_{\beta^+} = S(\log_\lambda \beta^-)$$

Hence, $S(\log_\lambda \beta^-) \leqslant S(\log_\lambda \beta) \leqslant S(\log_\lambda \beta^+)$. Now, we discuss the three cases:

Case 1: If $S(\log_\lambda \beta^-) < S(\log_\lambda \beta) < S(\log_\lambda \beta^+)$, then result holds.
Case 2: If $S(\log_\lambda \beta^+) = S(\log_\lambda \beta)$ then $\zeta_\beta - \kappa_\beta - \varphi_\beta = \zeta_{\beta^+} - \kappa_{\beta^-} - \varphi_{\beta^-}$ which implies that $\zeta_\beta = \zeta_{\beta^+}, \kappa_\beta = \kappa_{\beta^-}$ and $\varphi_\beta = \varphi_{\beta^-}$ and hence $H(\log_\lambda \beta^+) = H(\log_\lambda \beta)$.
Case 3: If $S(\log_\lambda \beta^-) = S(\log_\lambda \beta)$ then $\zeta_\beta - \kappa_\beta - \varphi_\beta = \zeta_{\beta^-} - \kappa_{\beta^+} - \varphi_{\beta^+}$ which implies that $\zeta_\beta = \zeta_{\beta^-}, \kappa_\beta = \kappa_{\beta^+}$ and $\varphi_\beta = \varphi_{\beta^+}$ and hence $H(\log_\lambda \beta^-) = H(\log_\lambda \beta)$.

Therefore, by combining all these cases, we get

$$\log_\lambda \beta^- \leqslant \text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \log_\lambda \beta^+ \quad \square$$

Property 3.3. *Let $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle$ and $\beta_i^* = \langle \zeta_i^*, \kappa_i^*, \varphi_i^* \rangle (i = 1, 2, \ldots, n)$ be two collections of SVNNs. If $\zeta_i \leqslant \zeta_i^*$, $\kappa_i \geqslant \kappa_i^*, \varphi_i \geqslant \varphi_i^*$, then*

$$\text{L-SVNWA}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \text{L-SVNWA}(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$$

Proof. Follows from the above, so we omit here. □

Definition 3.4. A logarithmic SVN ordered weighted average (L-SVNOWA) operator is a mapping L-SVNOWA : $\Theta^n \to \Theta$, such that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, with $\omega_i > 0$ and $\sum_{i=1}^{n}\omega_i = 1$, and

$$\begin{aligned}
&\text{L-SVNOWA}(\beta_1, \beta_2, \ldots, \beta_n)\\
&= \left(\omega_1 \log_{\lambda_{\sigma(1)}} \beta_{\sigma(1)}\right) \oplus \left(\omega_2 \log_{\lambda_{\sigma(2)}} \beta_{\sigma(2)}\right) \oplus \cdots\\
&\quad \oplus \left(\omega_n \log_{\lambda_{\sigma(n)}} \beta_{\sigma(n)}\right) \qquad (11)
\end{aligned}$$

where $0 < \lambda_{\sigma(i)} \leqslant \min\{\zeta_{\sigma(i)}, 1 - \kappa_{\sigma(i)}, 1 - \varphi_{\sigma(i)}\} \leqslant 1, \lambda_{\sigma(i)} \neq 1$ and σ is the permutation of $(1, 2, \ldots, n)$ such that $\beta_{\sigma(i-1)} \geqslant \beta_{\sigma(i)}$ for $i = 2, 3, \ldots, n$.

Theorem 3.9. *For a collection of SVNNs $\beta_i = \langle \zeta_i, \kappa_i, \varphi_i \rangle (i = 1, 2, \ldots, n)$, the aggregated value by using L-SVNOWA operator is still SVNN and given by*

$$\text{L-SVNOWA}(\beta_1,\beta_2,\ldots,\beta_n) = \begin{cases} \left\langle 1-\prod_{i=1}^{n}\left(\log_{\lambda_{\sigma(i)}}\zeta_{\sigma(i)}\right)^{\omega_i}, \prod_{i=1}^{n}\left(\log_{\lambda_{\sigma(i)}}(1-\kappa_{\sigma(i)})\right)^{\omega_i}, \prod_{i=1}^{n}\left(\log_{\lambda_{\sigma(i)}}(1-\varphi_{\sigma(i)})\right)^{\omega_i}\right\rangle; & 0<\lambda_{\sigma(i)}\leqslant \min_i\begin{Bmatrix}\zeta_{\sigma(i)}\\ 1-\kappa_{\sigma(i)}\\ 1-\varphi_{\sigma(i)}\end{Bmatrix}\leqslant 1, \lambda_{\sigma(i)}\neq 1 \\ \left\langle 1-\prod_{i=1}^{n}\left(\log_{\frac{1}{\lambda_{\sigma(i)}}}\zeta_{\sigma(i)}\right)^{\omega_i}, \prod_{i=1}^{n}\left(\log_{\frac{1}{\lambda_{\sigma(i)}}}(1-\kappa_{\sigma(i)})\right)^{\omega_i}, \prod_{i=1}^{n}\left(\log_{\frac{1}{\lambda_{\sigma(i)}}}(1-\varphi_{\sigma(i)})\right)^{\omega_i}\right\rangle; & 0<\frac{1}{\lambda_{\sigma(i)}}\leqslant \min_i\begin{Bmatrix}\zeta_{\sigma(i)}\\ 1-\kappa_{\sigma(i)}\\ 1-\varphi_{\sigma(i)}\end{Bmatrix}\leqslant 1, \lambda_{\sigma(i)}\neq 1 \end{cases} \tag{12}$$

Proof. The proof follows from Theorem 3.8. □

Definition 3.5. Let $\beta_i = \langle\zeta_i,\kappa_i,\varphi_i\rangle (i = 1,2,\ldots,n)$ be a collection of SVNNs, $0 \leqslant \lambda_i \leqslant \min_i\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leqslant 1$, $\lambda_i \neq 1$ and let L-SVNWG : $\Theta^n \to \Theta$. If

$$\text{L-SVNWG}(\beta_1,\beta_2,\ldots,\beta_n) = \left(\log_{\lambda_1}\beta_1\right)^{\omega_1} \otimes \left(\log_{\lambda_2}\beta_2\right)^{\omega_2} \otimes \ldots \otimes \left(\log_{\lambda_n}\beta_n\right)^{\omega_n} \tag{13}$$

then the function L-SVNWG is called logarithmic SVN weighted geometric operator, where $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\log_{\lambda_i}\beta_i$ with $\omega_i > 0$ and $\sum_{i=1}^{n}\omega_i = 1$.

