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COMPARISON OF OPTIMAL HOMOTOPY ASYMPTOTIC METHOD WITH ADOMIAN DECOMPOSITION AND HOMOTOPYPERTURBATION METHODSTO SOLVE WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS OF ABEL TYPE

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ABSTRACT

In the present paper , we obtain analytical-approximate solution of Abel Volterra integral equations by using Optimal Homotopy Asymptotic method (OHAM).This approach has been compared with some other powerful and efficient methods such as He's homotopy perturbation method (HPM) and Adomian decomposition method (ADM). This method uses simple computations with quite acceptable approximate solutions, which has close agreement with exact solutions. The accuracy and efficiency of OHAM approach is compared and illustrated by presenting four test examples that satisfy the power of OHAM compared to ADM and HPM methods.

Keywords: Nonlinear singular Volterra integral equations of Abel type, Optimal Homotopy Asymptotic method , Least square method , Homotopy Perturbation method, Asymptotic behavior, Adomian Decomposition method.

2000 AMS Classification: 65R20, 45E10, 45D05

1 INTRODUCTION

Many problems in science and engineering such as solid state physics, plasma physics, fluid mechanics, chemical kinetics and mathematical biology lead to nonlinear singular Volterra integral equations of Abel type as:

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$$u(x) = f(x) + \int_0^x (x-t)^{-\alpha} g(x,t,u(t))dt \qquad (1)$$

in which $f(x)$ is leading term, $u(x)$ is known function , $g(x,t,u(t))$ is unknown nonlinear functional and $0 < \alpha \leq 1$.

In recent years researchers have turned their attention towards solving Volterra integral equations with $0<\alpha\leq 1$ and have represented different methods [10, 11, 30,31]. Also, many powerful and applicable methods have been proposed and applied successfully to approximate many types of non-linear singular integral equations with a wide range of applications [19, 20, 42, 44]. Also, the generalized Abel integral equation on a finite interval was studied by zeilon [53].

Abel Volterra integral equations have been proposed in first and second types. This equation was applied by Niels Abel in 1823 to describe a sliding point mass in a vertical plane on a unknown curve under gravitational force. The point mass starts its motion without initial velocity from a point which has a vertical distance x from the lowest point of the curve [45]. Using the work-energy theorem, the equation of the unknown curve that obtained is the well-known Abel integral equation

$$\int_0^x \frac{g(t)}{\sqrt{x-t}} dt = f(x), \qquad (2)$$

where $f(x)$ is a given function and $g(t)$ is an unknown function.

In this paper, we articulate the concept of OHAM to express a reasonable and reliable method to solve weakly singular integral equations of Abel type. This approach was established by Marinca and Herisanu [22, 32]. Afterwards, they published some papers presented in [33, 34, 35, 36] to show the ability of OHAM to expand their ideas in order to implement it to solve a vast domain of non linear problems. The advantage of OHAM is built in convergence criteria which are similar to HAM but more flexible. Also ,a series of papers by Iqbal et al [23] Iqbal and Javed [24] and Haq [15] have proved the effectiveness, generalization and reliability of this method and obtained solutions of currently important application in science and engineering. In order to explain reliability of the method, we deal with different examples in the subsequent section.

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Finally, numerical comparison between OHAM and other existing methods shows the efficiency of OHAM. Comparison graphs of exact solutions and approximate solutions are also plotted to visualize the performance of OHAM .Since there is a paucity of exact solution of a non linear problems, we may go for approximate analytic solutions.

Many asymptotic techniques are used for solving non linear problems. So by keeping this fact in mind, we have presented a powerful technique OHAM which is generalized form of HPM and HAM. OHAM is simple, straightforward technique and does not require the existence of any small or large parameter as do traditional perturbation methods. OHAM has successfully applied to a number of non-linear problems arising in the science and engineering by various researchers. This proves the validity and acceptability of OHAM as a useful technique [25, 38, 39, 43].

In this paper , we propose Semi-Analytical methods respectively as follows:

(1) Adomian decomposition method(ADM).

(2) Homotopy perturbation method(HPM).

(3) Optimal Homotopy Asymptotic method (OHAM).

