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Numerical solution of fractional optimal control problems via Lagrange polynomials

Sedigheh Sabermahani, Yadollah Ordokhani
Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University
Tehran, Iran
s.saber@alzahra.ac.ir; ordokhani@alzahra.ac.ir

ABSTRACT

A numerical method for solving a class of fractional optimal control problems (FOCPs) is presented. First, the FOCP is transformed into an equivalent variational problem, then using Lagrange polynomials, the problem is reduced to the problem of solving a system of algebraic equations. With the aid of an operational matrix of fractional integration, Gauss quadrature formula and Newton's iterative method for solving a system of algebraic equations, the problem is solved approximately. Approximate solutions are derived by the method satisfy all the initial conditions of the problem. Finally some illustrative examples are included to demonstrate the applicability of the present technique.

KEYWORDS: Fractional optimal control problems, Lagrange polynomials, Operational matrix of Riemann–Liouville fractional integration, Numerical method

1 INTRODUCTION

In the last four decades, it has been shown that the dynamic behavior of many physical systems can be described more accurately with a fractional order model than an integer order one. The modeling of many phenomenon leads to a set of fractional differential equations. Also fractional order dynamics appear in some problems in science and engineering such as, viscoelasticity, bioengineering, etc. When the fractional differential equations are used in conjunction with the performance index and a set of initial conditions, they lead to fractional optimal control problems. Thus, during the last decades many numerical techniques have been developed in this field. The existing numerical methods to solve these problems include Legendre multiwavelet collocation method (Yousefi et al., 2011), method based on Bernoulli polynomials (Keshavarz et al. 2015), Boubaker polynomials (Rabiei et al., 2017), etc.

2 PRELIMINARIES AND NOTATIONS

DEFINITION 2.1. Let $x:[a,b]\to R$ be a function, $\alpha>0$ a real number and $n=\lceil\alpha\rceil$, where $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α, the Riemann-Liouville integral of fractional order is defined as (Sabermahani et al. 2018)

$$I^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}x(\tau)d\tau,$$

For the Riemann-Liouville fractional integrals, we have

$$I^{\alpha}t^{n}=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{\alpha+n},\ \ n>-1, \qquad (1)$$

$$(D^{\alpha}I^{\alpha}x)(t) = x(t),$$

$$(I^{\alpha}D^{\alpha}x)(t) = x(t) - \sum_{i=0}^{\lceil\alpha\rceil-1} x^{(i)}(0)\frac{t^{i}}{i!}.$$

D^{α} denotes fractional derivative in Caputo sense which is present in (Sabermahani et al. 2018).

2.1 Lagrange polynomials

Let the set of nodes be given by $t_i \in [0,1], i = 0,1,\ldots,n$. Lagrange polynomial based on these points can be defined as follows

$$L_i(t) = \prod_{\substack{j=0\\ i\neq j}}^{n} \frac{(t-t_j)}{(t_i-t_j)}.$$

Lemma 1. Let $L_i(t), i = 0,1,\ldots,n$ are the Lagrange polynomials on the set of nodes $t_i \in [0,1]$. Lagrange polynomials in these points are described by (Sabermahani et al. 2018)

$$L_i(t) = \sum_{s=0}^{n} \beta_{is}t^{n-s}, i = 0,1,\ldots,n, \qquad (2)$$

where

$$\beta_{i0} = \frac{1}{\prod_{\substack{j=0\\ i\neq j}}^{n}(t_i-t_j)}$$

$$\beta_{is} = \frac{(-1)^{s}}{\prod_{\substack{j=0\\ i\neq j}}^{n}(t_i-t_j)} \sum_{k_s=k_{s-1}+1}^{n} \cdots \sum_{k_1=0}^{n-s+1} \prod_{r=1}^{s} t_{k_r}, \quad s = 1,2,\ldots,n, \qquad i \neq k_1 \neq \cdots \neq k_s.$$

3 FUNCTION APPROXIMATION

Suppose that $f \in L^2[0,1]$ can be expanded in term of the Lagrange polynomials as

$$f(t) = \sum_{i=0}^{\infty} c_i L_i(t),$$

We can consider the following truncated series for f

$$f(t) \simeq \sum_{i=0}^{n} c_i L_i(t) = C^T L(t), \qquad (3)$$

where $C, L(t)$ are $1 \times (n+1)$ vectors given by

$$C = [c_0, c_1, \ldots, c_n]^T, \qquad L(t) = [L_0(t), L_1(t), \ldots, L_n(t)]^T. \qquad (4)$$