Theorem 3.10. *The aggregated value by L-SVNWG operator for SVNNs $\beta_i = \langle\zeta_i,\kappa_i,\varphi_i\rangle$ is also SVNN and is given by*

$$\text{L-SVNWG}(\beta_1,\beta_2,\ldots,\beta_n) = \begin{cases} \left\langle \prod_{i=1}^{n}(1-\log_{\lambda_i}\zeta_i)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\log_{\lambda_i}(1-\kappa_i))^{\omega_i}, 1-\prod_{i=1}^{n}(1-\log_{\lambda_i}(1-\varphi_i))^{\omega_i}\right\rangle; & 0<\lambda_i\leqslant \min_i\begin{Bmatrix}\zeta_i\\ 1-\kappa_i\\ 1-\varphi_i\end{Bmatrix}\leqslant 1, \lambda_i\neq 1 \\ \left\langle \prod_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_i}}\zeta_i\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_i}}(1-\kappa_i)\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_i}}(1-\varphi_i)\right)^{\omega_i}\right\rangle; & 0<\frac{1}{\lambda_i}\leqslant \min_i\begin{Bmatrix}\zeta_i\\ 1-\kappa_i\\ 1-\varphi_i\end{Bmatrix}\leqslant 1, \lambda_i\neq 1 \end{cases} \tag{14}$$

Definition 3.6. A logarithmic SVN ordered weighted geometric (L-SVNOWG) operator is a mapping L-SVNOWG : $\Theta^n \to \Theta$, such that $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$, with $\omega_i > 0$ and $\sum_{i=1}^{n}\omega_i = 1$, and

$$\text{L-SVNOWG}(\beta_1,\beta_2,\ldots,\beta_n) = \left(\log_{\lambda_{\sigma(1)}}\beta_{\sigma(1)}\right)^{\omega_1} \otimes \left(\log_{\lambda_{\sigma(2)}}\beta_{\sigma(2)}\right)^{\omega_2} \otimes \cdots \otimes \left(\log_{\lambda_{\sigma(n)}}\beta_{\sigma(n)}\right)^{\omega_n} \tag{15}$$

where $0 < \lambda_{\sigma(i)} \leqslant \min\{\zeta_{\sigma(i)}, 1-\kappa_{\sigma(i)}, 1-\varphi_{\sigma(i)}\} \leqslant 1, \lambda_{\sigma(i)} \neq 1$ and σ is the permutation of $(1,2,\ldots,n)$ such that $\beta_{\sigma(i-1)} \geqslant \beta_{\sigma(i)}$ for $i = 2,3,\ldots,n$.

Theorem 3.11. *For a collection of SVNNs $\beta_i = \langle\zeta_i,\kappa_i,\varphi_i\rangle (i = 1,2,\ldots,n)$, the aggregated value by using L-SVNOWG operator is still SVNN and given by*

$$\text{L-SVNOWG}(\beta_1,\beta_2,\ldots,\beta_n) = \begin{cases} \left\langle \prod_{i=1}^{n}\left(1-\log_{\lambda_{\sigma(i)}}\zeta_{\sigma(i)}\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-\log_{\lambda_{\sigma(i)}}(1-\kappa_{\sigma(i)})\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-\log_{\lambda_{\sigma(i)}}(1-\varphi_{\sigma(i)})\right)^{\omega_i}\right\rangle; & 0<\lambda_{\sigma(i)}\leqslant \min_i\begin{Bmatrix}\zeta_{\sigma(i)}\\ 1-\kappa_{\sigma(i)}\\ 1-\varphi_{\sigma(i)}\end{Bmatrix}\leqslant 1, \lambda_i\neq 1 \\ \left\langle \prod_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_{\sigma(i)}}}\zeta_{\sigma(i)}\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_{\sigma(i)}}}(1-\kappa_{\sigma(i)})\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_{\sigma(i)}}}(1-\varphi_{\sigma(i)})\right)^{\omega_i}\right\rangle; & 0<\frac{1}{\lambda_{\sigma(i)}}\leqslant \min_i\begin{Bmatrix}\zeta_{\sigma(i)}\\ 1-\kappa_{\sigma(i)}\\ 1-\varphi_{\sigma(i)}\end{Bmatrix}\leqslant 1, \lambda_i\neq 1 \end{cases} \tag{16}$$

Proof. Similar to Theorem 3.8. □

As similar to L-SVNWA operator, the L-SVNOWA, L-SVNOWG and L-SVNWG operators also have the same properties. Furthermore, if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda, 0 < \lambda \leqslant \min_i\{\zeta_{\sigma(i)}, 1-\kappa_{\sigma(i)}, 1-\varphi_{\sigma(i)}\} \leqslant 1$, then L-SVNOWA, LSVNOWG operators becomes

$$\text{L-SVNOWA}(\beta_1, \beta_2, \ldots, \beta_n) = \left\langle 1 - \prod_{i=1}^{n}(\log_\lambda \zeta_{\sigma(i)})^{\omega_i}, \prod_{i=1}^{n}(\log_\lambda(1-\kappa_{\sigma(i)}))^{\omega_i}, \prod_{i=1}^{n}\Big(\log_\lambda(1-\varphi_{\sigma(i)})\Big)^{\omega_i} \right\rangle$$

and

$$\text{L-SVNOWG}(\beta_1, \beta_2, \ldots, \beta_n) = \left\langle \prod_{i=1}^{n}(1-\log_\lambda \zeta_{\sigma(i)})^{\omega_i}, 1 - \prod_{i=1}^{n}(1-\log_\lambda(1-\kappa_{\sigma(i)}))^{\omega_i}, 1 - \prod_{i=1}^{n}\Big(1-\log_\lambda(1-\varphi_{\sigma(i)})\Big)^{\omega_i} \right\rangle$$

4. Proposed MADM method

In this section, a decision making method present under SVN information based on the proposed operators. For it, consider a MADM problem with 'm' different alternatives denoted by $A_1, A_2, \ldots, A_m$ and are evaluated under the set of 'n' different attribute $C_1, C_2, \ldots, C_n$ with weight vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^{n}\omega_j = 1$. An expert has evaluated these alternatives and gives their preferences as SVNNs $\beta_{ij} = \langle \zeta_{ij}, \kappa_{ij}, \varphi_{ij}\rangle$ such that $0 \leqslant \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \leqslant 1$ and $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \leqslant 3$. The collection information of all the alternatives are summarized in decision-matrix D as

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \ldots & \beta_{mn} \end{pmatrix} \end{array}$$

On the other hand, the logarithm base index for these SVNNs are denoted by λ_{ij} where $0 < \lambda_{ij} \leqslant \min\{\zeta_{ij}, 1-\kappa_{ij}, 1-\varphi_{ij}\} \leqslant 1, \lambda_{ij} \neq 1$ for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ and are summarized in the matrix format $\Lambda = (\lambda_{ij})_{m\times n}$ as

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m1} & \lambda_{m2} & \ldots & \lambda_{mn} \end{pmatrix}$$

By using the logarithm operation on each SVNN β_{ij}, we convert the given decision matrix D into its equivalent logarithm score matrix S as

$$S = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} S(\log_{\lambda_{11}}\beta_{11}) & S(\log_{\lambda_{12}}\beta_{12}) & \ldots & S(\log_{\lambda_{1n}}\beta_{1n}) \\ S(\log_{\lambda_{21}}\beta_{21}) & S(\log_{\lambda_{22}}\beta_{22}) & \ldots & S(\log_{\lambda_{2n}}\beta_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ S(\log_{\lambda_{m1}}\beta_{m1}) & S(\log_{\lambda_{m2}}\beta_{m2}) & \ldots & S(\log_{\lambda_{mn}}\beta_{mn}) \end{pmatrix} \end{array} \tag{17}$$

where $S(\log_{\lambda_{ij}}\beta_{ij}) = 1 - \log_{\lambda_{ij}}\zeta_{ij} - \log_{\lambda_{ij}}(1-\kappa_{ij}) - \log_{\lambda_{ij}}(1-\varphi_{ij})$ is the score function of $\log_{\lambda_{ij}}\beta_{ij}$.