We compare Optimal Asymptotic Homotopy Perturbation method to two other different methods namely, He's homotopy perturbation method (HPM) and Adomian Decomposition method (ADM) to solve weakly singular Abel Volterra integral equations of the second kind.

Then , we present four different test examples to show the ability of OHAM method rather to classic ADM and HPM. Also, the results have been compared to each other to show the power of OHAM rather to other two compared methods. Its noticed that using modified versions of ADM and HPM are not possible to be used all the time. However, it seems that for some special cases, these methods give us the closed form of the equation in just one or two iterations. But, it is not applicable to all types of these equations.

To empower ADM and HPM , researchers usually use a combination of Classic Semi-Analytical methods along with some tools such as pad´*e* approximant, Laplace transformations and so on in order to reach to the best approximation just by modifying them in one or two iterations. But, OHAM uses a direct method in two steps by adding

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optimization parameters to homotopy equation that enables us to use it for all types of linear and non linear problems too.

2 ADOMIAN DECOMPOSITION METHOD

The Adomian decomposition method (ADM) [46, 47, 48] is a well-known systematic method for solution of linear or non-linear and deterministic or stochastic operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. The accuracy of the analytic approximate solutions obtained by ADM, can be verified by direct substitution. Advantages of the ADM over Picard's iterated method were demonstrated in [41]. More advantages of the ADM over the variational iteration method were presented in [49, 50]. Adomian and co-workers have solved nonlinear differential equations for a wide class of nonlinearities, including product [1], polynomial [2], exponential [3], trigonometric [4], hyperbolic [5], composite [6], negative-power [7], radical [8] and even decimal-power nonlinearities [9].

We find that the ADM solves nonlinear operator equations for any analytic nonlinearity, providing us with an easily computable and rapidly convergent sequence of analytic approximate functions.

In Adomian decomposition method, we consider the functional equation of Abel integral equation of the form

$$u = f(x) + N(u) \qquad (3)$$

where N is a nonlinear operator and f is a given function. We assume the solution as infinite series for the unknown function $u(x)$, given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4)$$

and then we decompose the non linear term Nu into a series

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$$Nu = \sum_{n=0}^{\infty} A_n \qquad (5)$$

where the A_n, depending on $u_0, u_1, u_2, \cdots, u_n$ are called the Adomian polynomials and are obtained for the nonlinearity $Nu = f(u)$ by the definitional formula:

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[f(\sum_{k=0}^{\infty} \lambda^k u_k)\right]_{\lambda=0}, n = 0,1,2,... \qquad (6)$$

We list the formulas of the first several Adomian polynomials for the one-variable simple analytic nonlinearity $Nu = f(u(x))$ from A_0 through A_4, inclusively, for convenient reference as

$$A_0 = f(u_0)$$
$$A_1 = f'(u_0)u_1$$
$$A_2 = f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}$$
$$A_3 = f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\frac{u_1^3}{3!}$$
$$A_4 = f'(u_0)u_4 + f''(u_0)\left(\frac{u_2^2}{2!} + u_1u_3\right) + f^{(3)}(u_0)\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!},$$

and so on.Substituting Eq.4 and Eq.5 into Abel's integral equation of the form Eq.3 , we get:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \sum_{n=0}^{\infty} A_n(x) \qquad (7)$$

The components $u_0(x), u_1(x), \ldots$ are usually determined by using the recurrence relation:

$$u_0(x) = f(x), u_{n+1}(x) = A_n\left(u_0, u_1, \ldots, u_n\right). \qquad (8)$$

Having determined the components $u_0, u_1, \cdots, u_n$, the solution $u(x)$ of Eq.4 is determined in the form of a rapid convergent power series by substituting the derived components in Eq.8. Thus in order to implement ADM on Abel integral equation, weuse this form of equation :

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$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt. \qquad (9)$$

Substituting Eq.4 into Eq.9, results:

$$\sum_n u_n(x) = f(x) + \int_0^x k(x,t)\sum_n u_n(t)dt. \qquad (10)$$

Then, we can use the following recursive relation to evaluate the various iterations $u_1, u_2, \cdots, u_{n+1}$ as follows :

$$u_0(x) = f(x), u_{n+1}(x) = \int_0^x k(x,t)u_n(t)dt. \qquad (11)$$

Here, we assume that the kernel $k(x,t)$ to be Abel's kernel i.e.

$$k(x,t) = \frac{1}{(x-t)^\alpha}, \quad a \le x \le b \qquad (12)$$

3 BASIC IDEA OF HOMOTOPY PERTURBATION METHOD

In this method, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a small parameter.