We can derive unknown vector C, as

$$C = D^{-1}\langle f, L\rangle, \qquad D = \langle L(t), L(t)\rangle$$

4 OPERATIONAL MATRIX OF RIEMANN–LIOUVILLE FRACTIONAL INTEGRATION

Let $L(t)$ be Lagrange polynomials vector defined in Eq. (4), then

$$I^{\alpha}L(t) = F^{(\alpha)}L(t),$$

where $F^{(\alpha)}$ is $(n+1)\times(n+1)$ operational of fractional integration of order in the Riemann-Liouville sense. Using Remark 1 and the properties of the operator I^{α}, for $i = 0, 1, \dots, n$, we have

$$I^{\alpha}L_i(t) = I^{\alpha}\left(\sum_{s=0}^{n}\beta_{is}t^{n-s}\right) = \sum_{s=0}^{n}\beta_{is}I^{\alpha}\,t^{n-s} = \sum_{s=0}^{n}w_{i,s}\,t^{n-s+\alpha}, \qquad (5)$$

$$w_{i,s} = \frac{\Gamma(n-s+1)}{\Gamma(n-s+1-\alpha)}\beta_{is}.$$

Now, we can expand $t^{n-s+\alpha}$ in terms of Lagrange polynomials as:

$$t^{n-s+\alpha} \simeq \sum_{j=0}^{n}c_{s,j}L_j(t), \qquad c_{s,j} = \frac{\langle t^{n-s+\alpha}, L_j(t)\rangle}{\langle L_j(t), L_j(t)\rangle},$$

and substitute in Eq. (5), we get

$$I^{\alpha}L_i(t) = \sum_{s=0}^{n}w_{i,s}\sum_{j=0}^{n}c_{s,j}L_j(t) = \sum_{j=0}^{n}(\sum_{s=0}^{n} c_{s,j}w_{i,s})L_j(t) = \sum_{j=0}^{n}\sum_{s=0}^{n}\theta_{i,j,s}L_j(t), \qquad \theta_{i,j,s} = c_{s,j}w_{i,s}\,.$$

We obtain

$$F^{(\alpha)} = \begin{bmatrix} \sum_{s=0}^{n}\theta_{0,0,s} & \sum_{s=0}^{n}\theta_{0,1,s} & \cdots & \sum_{s=0}^{n}\theta_{0,n,s} \\ \sum_{s=0}^{n}\theta_{1,0,s} & \sum_{s=0}^{n}\theta_{1,1,s} & \cdots & \sum_{s=0}^{n}\theta_{1,n,s} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{n}\theta_{n,0,s} & \sum_{s=0}^{n}\theta_{n,1,s} & \cdots & \sum_{s=0}^{n}\theta_{n,n,s} \end{bmatrix}.$$

5 NUMERICAL METHOD

In this study, we focus on the following fractional optimal control problems.

$$\min J(u) = \int_0^1 f\big(t, x(t), u(t)\big)dt, \qquad (6)$$

subject to

$$D^{\alpha}x(t) = g\big(t, x(t)\big) + b(t)u(t), \qquad m-1 < \alpha \le m\,, \qquad (7)$$

with initial conditions

$x(0) = x_0,\ x'(0) = x_1,\ \dots,\ x^{\lceil\alpha\rceil-1}(0) = x_{\lceil\alpha\rceil-1}$,

where f, g and $b \neq 0$ are smooth functions of their arguments. Now, we solve the optimization problem (6, 7) via Lagrange polynomials approximation where $t_i = \frac{i}{n}, i = 0, 1, \dots, n$ are nodes in $[0,1]$. First, we approximate $D^{\alpha}x(t)$ as

$$D^{\alpha}x(t) \simeq C^T L(t),$$

Using operational matrix of fractional integration and property of Riemann-Liouville of integration, we have

$$x(t) = C^T F^{(\alpha)} L(t) + \sum_{i=0}^{\lceil\alpha\rceil-1} x_i \frac{t^i}{i!},$$