The attribute weights plays a significant role during the ranking order of the alternatives and hence in the decision making process, a weighted sum of each alternative, called as suitability function $Q(A_i)$, is obtained as

$$Q(A_i) = \sum_{j=1}^{n}\omega_j S_{ij}; \quad i = 1, 2, \ldots, m \tag{18}$$

Based on this function, a mathematical programming model for determining the weight vector is formulated as below.

$$\begin{aligned} &\max f = \sum_{i=1}^{m} Q(A_i) \\ &\quad \text{s.t.} \sum_{j=1}^{n}\omega_j = 1 \\ &\qquad \omega_j \geqslant 0; \quad \omega \in H \end{aligned} \tag{19}$$

where $Q(A_i) = \sum_{j=1}^{n}\omega_j S_{ij}$

Here, $Q(A_i)$ represents the overall score function for each alternative $A_i (i = 1, 2, \ldots, m)$. After solving this model, we get the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. Now, based on these weight vectors, aggregate all the preference values either by using L-SVNWA or L-SVNWG or L-SVNOWA or L-SVNOWG and get the collective one β_i. Finally, rank the alternative based on the score value of the aggregated number β_i and chose the best choice according to the highest value.

In the nutshells, after combination all the above demonstrations, our proposed decision making method under SVNN environment has been summarized as follows.

Step 1: Formulate the neutrosophic decision matrix D of rating values of the alternative A_i with respect to attribute G_j denoted by $\beta_{ij} = \langle \zeta_{ij}, \kappa_{ij}, \varphi_{ij}\rangle$ and the parameters λ_{ij} for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

Step 2: Convert the above matrix into the score matrix S by using Eq. (17).

Step 3: Solve the optimization model (19) based on the partial known weight information H about the attribute and get weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Step 4: Utilize the appropriate aggregation operator either L-SVNWA, or L-SVNOWA, or L-SVNWG or L-SVNOWG to aggregate the different preferences of the decision maker.

Step 5: Rank the alternative by using score function and chose the best alternative(s).

$$D = \begin{matrix} & C_1 & C_2 & C_3 & C_4 & C_5 \\ A_1 & \langle 0.5,0.3,0.4\rangle & \langle 0.5,0.2,0.3\rangle & \langle 0.2,0.2,0.6\rangle & \langle 0.3,0.2,0.4\rangle & \langle 0.3,0.3,0.4\rangle \\ A_2 & \langle 0.7,0.1,0.3\rangle & \langle 0.7,0.2,0.3\rangle & \langle 0.6,0.3,0.2\rangle & \langle 0.6,0.4,0.2\rangle & \langle 0.7,0.1,0.2\rangle \\ A_3 & \langle 0.5,0.3,0.4\rangle & \langle 0.6,0.2,0.4\rangle & \langle 0.6,0.1,0.2\rangle & \langle 0.5,0.1,0.3\rangle & \langle 0.6,0.4,0.3\rangle \\ A_4 & \langle 0.7,0.3,0.2\rangle & \langle 0.7,0.2,0.2\rangle & \langle 0.4,0.5,0.2\rangle & \langle 0.5,0.2,0.2\rangle & \langle 0.4,0.5,0.4\rangle \\ A_5 & \langle 0.4,0.1,0.3\rangle & \langle 0.5,0.1,0.2\rangle & \langle 0.4,0.1,0.5\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.3,0.2,0.4\rangle \end{matrix}$$

5. Illustrative example

The proposed approach is illustrating with a practical example which is stated as below.

5.1. Case study

Goods and Services Tax (GST) is an indirect tax which is designed to make India an integrated common market. While GST promises to user in an era of unified indirect tax regime, integrating India into a single homogenous market, it comes with certain complications inherited from the legacy tax regime. With the government gearing up to enforce the GST in Punjab from July 1, the issue of traders having limited computer knowledge and poor connectivity. In order to counter this, the state government has planned to train more than 2000 youths as 'GST Mitra' to cater the traders. Punjab GST Mitra Scheme, which has to be started as pilot project from Patiala, proposes to assist tax payers in furnishing the details of outward supplies, inward supplies and returns, filing claims or refunds, filing any other applications etc., in GST Regime. It aims to create a group of Tax professionals available in the locality or at the doorstep of tax payer, at affordable costs throughout State of Punjab.

The poor internet connectivity in far-flung areas has emerged as a big stumbling block in the success of 'GST Mitra' scheme. In order to provide the online services to run this scheme, state government is planning to give contract combinedly to private mobile service provider along with state-owned BSNL. For this, the Indian government had been issued the global tender to select the contractor for these projects in the newspaper and considered the five attribute required for its namely, Technology Expertise (C_1), Service quality (C_2), Bandwidth (C_3), Internet speed (C_4) and Customer Services (C_5). The importance of these attribute is taken as partially known. The five contractors (i.e. alternatives) namely, "Jaihind Road Builders private (Pvt.) limited (Ltd.)" (A_1), "J.K. Construction" (A_2), "Build quick Infrastructure Pvt. Ltd." (A_3), "Relcon Infra projects Ltd." (A_4), and "Tata Infrastructure Ltd." (A_5) bid for these projects. Then, the aim of the government is to recognize the best internet service to their own citizens. The procedure for selecting the best internet service provider is summarized in the following steps.

Step 1: The rating values of the expert towards the five alternatives $A_i (i = 1, 2, 3, 4, 5)$ are listed as

In this matrix, corresponding to alternative A_1 under criterion C_1, when we ask the opinion of an expert about the alternative A_1 with respect to the criterion C_1, he or she may that the possibility degree in which the statement is good is 0.5, the statement is false is 0.4 and the degree in which he or she is unsure is 0.3. The other values in the matrix have similar meanings. Furthermore, the preferences of the logarithm base λ_{ij} are summarized as

$$\Lambda = \begin{pmatrix} 0.4 & 0.3 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.3 & 0.2 \end{pmatrix}$$

Step 2: By using Eq. (17), the score matrix M is

$$M = \begin{matrix} & C_1 & C_2 & C_3 & C_4 & C_5 \\ A_1 & -0.7032 & -0.0573 & -0.1938 & -0.2041 & -0.2871 \\ A_2 & 0.1065 & 0.4181 & 0.0941 & -0.3585 & 0.4309 \\ A_3 & -0.2962 & -0.3585 & 0.3029 & 0.2822 & -0.5042 \\ A_4 & -0.0220 & 0.5011 & -0.5221 & -0.2435 & 0.0792 \\ A_5 & -0.1448 & -0.1150 & -0.0655 & -0.8184 & -0.2041 \end{matrix}$$