This method became very popular among the scientists and engineers, even though it involves continuous deformation of a simple problem into a more difficult problem under consideration. Most of the perturbation methods depend on the existence of a small perturbation parameter but many nonlinear problems have no small perturbation parameter at all. Many new methods have been proposed in the late nineties to solve such nonlinear equation devoid of such small parameters, Dehghan and Shakeri [12, 13], Ganji and Rajabi [14], He [16, 17], Liao [27, 28]. The homotopy perturbation method is considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted to small parameters as occur with traditional perturbation methods. This method can be done in few iterations to obtain highly accurate solutions. When the homotopy theory is coupled with perturbation theory ,it provides a powerful mathematical tool to solve non linear problems. A review of recently developed methods of nonlinear

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analysis can be found in He [18]. To figure out the basic concept of HPM, consider the following nonlinear functional equation

$$A(u) - f(r) = 0, r \in \Omega \qquad (13)$$

with the following boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Omega \qquad (14)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytic function, and Γ is the domain boundary for Ω. A can be divided into two operators L and N, where L is linear and N is non linear so that Eq.13 can be rewritten as

$$L(u) + N(u) - f(r) = 0 \qquad (15)$$

Generally, a homotopy function can be constructed as

$$H(U,p) = (1-p)\left[L(U) - L(u_0)\right] + p\left[L(U) + N(u) - f(r)\right] = 0 \qquad (16)$$

or

$$H(U,p) = L(U) - L(u_0) + p\left[L(U) + N(u) - f(r)\right] = 0 \qquad (17)$$

where p is a homotopy parameter, whose values are within range of 0 and 1, and u_0 is the first approximation for the solution of Eq.13 that satisfies the boundary conditions.

Assuming that solution for Eq.13 or Eq.15 can be written as a power series of p

$$U = v_0 + pv_1 + p^2 v_2 + \cdots \qquad (18)$$

Substituting Eq.18 into Eq.16 or Eq.17 and equating identical powers of p term, there can be found values for the sequence $v_0, v_1, v_2, \cdots$. When $p \to 1$, it yields in the approximate solution for Eq.13 in the form

$$U = v_0 + v_1 + v_2 + \cdots \qquad (19)$$

4 BASIC FORMULATION OF OPTIMAL HOMOTOPY ASYMPTOTIC METHOD (OHAM)

We apply the OHAM to Eq.13 as follows:

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$$L(u(x))+f(x)+N(u(x))=0,\ \ B\left(u,\frac{du}{dx}\right)=0 \qquad (20)$$

where L is a linear operator, N is unknown function and $f(x)$ is known function, $N(u(x))$ is a non-linear operator and B is a boundary operator.

By means of OHAM , one first constructs a family of equation [37];

$$(1-p)\left[L(u(x,p))+f(x)\right]=H(p)\left[L(u(x,p))+f(x)+N(u(x,p))\right]$$
$$B\left(u(x,p),\frac{du(x,p)}{dx}\right)=0 \qquad (21)$$

where $p\in[0,1]$ is an embedding parameter, $H(p)$ is a non-zero auxiliary function for $p\neq 0$, $H(0)=0$ and $u(x,p)$ is an unknown function. Obviously when $p = 0$ and $p = 1$, then it holds

$$u(x,0)=u_0(x), u(x,1)=u_1(x) \qquad (23)$$

respectively. Thus,as$p$ increases from 0 to 1, the solution $u(x,p)$ varies from $u_0(x)$ to the solution $u(x)$, where $u_0(x)$ is obtained from Eq.21 for $p = 0$.