and

$$\sum_{i=0}^{\lceil\alpha\rceil-1} x_i \frac{t^i}{i!} \simeq A^T L(t),$$

then, we have

$$x(t) = C^T F^{(\alpha)} L(t) + A^T L(t).$$

Also, using Eq. (7) and above discussion, we have

$$u(t) = \frac{1}{b(t)}\left(D^{\alpha}x(t) - g\big(t, x(t)\big)\right) = \frac{1}{b(t)}\left(C^T L(t) + g(t, C^T F^{(\alpha)} L(t) + \sum_{i=0}^{\lceil\alpha\rceil-1} x_i \frac{t^i}{i!})\right). \quad (8)$$

Then, the following optimization problem is given

$$\min \quad \tilde{J}(u) = \int_0^1 f\left(t, C^T F^{(\alpha)} L(t) + \sum_{i=0}^{\lceil\alpha\rceil-1} x_i \frac{t^i}{i!}, \frac{1}{b(t)}(C^T L(t) + g(t, C^T F^{(\alpha)} L(t) + \sum_{i=0}^{\lceil\alpha\rceil-1} x_i \frac{t^i}{i!}))\right) dt. \quad (9)$$

According to differential calculus, we have the following necessary conditions of optimization

$$\frac{\partial \tilde{J}}{\partial c_i} = 0, \quad i = 0, 1, \dots, n. \quad (10)$$

We solve the system (10), using Newton's iterative method.
System (9) is to some extent complex in view of calculations. In order to simplify calculations, we can solve numerically by Gauss–Legendre integration method.

6 NUMERICAL RESULTS

In this section, we apply the method presented in this paper to solve the following test examples.

Example 1: Assume that we wish to minimize the functional

$$J(u) = \int_0^1 \left(0.625\, x^2(t) + 0.5\, x(t)u(t) + 0.5\, u^2(t)\right)dt,$$

where

$$D^\alpha x(t) = 0.5\, x(t) + u(t), \quad t \in [0,1], 0 < \alpha \le 1,$$

and the condition $x(0) = 1$. The exact solution for $\alpha = 1$, is $\mathrm{J} = 0.3807971$, and the exact value of control variable is

$$\mathrm{u(t)} = \frac{-(\tanh(1-\mathrm{t}) + 0.5)\cosh(1-\mathrm{t})}{\cosh(1)}.$$

Here, we solve this problem by using the present method with $\mathrm{n} = 3$. We present the results for different values of α, in Table 1 and see that as approaches to 1, the numerical values of J converge to the objective value of $\alpha = 1$. Also, Fig. 1 shows the curves for the exact values of control variable and the numerical values of $u(t)$ and $x(t)$ for different values of α. This problem is solved in (El-Kady, 2003) for $\alpha = 1$, with Chebyshev finite difference method for $n = 7$, with the Boubaker polynomials in (Rabiei et al. 2017) for $n = 5$ and the result for these n, are as accurate as our values for $n = 3$.