Step 3: Assume that the partial weight information about the attribute weights as given by decision maker is $H = \{0.15 \leqslant \omega_1 \leqslant 0.20, 0.2 \leqslant \omega_2 \leqslant 0.3, 0.2 \leqslant \omega_3 \leqslant 0.4, 0.22 \leqslant \omega_4 \leqslant 0.25, 0.15 \leqslant \omega_5 \leqslant 0.20\}$. Based on this information, an optimization model has been formulated as

$$\begin{aligned} \max f = & -0.7032\omega_1 - 0.0573\omega_2 - 0.1938\omega_3 - 0.2041\omega_4 \\ & - 0.2871\omega_5 + 0.1065\omega_1 + 0.4181\omega_2 + 0.0941\omega_3 \\ & - 0.3585\omega_4 + 0.4309\omega_5 - 0.2962\omega_1 - 0.3585\omega_2 \\ & + 0.3029\omega_3 + 0.2822\omega_4 - 0.5042\omega_5 - 0.0220\omega_1 \\ & + 0.5011\omega_2 - 0.5221\omega_3 - 0.2435\omega_4 + 0.0792\omega_5 \\ & - 0.1448\omega_1 - 0.1150\omega_2 - 0.0655\omega_3 - 0.8184\omega_4 \\ & - 0.2041\omega_5 \end{aligned}$$

i.e. $f = -1.0598\omega_1 + 0.3884\omega_2 - 0.3844\omega_3 - 1.3423\omega_4 - 0.4854\omega_5$

$$\begin{aligned} s.t.\ & 0.15 \leqslant \omega_1 \leqslant 0.20 \\ & 0.20 \leqslant \omega_2 \leqslant 0.30 \\ & 0.20 \leqslant \omega_3 \leqslant 0.40 \\ & 0.22 \leqslant \omega_4 \leqslant 0.25 \\ & 0.15 \leqslant \omega_5 \leqslant 0.20 \\ & \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 1 \\ & \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \geqslant 0 \end{aligned}$$

and hence after solving we get $\omega = (0.15, 0.28, 0.20, 0.22, 0.15)^T$.

Step 4a: Utilize the L-SVNWA aggregation operator, as given in Eq. (9), to aggregate all the preference value $\beta_{ij}(j = 1, 2, 3, 4)$ corresponding to each alternative $A_i(i = 1, 2, 3, 4, 5)$, we get the collective values β_i as

$$\begin{aligned} \beta_1 &= \text{L-SVNWA}(\beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}) \\ &= \left\langle 1 - \prod_{j=1}^{5} \left(\log_{\lambda_{1j}} \zeta_{1j}\right)^{\omega_j}, \prod_{j=1}^{5} \left(\log_{\lambda_{1j}} (1 - \kappa_{1j})\right)^{\omega_j}, \right. \\ &\quad \left. \prod_{j=1}^{5} \left(\log_{\lambda_{1j}} (1 - \varphi_{1j})\right)^{\omega_j} \right\rangle \\ &= \left\langle 1 - (\log_{0.4}(0.5))^{0.15} \times (\log_{0.3}(0.5))^{0.28} \right. \\ &\quad \times (\log_{0.1}(0.2))^{0.20} \times (\log_{0.2}(0.3))^{0.22} \times (\log_{0.2}(0.3))^{0.15}, \\ &\quad (\log_{0.4}(0.7))^{0.15} \times (\log_{0.3}(0.8))^{0.28} \times (\log_{0.1}(0.8))^{0.20} \\ &\quad \times (\log_{0.2}(0.8))^{0.22} \times (\log_{0.2}(0.7))^{0.15}, (\log_{0.4}(0.6))^{0.15} \\ &\quad \times (\log_{0.3}(0.7))^{0.28} \times (\log_{0.1}(0.4))^{0.20} \\ &\quad \left. \times (\log_{0.2}(0.6))^{0.22} \times (\log_{0.2}(0.6))^{0.15} \right\rangle \\ &= \langle 0.3130, 0.1753, 0.3544 \rangle \end{aligned}$$

$$\begin{aligned} \beta_2 &= \text{L-SVNWA}(\beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}, \beta_{25}) \\ &= \left\langle 1 - (\log_{0.4}(0.7))^{0.15} \times (\log_{0.2}(0.7))^{0.28} \right. \\ &\quad \times (\log_{0.3}(0.6))^{0.20} \times (\log_{0.4}(0.6))^{0.22} \times (\log_{0.3}(0.7))^{0.15}, \\ &\quad (\log_{0.4}(0.9))^{0.15} \times (\log_{0.2}(0.8))^{0.28} \times (\log_{0.3}(0.7))^{0.20} \\ &\quad \times (\log_{0.4}(0.6))^{0.22} \times (\log_{0.3}(0.9))^{0.15}, (\log_{0.4}(0.7))^{0.15} \\ &\quad \times (\log_{0.2}(0.7))^{0.28} \times (\log_{0.3}(0.8))^{0.20} \times (\log_{0.4}(0.8))^{0.22} \\ &\quad \left. \times (\log_{0.3}(0.8))^{0.15} \right\rangle = \langle 0.6486, 0.1989, 0.2313 \rangle \end{aligned}$$

$$\begin{aligned} \beta_3 &= \text{L-SVNWA}(\beta_{31}, \beta_{32}, \beta_{33}, \beta_{34}, \beta_{35}) \\ &= \left\langle 1 - (\log_{0.3}(0.5))^{0.15} \times (\log_{0.4}(0.6))^{0.28} \right. \\ &\quad \times (\log_{0.3}(0.6))^{0.20} \times (\log_{0.2}(0.5))^{0.22} \times (\log_{0.4}(0.6))^{0.15}, \\ &\quad (\log_{0.3}(0.7))^{0.15} \times (\log_{0.4}(0.8))^{0.28} \times (\log_{0.3}(0.9))^{0.20} \\ &\quad \times (\log_{0.2}(0.9))^{0.1} \times (\log_{0.4}(0.6))^{0.15}, \ (\log_{0.3}(0.6))^{0.22} \\ &\quad \times (\log_{0.4}(0.6))^{0.28} \times (\log_{0.3}(0.8))^{0.20} \times (\log_{0.2}(0.7))^{0.22} \\ &\quad \left. \times (\log_{0.4}(0.7))^{0.15} \right\rangle = \langle 0.4989, 0.1733, 0.3321 \rangle \end{aligned}$$

$$\begin{aligned} \beta_4 &= \text{L-SVNWA}(\beta_{41}, \beta_{42}, \beta_{43}, \beta_{44}, \beta_{45}) \\ &= \left\langle 1 - (\log_{0.4}(0.7))^{0.15} \times (\log_{0.2}(0.7))^{0.28} \right. \\ &\quad \times (\log_{0.3}(0.4))^{0.20} \times (\log_{0.4}(0.5))^{0.22} \times (\log_{0.1}(0.4))^{0.15}, \\ &\quad (\log_{0.4}(0.7))^{0.15} \times (\log_{0.2}(0.8))^{0.28} \times (\log_{0.3}(0.5))^{0.20} \\ &\quad \times (\log_{0.4}(0.8))^{0.22} \times (\log_{0.1}(0.5))^{0.15}, \ (\log_{0.4}(0.8))^{0.15} \\ &\quad \times (\log_{0.2}(0.8))^{0.28} \times (\log_{0.3}(0.8))^{0.20} \times (\log_{0.4}(0.8))^{0.22} \\ &\quad \left. \times (\log_{0.1}(0.6))^{0.15} \right\rangle = \langle 0.5585, 0.2736, 0.1942 \rangle \end{aligned}$$