$$L(u_0(x))+f(x)=0 \quad B(u,\frac{du}{dx})=0 \qquad (24)$$

We choose auxiliary function $H(p)$ in the form

$$H(p)=pc_1+p^2c_2+p^3c_3+\cdots, \qquad (25)$$

where $c_1,c_2,\cdots$are constants, which can be determined latter. Let us consider the solution of Eq.21 in the form

$$u(x;p,c_i)=u_0(x)+\sum_{k\geq 1}u_k(x,c_i)p^k, i=1,2,\cdots \qquad (26)$$

Now substituting Eq.26 into Eq.21 and equating the coefficients of like powers of p we obtain the governing equation of $u_0(x)$, given by Eq.24, and the governing equation of $u_k(x)$ as follows :

$$L(u_1(x))=c_1N(u_0(x)),\ B(u_1,\frac{du_1}{dx})=0 \qquad (27)$$

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So, we can obtain general iterative relation as follows :

$$L\left(u_k(x)-u_{k-1}(x)\right)=c_k N_0\left(u_0(x)\right)+\sum_{i=1}^{k-1} c_i[L\left(u_{k-i}(x)\right) \\ +N_{k-i}\left(u_0(x),u_1(x),\cdots,u_{k-1}(x)\right),\ B(u_k,\frac{du_k}{dx})=0,\ k=2,3,\cdots \quad (28)$$

where $N_m\left(u_0(x),u_1(x),\cdots,u_m(x)\right)$ is the coefficient of p^m, obtained by expanding $N\left(u;p,c_i\right)$ in series with respect to the embedding parameter p :

$$N\left(u\left(x;p,c_i\right)\right)=N_0\left(u_0(x)\right)+\sum_{m\geq 1} N_m\left(u_0,u_1,\cdots,u_m\right)p^m, i=1,2,\cdots,m \quad (29)$$

where $u(x;p,c_i)$ is given by Eq.26. It should be emphasized that u_k for $k>0$ are governed by the linear Eqns.24, 27 and Eq.29 with the linear boundary conditions that come from original problem, which can be easily solved. The convergence of the series equation 4.6 depends upon the auxiliary constants $c_1,c_2,\cdots,c_m$. If it is convergent at $p=1$, one has

$$u(x,c_i)=u_0(x)+\sum_{k\geq 1} u_k(x,c_i) \quad (30)$$

Therefore, the solution of Eq.20 can be determined approximately in the form:

$$u^m\left(x,c_i\right)=u_0(x)+\sum_{k=1}^{m} u_k\left(x,c_i\right) \quad (31)$$

Substituting Eq.31 into Eq.20, it results the following residual:

$$R(x;c_i)=L\left(u^m(x,c_i)\right)+g(x)+N\left(u^m(x,c_i)\right), i=1,2,\cdots,m \quad (32)$$

If $R(x;c_i)=0$ then $u_m(x,c_i)$ happens to be the exact solution. For the determination of auxiliary constants $c_i, i=1,2,\cdots,m$, there are different methods like Galerkin's Method, Ritz Method, Least Squares Method and Collocation Method. Generally such case will not arise for nonlinear problems. So in this case, we can minimize the functional by using Least square method as follows:

$$J\left(c_i\right)=\int_a^b R^2\left(x;c_i\right)dx, \quad (33)$$

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where a and b are two values, depending on the given problem. The unknown constants $c_i, i=1,2,\cdots,m$ can be optimally identified from the conditions

$$\frac{\partial J}{\partial c_1}=\frac{\partial J}{\partial c_2}=\cdots=\frac{\partial J}{\partial c_m}=0. \qquad (34)$$

With these known constants, the approximate solution (of order m) Eq.31 is well-determined. The constants c_i can be determined in another forms, for example, if $k\in[a,b]$, $i=1,2,\cdots,m$ and substituting k_i into Eq.32, we obtain the equation :

$$R(k_1,c_i)=R(k_2,c_i)=\cdots=R(k_m,c_i), i=1,2,\cdots,m \qquad (35)$$

Therefore, we can propose advantages and disadvantages of the method as follows:

(1) OHAM sometimes consume a lot of time to evaluate the residual when there are increasing number of convergence constants in the auxiliary function.

Hence, computing of more than three or four convergence constants is not feasible in such cases. As a computational point, time-consuming problems have also been observed by [29] and [40, 51].

(2) Although OHAM gives best approximations but it will not give the closed form solution because of the involvement of the convergence constants c_i in the auxiliary function $H(p)$.