Table 1: Numerical values of J for different values of α, in Example .1

α	0.5	0.8	0.9	0.99	1
J	0.309498	0.352314	0.366704	0.379407	0.380797

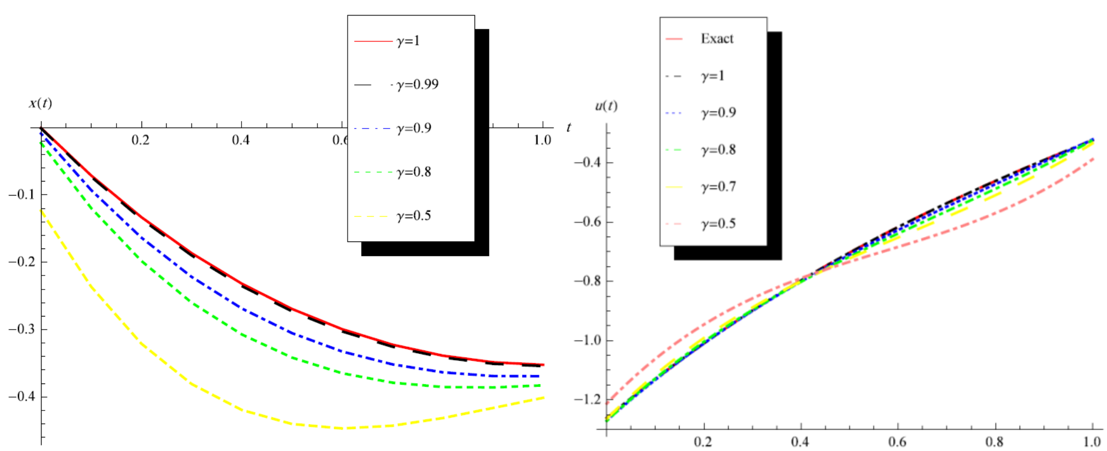

Figure 1: Curves of the exact and numerical values of $x(t)$ and $u(t)$ different values of α, in Example 1.

Example 2: Consider the following time invariant problem

$$J(u) = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \qquad (11)$$

subject to the system dynamics

$$D^\alpha x(t) = -x(t) + u(t), \quad t \in [0,1], \quad 0 < \alpha \le 1,$$

and $x(0) = 1$.
The problem is to find the control $u(t)$, which minimizes the quadratic performance index Eq. (11). For this problem, the exact solution in the case of $\alpha = 1$ is given by J=0.1929092980932,

$$x(t) = cosh(\sqrt{2}t) + wsinh(\sqrt{2}t), \quad u(t) = (1+\sqrt{2}w)\cosh(\sqrt{2}t) + (\sqrt{2}+w)\sinh(\sqrt{2}t),$$

where

$$w = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}.$$

From above equations, we apply our method. Table 2 shows the comparison between the approximation of J obtained using proposed method and Bessel collocation method (Tohidi et al. 2015) for $\alpha = 1$ and different values of m.

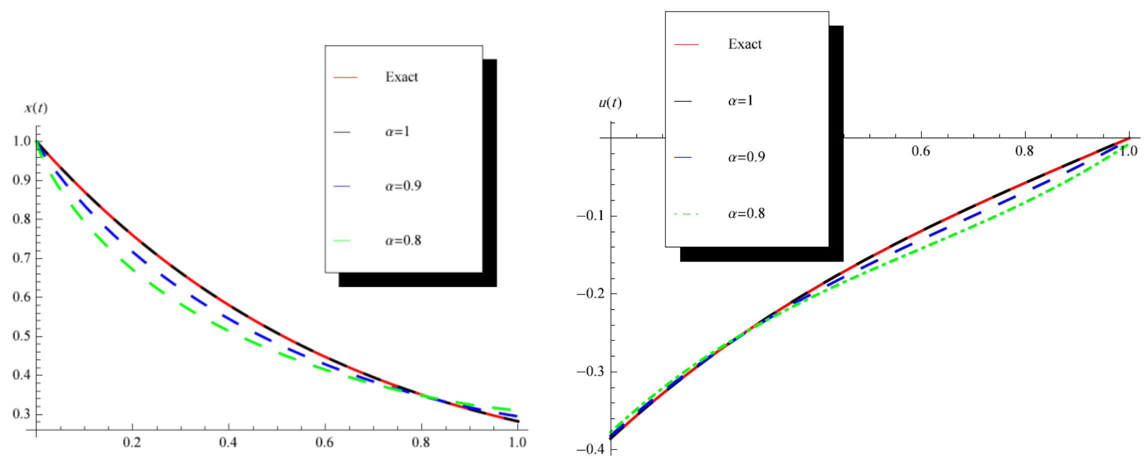

Figure 2: Curves of exact and numerical values of $x(t)$ and $u(t)$ for various of α , in Example 2.

Table 2: Values of J with $\alpha = 1$, for Example 2.

n	Present method	Bessel Collocation Method
4	0.1929092982257	0.1929041515
5	0.1929092980929	0.1929065847

7 CONCLUSION

In this article, a new method for solving a class of fractional optimal control problems is introduced. First the given problem is transformed into an equivalent variational problem, then the variational problem is solved approximately by utilizing the Lagrange polynomials, operational matrix of Riemann–Liouville fractional integration, Gauss quadrature formula and Newton's iterative formula for solving the system of equations. Several illustrative test examples are included to demonstrate the validity and applicability of the new technique.

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