$$\begin{aligned} \beta_5 &= \text{L-SVNWA}(\beta_{51}, \beta_{52}, \beta_{53}, \beta_{54}, \beta_{55}) \\ &= \left\langle 1 - (\log_{0.3}(0.4))^{0.15} \times (\log_{0.4}(0.5))^{0.28} \right. \\ &\quad \times (\log_{0.2}(0.4))^{0.20} \times (\log_{0.3}(0.4))^{0.22} \times (\log_{0.2}(0.3))^{0.15}, \\ &\quad (\log_{0.3}(0.9))^{0.15} \times (\log_{0.4}(0.9))^{0.28} \times (\log_{0.2}(0.9))^{0.20} \\ &\quad \times (\log_{0.3}(0.7))^{0.22} \times (\log_{0.2}(0.8))^{0.15}, \ (\log_{0.3}(0.7))^{0.15} \\ &\quad \times (\log_{0.4}(0.8))^{0.28} \times (\log_{0.2}(0.5))^{0.20} \times (\log_{0.3}(0.4))^{0.22} \\ &\quad \left. \times (\log_{0.2}(0.6))^{0.15} \right\rangle = \langle 0.2849, 0.1249, 0.3758 \rangle \end{aligned}$$

Step 4b: On the other hand, if we utilize L-SVNWG aggregation operator, given in Eq. (14) to aggregate the decision information, then the values are

$$\begin{aligned} \beta_1 &= \text{L-SVNWG}(\beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}) \\ &= \left\langle (1 - \log_{0.4}(0.5))^{0.15} \times (1 - \log_{0.3}(0.5))^{0.28} \right. \\ &\quad \times (1 - \log_{0.1}(0.2))^{0.20} \times (1 - \log_{0.2}(0.3))^{0.22} \\ &\quad \times (1 - \log_{0.2}(0.3))^{0.15}, 1 - (1 - \log_{0.4}(0.7))^{0.15} \\ &\quad \times (1 - \log_{0.3}(0.8))^{0.28} \times (1 - \log_{0.1}(0.8))^{0.20} \\ &\quad \times (1 - \log_{0.2}(0.8))^{0.22} \times (1 - \log_{0.2}(0.7))^{0.15}, \\ &\quad 1 - (1 - \log_{0.4}(0.6))^{0.15} \times (1 - \log_{0.3}(0.7))^{0.28} \\ &\quad \times (1 - \log_{0.1}(0.4))^{0.20} \times (1 - \log_{0.2}(0.6))^{0.22} \\ &\quad \left. \times (1 - \log_{0.2}(0.6))^{0.15} \right\rangle = \langle 0.3006, 0.1992, 0.3709 \rangle \end{aligned}$$

Similarly, for other alternatives, we can obtain $\beta_2 = \langle 0.6147, 0.2764, 0.2421 \rangle$, $\beta_3 = \langle 0.4899, 0.2490, 0.3819 \rangle$, $\beta_4 = \langle 0.4416, 0.3313, 0.2004 \rangle$, and $\beta_5 = \langle 0.2724, 0.1488, 0.4597 \rangle$.

Step 5: The score values of the aggregated number $\beta_i(i = 1, 2, 3, 4, 5)$ corresponding to L-SVNWA operator are $S(\beta_1) = -0.2168, S(\beta_2) = 0.2185, S(\beta_3) = -0.0066, S(\beta_4) = 0.0907$ and $S(\beta_5) = -0.2158$. Thus, the ranking order of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$. On the other hand, these score values of the aggregated number corresponding to L-SVNWG operator are $S(\beta_1) = -0.2695, S(\beta_2) = 0.0962, S(\beta_3) = -0.1410, S(\beta_4) = -0.0900$ and $S(\beta_5) = -0.3362$. Thus, the ranking order of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1 \succ A_5$ in which $\succ$ means "preferred to".

From these ranking order, we see that the best alternative remains A_2 by both the operators while the alternative $A_5 \succ A_1$ for averaging and $A_1 \succ A_5$ for geometric aggregation operator. Thus, according to the decision maker behavior either as an optimistic or pessimistic, they can select the alternatives accordingly and reach their desired goals.

5.2. Validity test

Wang and Triantaphyllou (2008) established the following testing criteria to evaluate the validity of MADM methods.

Test criterion 1: "An effective MADM method does not change the index of the best alternative by replacing a non-optimal alternative with a worse alternative without shifting the corresponding importance of every decision attribute".

Test criterion 2: "To an effective MADM method must be satisfied transitive property".

Test criterion 3: "If we decomposed a MADM problem into the sub DM problems and same MADM method is utilized on subproblems to rank alternatives, the collective ranking of alternatives must be identical to ranking of un-decomposed DM problem".

5.2.1. Validity test by test criterion 1

Under the test criterion 1, we change the rating value for non-optimal alternative A_1 by an arbitrary worse alternative A_1' as $A_1' = \{(C_1, 0.4, 0.4, 0.6), (C_2, 0.3, 0.4, 0.5), (C_3, 0.1, 0.5, 0.4), (C_4, 0.2, 0.3, 0.5), (C_5, 0.2, 0.4, 0.4)\}$ which are summarized as

$$D = \begin{array}{c} \\ A_1' \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{pmatrix} \langle 0.4,0.4,0.6\rangle & \langle 0.3,0.4,0.5\rangle & \langle 0.1,0.5,0.4\rangle & \langle 0.2,0.3,0.5\rangle & \langle 0.2,0.4,0.4\rangle \\ \langle 0.7,0.1,0.3\rangle & \langle 0.7,0.2,0.3\rangle & \langle 0.6,0.3,0.2\rangle & \langle 0.6,0.4,0.2\rangle & \langle 0.7,0.1,0.2\rangle \\ \langle 0.5,0.3,0.4\rangle & \langle 0.6,0.2,0.4\rangle & \langle 0.6,0.1,0.2\rangle & \langle 0.5,0.1,0.3\rangle & \langle 0.6,0.4,0.3\rangle \\ \langle 0.7,0.3,0.2\rangle & \langle 0.7,0.2,0.2\rangle & \langle 0.4,0.5,0.2\rangle & \langle 0.5,0.2,0.2\rangle & \langle 0.4,0.5,0.4\rangle \\ \langle 0.4,0.1,0.3\rangle & \langle 0.5,0.1,0.2\rangle & \langle 0.4,0.1,0.5\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.3,0.2,0.4\rangle \end{pmatrix} \end{array}$$

Then, by utilizing the proposed approach using L-SVNWA operator to this transform data, we get the score values of the alternatives $A_i (i = 1, 2, 3, 4, 5)$ as $-0.7859, 0.2185, -0.0066, 0.0907$ and -0.2158. Thus, ranking order of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1'$ and it validate the test criterion 1.

5.2.2. Validity test by criteria 2 and 3

Under these test, we have decomposed original DM problem into three sub problems with alternatives $\{A_1, A_2, A_4, A_5\}, \{A_1, A_3, A_4, A_5\}$ and $\{A_2, A_3, A_4, A_5\}$. Now, by applying proposed MADM approach on these alternatives by using L-SVNWA approach then we get $H_2 \succ A_4 \succ A_5 \succ A_1, A_4 \succ A_3 \succ A_5 \succ A_1$ and $A_2 \succ A_4 \succ A_3 \succ A_5$ respectively. Therefore, from these, we get the final ranking order as $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$ which is same as the original ranking. Hence, it validates test criteria 2 and 3.