5 APPLICATION OF OHAM TO SOLVE ABEL VOLTERRA WEAKLY SINGULAR INTEGRAL EQUATIONS

In this section, we implement OHAM on general form of weakly singular integral equations of Abel type. Let us consider the equation in the form:

$$u(x)=f(x)+\int_0^x \frac{u(t)}{(x-t)^{\alpha}}\,dt\ ,\ x\in[0,X] \qquad (36)$$

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Now, we construct an optimal homotopy function $u(x,p):\Omega\times[0,1]\to R$ which satisfies in:

$$H(u,p)=(1-p)\{L(u(x,p))+f(x)\}+H(p)\{L(u(x,p))+N(u(x,p))+f(x)\} \qquad (37)$$

where $p\in[0,1]$ is an embedding parameter, $H(p)=\sum_{i=1}^{n}p^i c_i$ and $c_i, i=1,2,\cdots,n$ are auxiliary constants in which must be identified through numerical optimization methods. So for finding the approximate solution of the problem, we use Taylor's series expansion around p as follows:

$$u(x;p;c_i)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x;c_i)p^k \qquad (38)$$

Substituting Eq.38 in Eq.37 and equating the coefficients of like powers of p, we get a series of general governing equations as follows:

$$O(p^0): u_0(x)=f(x)$$

$$O(p^1): u_1(x)=c_1\int_0^x \frac{u_0(t)}{(x-t)^\alpha}\,dt$$

$$\vdots$$

$$O(p^j): u_j(x)=(1+c_1)u_{j-1}(x)+\sum_{i=2}^{j-1}c_i u_{j-i}(x)+\sum_{k=1}^{j}c_k\int_0^x \frac{u_{j-k}(t)}{(x-t)^\alpha}dt$$

By substituting the solutions of the above equations in Eq.36, we obtain an analytic approximate solution of our problem. Here, we note that the constants $c_i, i=1,2,\cdots$ are present and unknown in this series of solution. Thus, for finding these constants as optimal parameters, we form the following residual equation:

$$R(x;c_i)=L(u(x;c_i))+f(x)+\int_0^x \frac{u(t;c_i)}{(x-t)^\alpha}\,dt. \qquad (39)$$

If $R(x;c_i)=0$ then $u_m(x;c_i)$ will be the exact solution. Generally, such a case will not arise for nonlinear problems, but we can minimize the functional

$$J(c_i)=\int_0^x R^2(x;c_i)\,dx. \qquad (40)$$

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The unknown constants $c_i, i=1,2,\cdots,m$ can be optimally identified from the conditions given in section (4).These conditions form a set of normal equations that can be solved by mathematical packages such as Mathematica, Maple and Matlab that provide us a set of complex and real values of constants c_i . So by choosing optimal real parameters from this set of obtained parameters c_i, we can achieve the best approximation for the series of solutions.

6 NUMERICAL EXAMPLES

In this section, we have presented four examples that can be solved by means of ADM, HPM and OHAM. The obtained numerical results based on mentioned methods are illustrated to compare proposed methods with each other. It's noticed that we have continued to computations till fourth iterations for each method to be compared with each other regarding accuracy of approximations by obtaining series of solutions.

6.1. Example 1. We consider the Linear Volterra integral equation with algebric singularity presented in [21]:

$$u(x)=\frac{1}{2}\pi x+\sqrt{x}+\int_0^x -\frac{u(t)}{\sqrt{x-t}}dt, 0\le t\le x\le 1. \qquad (41)$$

The exact solution is $u(x)=\sqrt{x}$.