5.3. Influence of logarithm operation and λ selection in practice

Here, we have investigated the influence of the logarithm operations for SVNNs and the selection of the logarithm base parameter λ in practice. From SVNN β, the operation of logarithm is defined as $\log_\lambda \beta = \langle 1 - \log_\lambda \zeta, \log_\lambda (1-\kappa), \log_\lambda (1-\varphi)\rangle$ and $0 < \lambda \leqslant \min\{\zeta, 1-\kappa, 1-\varphi\} < 1$. When $0 < \lambda < 1, \log_\lambda \gamma$ increases as λ increases, however, the larger the value of a real number γ, the smaller the value of $\log_\lambda \gamma$ is.

(P1) From the properties of the logarithm function of real numbers, and from the definition of $\log_\lambda \beta$, we observe that

(a) There always exists a real number $\lambda_1 = \zeta^{\frac{1}{1-\zeta}}$ such that

- if $\lambda = \lambda_1$, then $1 - \log_\lambda \zeta = \zeta$; and
- if $\lambda > \lambda_1$, then $1 - \log_\lambda \zeta < \zeta$; and
- if $\lambda < \lambda_1$, then $1 - \log_\lambda \zeta > \zeta$.

(b) There always exists a real number $\lambda_2 = (1-\kappa)^{\frac{1}{\kappa}}$ such that

- If $\lambda = \lambda_2$, then $\log_\lambda (1-\kappa) = \kappa$;
- If $\lambda > \lambda_2$, then $\log_\lambda (1-\kappa) > \kappa$;
- If $\lambda < \lambda_2$, then $\log_\lambda (1-\kappa) < \kappa$.

(c) There exists a real number $\lambda_3 = (1-\varphi)^{\frac{1}{\varphi}}$ such that

- If $\lambda = \lambda_3$, then $\log_\lambda (1-\varphi) = \varphi$;
- If $\lambda > \lambda_3$, then $\log_\lambda (1-\varphi) > \varphi$;
- If $\lambda < \lambda_3$, then $\log_\lambda (1-\varphi) < \varphi$.

(P2) If we choose a relative small number λ such that $\lambda < \lambda_1 < \lambda_2 < \lambda_3$ and $\lambda < \zeta$ then by part (P1), we have $1 - \log_\lambda \zeta > \zeta, \log_\lambda (1-\kappa) < \kappa$ and $\log_\lambda (1-\varphi) < \varphi$. From this, it implies that $\log_\lambda \beta > \beta$, i.e., the value of SVNN β will be increased after applying the logarithmic operator. In other words, the logarithm operator will enhanced the values of SVNN.

(P3) If we choose the parameter λ in such a way that $\lambda_1 < \lambda < \lambda_2 < \lambda_3$, then we get $1 - \log_\lambda \zeta < \zeta, \log_\lambda (1-\varphi) < \varphi$ and $\log_\lambda (1-\kappa) < \kappa$

which suggests that the value of the truth, indeterminacy and the falsity degrees are decreased after applying the logarithm operator.

(P4) If we choose the parameter λ in such a way that $\lambda_1 < \lambda_2 < \lambda < \lambda_3$, then we get $1 - \log_\lambda \zeta < \zeta$, $\log_\lambda(1-\kappa) > \kappa$ and $\log_\lambda(1-\varphi) < \varphi$ which suggests that the value of the truth and falsity degrees decreases while indeterminacy degree increases after applying the logarithm operator.

(P5) If we choose a relatively large number λ such that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda$ then we get $\log_\lambda \beta < \beta$. That is, the logarithm operator will reduce the value of SVNN.

Therefore, based on this comprehensive evaluation, the decision maker can select the desired value of λ for their suitable task. For instance, if we want to enhance the SVNN which is undervalued for poor information, then we can choose the logarithm operator with a small number λ, and vice versa. This task has been illustrated with the following example.

Example 5.1. Consider $\beta_1 = \langle 0.3, 0.5, 0.3\rangle, \beta_2 = \langle 0.7, 0.2, 0.4\rangle, \beta_3 = \langle 0.2, 0.7, 0.1\rangle$ and $\beta_4 = \langle 0.4, 0.2, 0.8\rangle$ be four SVNNs, which is the achievement of the employ evaluated by the their senior administrator during their promotion interview. In the following, we show the different comprehensive evaluation based on the above results.

(1) If we utilized traditional SVNWA operator with senior administrator weight $\omega = (0.25, 0.25, 0.25, 0.25)^T$ towards the rating, then

$$
\begin{aligned}
S_1 &= \text{SVNWA}(\beta_1,\beta_2,\beta_3,\beta_4) \\
&= \left\langle 1 - \prod_{j=1}^{4}(1-\zeta_j)^{\omega_j}, \prod_{j=1}^{4}(\kappa_j)^{\omega_j}, \prod_{j=1}^{4}(\varphi_j)^{\omega_j} \right\rangle \\
&= \Big\langle 1 - (0.7\times 0.3\times 0.8\times 0.6)^{0.25}, \\
&\quad (0.5\times 0.2\times 0.7\times 0.2)^{0.25}, (0.3\times 0.4\times 0.1\times 0.8)^{0.25}\Big\rangle \\
&= \langle 0.4365, 0.3440, 0.3130\rangle
\end{aligned}
$$

(2) Assume that the values of $\beta_j (j = 1,2,3,4)$ are all undervalued for evaluator's preferences and have re-evaluated according to (P1). In order to get more reasonable results, expert will agree to adjust their scores by using logarithm operator and then by (P1), we get $(\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}) = (0.1791, 0.3046, 0.1337, 0.2172)$, $(\lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{24}) = (0.2500, 0.3277, 0.1791, 0.3277)$ and $(\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34}) = (0.3046, 0.2789, 0.3487, 0.1337)$ be the threshold values of $\beta_j (j = 1,2,3,4)$. Here, we assume that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.15$, then

$$
\begin{aligned}
S_2 &= \text{L-SVNWA}(\beta_1,\beta_2,\beta_3,\beta_4) \\
&= \left\langle 1 - \prod_{j=1}^{4}\left(\log_\lambda \zeta_j\right)^{\omega_j}, \prod_{j=1}^{4}\left(\log_\lambda(1-\kappa_j)\right)^{\omega_j}, \prod_{j=1}^{4}\left(\log_\lambda(1-\varphi_j)\right)^{\omega_j} \right\rangle \\
&= \Big\langle 1 - (\log_{0.15}0.3)^{0.25}\times(\log_{0.15}0.7)^{0.25}\times(\log_{0.15}0.2)^{0.25} \\
&\quad \times(\log_{0.15}0.4)^{0.25}, (\log_{0.15}0.5\times\log_{0.15}0.8\times\log_{0.15}0.3 \\
&\quad \times\log_{0.15}0.8)^{0.25}, (\log_{0.15}0.7\times\log_{0.15}0.6\times\log_{0.15}0.9 \\
&\quad \times\log_{0.15}0.2)^{0.25}\Big\rangle = \langle 0.5298, 0.2380, 0.2210\rangle
\end{aligned}
$$