Case 1 : Adomian decomposition method:

From the recursive relation (11), we obtain

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$$u_0(x) = f(x) = \frac{\pi x}{2} + \sqrt{x}$$
$$u_1(x) = -\int_0^x \frac{u_0(t)}{\sqrt{x-t}}dt = -\frac{1}{6}\pi\left(4\sqrt{x}+3\right)x,$$
$$u_2(x) = \frac{1}{12}\pi\left(3\pi\sqrt{x}+8\right)x^{\frac{3}{2}},$$
$$u_3(x) = -\frac{1}{60}\pi^2\left(16\sqrt{x}+15\right)x^2,$$
$$u_4(x) = \frac{1}{60}\pi^2\left(5\pi\sqrt{x}+16\right)x^{\frac{5}{2}}$$

Then, we have:

$$\begin{aligned} u(x) = \sum_{i=0}^{4} u_i(x) &= \frac{1}{60}\pi^2\left(5\pi\sqrt{x}+16\right)x^{\frac{5}{2}} + \frac{1}{12}\pi\left(3\pi\sqrt{x}+8\right)x^{\frac{3}{2}} \\ &-\frac{1}{60}\pi^2\left(16\sqrt{x}+15\right)x^2 - \frac{1}{6}\pi\left(4\sqrt{x}+3\right)x + \frac{\pi x}{2} + \sqrt{x} \end{aligned} \tag{42}$$

Case 2: Homotopy Perturbation method

A homotopy perturbation equation can be written for Eq.41 as follows:

$$H(u,p) = u(x) - \sqrt{x} - \frac{\pi x}{2} + p\int_0^x \frac{u(t)}{\sqrt{x-t}}dt \tag{43}$$

Now, we can try to obtain a solution for Eq.41 in the form of

$$u(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \cdots \tag{44}$$

Where $u_i(x), i = 0,1,\cdots$ are functions which must be determined. From the Eqns.43 and 44, the approximations $u_i(x)$ are as follows :

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$$O\left(p^0\right): u_0(x)=\sqrt{x}+\frac{\pi}{2}x$$

$$O\left(p^1\right): u_1(x)+\int_0^x \frac{u_0(t)}{\sqrt{x-t}}dt=0 \Rightarrow u_1(x)=-\frac{1}{2}\pi x-\frac{2}{3}\pi x^{\frac{3}{2}}$$

$$O\left(p^2\right): u_2(x)=\frac{1}{4}\pi^2 x^2+\frac{2}{3}\pi x^{\frac{3}{2}}$$

$$O\left(p^3\right): u_3(x)=-\frac{1}{4}\pi^2 x^2-\frac{4}{15}\pi^2 x^{\frac{5}{2}}$$

$$O\left(p^4\right): u_4(x)=\frac{1}{60}\pi^2\left(5\pi\sqrt{x}+16\right)x^{\frac{5}{2}}$$

Then, the series solution is given by :

$$u(x)=\sum_{i=0}^{4}u_i(x)=\frac{1}{60}\pi^2\left(5\pi\sqrt{x}+16\right)x^{\frac{5}{2}}-\frac{4}{15}\pi^2 x^{\frac{5}{2}}+\sqrt{x}. \qquad (45)$$

Case 3: Optimal Homotopy Asymptotic method:

The OHAM formulation of the above example is:

The linear section is considered as:

$$L\left(u(x;p)\right)=u(x)-\frac{1}{2}\pi x-\sqrt{x} \qquad (46)$$

and the non-linear section is defined as:

$$N(u(x,p))=\int_0^x \frac{u(t)}{\sqrt{x-t}}dt \qquad (47)$$

and the leading term $f(x)$ is:

$$f(x)=\frac{1}{2}\pi x+\sqrt{x} \qquad (48)$$

which satisfies in the homotopy function as follows:

$$H(u,p)=(1-p)\left\{u(x)-\frac{1}{2}\pi x-\sqrt{x}\right\}=$$
$$\left(pc_1+p^2c_2+p^3c_3+\cdots\right)\left\{u(x)-\frac{1}{2}\pi x-\sqrt{x}+\int_0^x \frac{u(t)}{\sqrt{x-t}}dt\right\} \qquad (49)$$