(3) In order to induce the values of $\beta_j (j = 1,2,3,4)$ reasonable and by taking the different weight of the senior administrator during evaluating the achievement of the employ, we suppose that $\lambda_1 = 0.2, \lambda_2 = 0.4, \lambda_3 = 0.1$ and $\lambda_4 = 0.2$ and $\omega_j = \frac{\lambda_j}{\sum_{j=1}^{5}\lambda_j}$, we get $\omega_1 = 0.2222, \omega_2 = 0.4444$, $\omega_3 = 0.1111$ and $\omega_4 = 0.2223$. Thus, by using L-SVNWA operator to aggregate the different preference, we have

$$
\begin{aligned}
S_3 &= \text{L-SVNWA}(\beta_1,\beta_2,\beta_3,\beta_4) \\
&= \left\langle 1 - \prod_{j=1}^{4}\left(\log_{\lambda_j} \zeta_j\right)^{\omega_j}, \prod_{j=1}^{4}\left(\log_{\lambda_j}(1-\kappa_j)\right)^{\omega_j}, \prod_{j=1}^{4}\left(\log_{\lambda_j}(1-\varphi_j)\right)^{\omega_j} \right\rangle \\
&= \Big\langle 1 - (\log_{0.2}0.3)^{0.2222}\times(\log_{0.4}0.7)^{0.4444}\times(\log_{0.1}0.2)^{0.1111} \\
&\quad \times(\log_{0.2}0.4)^{0.2223}, (\log_{0.2}0.5)^{0.2222}\times(\log_{0.4}0.8)^{0.4444} \\
&\quad \times(\log_{0.1}0.3)^{0.1111}\times(\log_{0.2}0.8)^{0.2223}, (\log_{0.2}0.7)^{0.2222} \\
&\quad \times(\log_{0.4}0.6)^{0.4444}\times(\log_{0.1}0.9)^{0.1111}\times(\log_{0.2}0.2)^{0.2223}\Big\rangle \\
&= \langle 0.4773, 0.2655, 0.3917\rangle
\end{aligned}
$$

(4) Consider the weight of the senior administrator which are going to evaluate the performance of the employs are not equally distributed. For it, we assume that during the evaluation, the weight of second administrator is double than the third ones and half than the one administrator. On the other hand, the importance of the first administrator is double than the fourth administrator. Thus, it implies that $\omega_1 = 4/9, \omega_2 = 2/9$, $\omega_3 = 1/9$ and $\omega_4 = 2/9$ for some acceptable reasons. Further, for the undervalued SVNNs, assume that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.15$, then

$$
\begin{aligned}
S_4 &= \text{L-SVNWA}(\beta_1,\beta_2,\beta_3,\beta_4) \\
&= \left\langle 1 - \prod_{j=1}^{4}\left(\log_\lambda \zeta_j\right)^{\omega_j}, \prod_{j=1}^{4}\left(\log_\lambda(1-\kappa_j)\right)^{\omega_j}, \prod_{j=1}^{4}\left(\log_\lambda(1-\varphi_j)\right)^{\omega_j} \right\rangle \\
&= \Big\langle 1 - (\log_{0.15}0.3)^{4/9}\times(\log_{0.15}0.7)^{2/9}\times(\log_{0.15}0.2)^{1/9} \\
&\quad \times(\log_{0.15}0.4)^{2/9}, (\log_{0.15}0.5)^{4/9}\times(\log_{0.15}0.8)^{2/9} \\
&\quad \times(\log_{0.15}0.3)^{1/9}\times(\log_{0.15}0.8)^{2/9}, (\log_{0.15}0.7)^{4/9} \\
&\quad \times(\log_{0.15}0.6)^{2/9}\times(\log_{0.15}0.9)^{1/9}\times(\log_{0.15}0.2)^{2/9}\Big\rangle \\
&= \langle 0.5293, 0.2347, 0.2486\rangle
\end{aligned}
$$

Hence, based on score function, we get $S_2 > S_4 > S_1 > S_3$ which is in accordance with our expectation.

5.4. Comparison with other existing methods

In order to verify the validity of our method, we make a comparison between our proposed operator with the weighted averaging and geometric aggregation operators as proposed by the authors in (Liu et al., 2014; Nancy & Garg, 2016b; Peng et al., 2016; Ye, 2014, 2016) for multi attribute decision making with SVN information. Under this, if we utilize weighted averaging aggregation operators such as SVNWA (Peng et al., 2016), SVNOWA (Peng et al., 2016), NWA (Ye, 2014), SVNHWA (Liu et al., 2014), and SVNFWA (Nancy & Garg, 2016b) under SVN environment to aggregate the information of each alternative into the collective one, then the aggregated values corresponding to these operators are summarized in Table 1 along with the proposed operators. On the other hand, the aggregated values by using some existing geometric aggregation operators which include the NWG (Ye, 2014), SVNWG(Peng et al.,

Table 1
Neutrosophic aggregated results by averaging operators.

	SVNWA (Peng et al., 2016)	SVNOWA (Peng et al., 2016)	NWA (Ye, 2014)	SVNFWA (Nancy & Garg, 2016b)
A_1	$\langle 0.3779, 0.2259, 0.4002\rangle$	$\langle 0.3820, 0.2449, 0.4071\rangle$	$\langle 0.3779, 0.2314, 0.4223\rangle$	$\langle 0.3755, 0.2262, 0.4018\rangle$
A_2	$\langle 0.6615, 0.2052, 0.2381\rangle$	$\langle 0.6663, 0.1801, 0.2430\rangle$	$\langle 0.6615, 0.2426, 0.2446\rangle$	$\langle 0.6611, 0.2072, 0.2385\rangle$
A_3	$\langle 0.5656, 0.1763, 0.3131\rangle$	$\langle 0.5597, 0.1838, 0.3122\rangle$	$\langle 0.5656, 0.2109, 0.3272\rangle$	$\langle 0.5652, 0.1779, 0.3141\rangle$
A_4	$\langle 0.5722, 0.2929, 0.2219\rangle$	$\langle 0.5706, 0.3145, 0.2219\rangle$	$\langle 0.5722, 0.3348, 0.2338\rangle$	$\langle 0.5692, 0.2956, 0.2225\rangle$
A_5	$\langle 0.4165, 0.1413, 0.3607\rangle$	$\langle 0.3960, 0.1373, 0.3696\rangle$	$\langle 0.4165, 0.1633, 0.4131\rangle$	$\langle 0.4159, 0.1422, 0.3646\rangle$
	SVNHWA (Liu et al., 2014)		L-SVNWA	L-SVNOWA
	$\gamma = 2$	$\gamma = 3$		
A_1	$\langle 0.3725, 0.2264, 0.4033\rangle$	$\langle 0.3693, 0.2266, 0.4048\rangle$	$\langle 0.3130, 0.1753, 0.3544\rangle$	$\langle 0.3229, 0.1926, 0.3607\rangle$
A_2	$\langle 0.6608, 0.2086, 0.2388\rangle$	$\langle 0.6604, 0.2099, 0.2390\rangle$	$\langle 0.6486, 0.1989, 0.2313\rangle$	$\langle 0.6549, 0.1719, 0.2368\rangle$
A_3	$\langle 0.5648, 0.1790, 0.3149\rangle$	$\langle 0.5645, 0.1800, 0.3157\rangle$	$\langle 0.4989, 0.1733, 0.3321\rangle$	$\langle 0.4896, 0.1823, 0.3303\rangle$
A_4	$\langle 0.5663, 0.2978, 0.2230\rangle$	$\langle 0.5635, 0.3000, 0.2234\rangle$	$\langle 0.5585, 0.2736, 0.1942\rangle$	$\langle 0.5561, 0.2975, 0.1942\rangle$
A_5	$\langle 0.4151, 0.1427, 0.3680\rangle$	$\langle 0.4143, 0.1432, 0.3714\rangle$	$\langle 0.2849, 0.1249, 0.3758\rangle$	$\langle 0.2442, 0.1209, 0.3834\rangle$

Table 2
Neutrosophic aggregated results by geometric operators.