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By equating the coefficients of like powers of p, we get a sequence of solutions as follows:

$$O\left(p^0\right): u_0(x)=\frac{1}{2}\pi x+\sqrt{x}$$

$$O\left(p^1\right): u_1(x)=c_1\int_0^x \frac{u_0(t)}{\sqrt{x-t}}dt=c_1\left(\frac{1}{6}\pi(3+4\sqrt{x})x\right)$$

$$O\left(p^2\right): u_2(x)=u_1(x)+c_1^2\left(\frac{1}{6}\pi(3+4\sqrt{x})x\right)+c_2\int_0^x \frac{u_0(t)}{\sqrt{x-t}}dt+c_1\int_0^x \frac{u_1(t)}{\sqrt{x-t}}dt=$$
$$c_1\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)+c_1^{\,2}\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)+c_1^{\,2}\left(\frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}\right)+c_2\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)$$

$$O(p^3): u_3(x)=u_2(x)-c_3\sqrt{x}+c_3\int_0^x \frac{u_0(t)}{\sqrt{x-t}}dt+c_2\int_0^x \frac{u_1(t)}{\sqrt{x-t}}dt+c_1\int_0^x \frac{u_2(t)}{\sqrt{x-t}}dt=$$
$$c_1\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)+c_1^{\,2}\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)+c_1^{\,2}\left(\frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}\right)+c_2\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)$$
$$-c_3\sqrt{x}+c_3\left(\frac{1}{6}\pi\left(3+4\sqrt{x}\right)x\right)+c_2c_1\left(\frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}\right)+c_1^{\,2}\left(\frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}\right)+$$
$$c_1^{\,3}\left(\frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}\right)+c_1^{\,3}\left(\frac{1}{60}\pi^2\left(15+16\sqrt{x}\right)x^2\right)+c_2c_1\left(\frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}\right)$$

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$$O(p^4): u_4(x) = u_3(x) - c_4\sqrt{x} + c_3\int_0^x \frac{u_1(t)}{\sqrt{x-t}}dt + c_2\int_0^x \frac{u_2(t)}{\sqrt{x-t}}dt + c_1\int_0^x \frac{u_3(t)}{\sqrt{x-t}}dt$$

$$= \frac{1}{6}\pi\left(3+4\sqrt{x}\right)xc_1 + \frac{1}{6}\pi\left(3+4\sqrt{x}\right)xc_1^2 + \frac{1}{6}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}c_1^2 + \frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}c_1^3$$
$$+\frac{1}{60}\pi^2\left(15+16\sqrt{x}\right)x^2c_1^3 + \frac{1}{6}\pi\left(3+4\sqrt{x}\right)xc_2 + \frac{1}{6}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}c_1c_2$$
$$+\frac{1}{60}\pi x^{3/2}c_2\left(5\left(8+3\pi\sqrt{x}\right)c_1 + 2\left(20+15\pi\sqrt{x}+8\pi x\right)c_1^2 + 5\left(8+3\pi\sqrt{x}\right)c_2\right)$$
$$-\sqrt{x}c_3 + \frac{1}{6}\pi\left(3+4\sqrt{x}\right)xc_3 + \frac{1}{12}\pi\left(8+3\pi\sqrt{x}\right)x^{3/2}c_3 + \frac{1}{60}\pi xc_1(\sqrt{x}\left(40+45\pi\sqrt{x}+32\pi x\right)c_1^2$$
$$+\pi x\left(15+32\sqrt{x}+5\pi x\right)c_1^3 + c_1\left(5\left(8\sqrt{x}+3\pi x\right)+2\pi\left(15+16\sqrt{x}\right)xc_2\right)$$
$$+5\left(\left(8\sqrt{x}+3\pi x\right)c_2 + \left(-6+8\sqrt{x}+3\pi x\right)c_3\right)) - \sqrt{x}c_4$$

Our experience shows that after fourth iteration, we will reach to favorite approximate solution. Then, the series solution is given by:

$$u(x) = \sum_{i=0}^{4} u_i(x) = \frac{1}{60}\sqrt{x}\left(\pi x\left(120+105\pi\sqrt{x}+64\pi x\right)\right)c_1^3$$
$$+\pi^2 x^{\frac{3}{2}}(15+32\sqrt{x}+5\pi x)c_1^4 + 2\pi\sqrt{x}c_1^2(45(1+4\sqrt{x}+\pi x)$$
$$+(20\sqrt{x}+30\pi x+24\pi x^{\frac{3}{2}})c_2) + 5\pi\sqrt{x}c_1(6+3+4\sqrt{x}) + 6(8\sqrt{x}+3\pi x)c_2$$
$$+(-6+8\sqrt{x}+3\pi x)c_3) + 5(6\pi(3\sqrt{x}+4x)c_2 + \pi(8+3\pi\sqrt{x})xc_2^2$$
$$+3(-8+4\pi\sqrt{x}+8\pi x+\pi^2 x^{\frac{3}{2}})c_3 + 4(3+3\pi\sqrt{x}+2\pi x-3c_4))),$$