	SVNWG(Peng et al., 2016)	SVNOWG(Peng et al., 2016)	NWG (Ye, 2014)	SVNHWG (Liu et al., 2014)	
				$\gamma = 2$	$\gamma = 3$
A_1	$\langle 0.3446, 0.2314, 0.4223\rangle$	$\langle 0.3516, 0.2517, 0.4222\rangle$	$\langle 0.3446, 0.2259, 0.4002\rangle$	$\langle 0.3491, 0.2305, 0.4183\rangle$	$\langle 0.3511, 0.2300, 0.4162\rangle$
A_2	$\langle 0.6561, 0.2426, 0.2446\rangle$	$\langle 0.6612, 0.2173, 0.2497\rangle$	$\langle 0.6561, 0.2052, 0.2381\rangle$	$\langle 0.6570, 0.2374, 0.2437\rangle$	$\langle 0.6576, 0.2338, 0.2430\rangle$
A_3	$\langle 0.5609, 0.2109, 0.3272\rangle$	$\langle 0.5548, 0.2257, 0.3233\rangle$	$\langle 0.5609, 0.1763, 0.3131\rangle$	$\langle 0.5617, 0.2059, 0.3250\rangle$	$\langle 0.5621, 0.2025, 0.3237\rangle$
A_4	$\langle 0.5344, 0.3348, 0.2338\rangle$	$\langle 0.5320, 0.3524, 0.2338\rangle$	$\langle 0.5344, 0.2929, 0.2219\rangle$	$\langle 0.5406, 0.3275, 0.2316\rangle$	$\langle 0.5439, 0.3232, 0.2302\rangle$
A_5	$\langle 0.4078, 0.1633, 0.4131\rangle$	$\langle 0.3882, 0.1555, 0.4065\rangle$	$\langle 0.4078, 0.1413, 0.3607\rangle$	$\langle 0.4091, 0.1603, 0.4046\rangle$	$\langle 0.4097, 0.1582, 0.3997\rangle$
	SVNFWG (Nancy & Garg, 2016b)	SNWEA (Ye, 2016)	L-SVNOWG	L-SVNWG	
A_1	$\langle 0.3471, 0.2311, 0.4204\rangle$	$\langle 0.0058, 0.8315, 0.9665\rangle$	$\langle 0.2771, 0.2359, 0.4040\rangle$	$\langle 0.3006, 0.1992, 0.3709\rangle$	
A_2	$\langle 0.6565, 0.2405, 0.2442\rangle$	$\langle 0.1399, 0.7168, 0.7589\rangle$	$\langle 0.6159, 0.2422, 0.2635\rangle$	$\langle 0.6147, 0.2764, 0.2421\rangle$	
A_3	$\langle 0.5612, 0.2089, 0.3263\rangle$	$\langle 0.0727, 0.6965, 0.8422\rangle$	$\langle 0.4909, 0.2743, 0.3657\rangle$	$\langle 0.4899, 0.2490, 0.3819\rangle$	
A_4	$\langle 0.5373, 0.3317, 0.2329\rangle$	$\langle 0.0362, 0.9206, 0.8428\rangle$	$\langle 0.4278, 0.3678, 0.2126\rangle$	$\langle 0.4416, 0.3313, 0.2004\rangle$	
A_5	$\langle 0.4085, 0.1622, 0.4092\rangle$	$\langle 0.0184, 0.6522, 0.9338\rangle$	$\langle 0.2727, 0.1333, 0.4240\rangle$	$\langle 0.2724, 0.1488, 0.4597\rangle$	

Table 3
Ordering of the alternatives

Existing operators	Ordering	Proposed operators	Ordering
NWA (Ye, 2014)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SVNWA (Peng et al., 2016)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$	L-SVNWA	$A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$
SVNOWA (Peng et al., 2016)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$	L-SVNOWA	$A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$
SVNWG (Peng et al., 2016)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$	L-SVNWG	$A_2 \succ A_4 \succ A_3 \succ A_1 \succ A_5$
SVNOWG (Peng et al., 2016)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$	L-SVNOWG	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$
SVNFWA (Nancy & Garg, 2016b)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SVNHWA (Liu et al., 2014) ($\gamma = 2$)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SVNHWA (Liu et al., 2014) ($\gamma = 3$)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
NWG (Ye, 2014)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SVNFWG (Nancy & Garg, 2016b)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SVNHWG (Liu et al., 2014) ($\gamma = 2$)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SVNHWG (Liu et al., 2014) ($\gamma = 3$)	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$		
SNWEA (Ye, 2016)	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$		

2016), SVNOWG (Peng et al., 2016), SVNFWG (Nancy & Garg, 2016b), SNWEA (Ye, 2016) and SVNHWG(Liu et al., 2014) operators are summarized in Table 2 along with the proposed geometric operators.

By using these collective values and the score functions, the ranking order of the alternatives are summarized in Table 3 in which $\succ$ means "preferred to". From these results, it has been seen that the best alternative is A_2 by all the operators while the different aggregation operators have different ranking strategies which are slightly different. Thus, we can conclude that decision maker reach different decisions based on their preference in terms aggregation operators.

6. Conclusion

In this paper, we present a novel logarithm operational laws (LOL) of SVNSs with the real base number λ which is a useful supplement to the existing operational laws. Also, we have examined their properties and correlations. Based on these LOLs, we developed the weighted averaging and geometric aggregation operators named as L-SVNWA, L-SVNOWA, L-SVNWG, and L-SVNOWG. Then, we utilized these operators to develop a multiattribute decision making approach for solving the practical problem with single-valued neutrosophic fuzzy information. The proposed approach has been verified by an illustrative example. A comparative study with several of the existing approaches is presented to show their superiority as well as the validity of the approach. At last, the influence of the logarithm operations, as well as the selection of the logarithm base λ, are discussed. From the study, it is concluded that the proposed operational laws and the aggregation operators can equivalently solve the decision making problem in a more efficient manner. Also, by assigning a different parameter to base λ, the decision maker can choose the best alternative according to his or her preferences. In the future, the result of this paper can be extended to some other fuzzy and uncertain environment (Garg & Arora, 2018a, 2018b; Peng & Yang, 2017; Rani & Garg, 2017).

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Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.cogsys.2018.09.001.

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