and by using Least square method presented in section 4, we find real optimal parameters $c_i, i = 1,2,3,4$ among a set of complex and real roots as follows:

$$c_1 = 0.0024038192517472404, c_2 = -0.5036184105906896$$
$$c_3 = -0.061276454393664505, c_4 = 0.2369720849104333$$

Therefore, the series solution till fourth term is given by:

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$$u(x) = \sum_{i=0}^{4} u_i(x) = -0.0000228307x^{\frac{5}{2}} - 0.924054x^{\frac{3}{2}}$$
$$+8.627314064854607.10^{-11}.x^3 + 0.456388x^2 + 0.587427x + 0.885581\sqrt{x} \tag{50}$$

So, the result of approximate solution obtained by OHAM is compared to solutions of ADM and HPM along with exact solution in Fig.1.

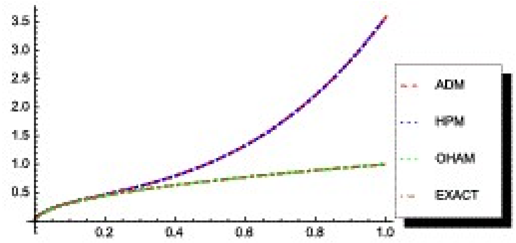

Fig. 1 - Graph of approximate and exact solutions of Example 1

6.2. Example 2. We consider the singular Volterra integral equation in the form [52]

$$u(x) = 2\sqrt{x} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt, 0 \le x \le 1 \tag{51}$$

which has $u(x) = 1 - e^{\pi x} erfc\left(\sqrt{\pi x}\right)$ as the exact solution, where the complementary error function $erfc$ is defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} dy.$$

Case 1: Adomian Decomposition Method:

For this problem, the components $u_i(x)$ are obtained as follows:

$$u_0(x) = 2\sqrt{x}, u_1(x) = -\pi x, u_2(x) = \frac{4}{3}\pi x^{\frac{3}{2}}, u_3(x) = -\frac{\pi^2 x^2}{2}, u_4(x) = \frac{8}{15}\pi^2 x^{\frac{5}{2}}$$

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The general series solution is obtained as:

$$u(x)=\sum_{i=0}^{\infty}u_i(x)=2\sqrt{x}-\pi x+\frac{4}{3}\pi x^{\frac{3}{2}}-\frac{1}{2}\pi^2x^2+\frac{8}{15}\pi^2x^{\frac{5}{2}}-\frac{1}{6}\pi^3x^3+\frac{16}{105}\pi^3x^{\frac{7}{2}}-\cdots$$
$$=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\pi x)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}+1\right)}=1-E_{\frac{1}{2}}\left(-\sqrt{\pi x}\right)=1-e^{\pi x}erfc\left(\sqrt{\pi x}\right), \tag{52}$$

where $u(x)$ is the exact solution. Thus, the series solution till fourth iteration is given by

$$u(x)=\sum_{i=0}^{4}u_i(x)=2\sqrt{x}-\pi x+\frac{4}{3}\pi x^{\frac{3}{2}}-\frac{1}{2}\pi^2x^2+\frac{8}{15}\pi^2x^{\frac{5}{2}} \tag{53}$$

Case 2: Homotopy perturbation method:

The homotopy perturbation function is constructed from Eq.51 as follows:

$$H(u,p)=u(x)-2\sqrt{x}+p\int_0^x\frac{u(t)}{\sqrt{x-t}} \tag{54}$$

The components $u_i(x), i=0,1,2,\cdots$ are computed sequentially as follows:

$$u_0(x)=2\sqrt{x},\ u_1(x)=-\pi x,\ u_2(x)=\frac{4}{3}\pi x^{\frac{3}{2}}, u_3(x)=-\frac{\pi^2x^2}{2}, u_4(x)=\frac{8}{15}\pi^2x^{\frac{5}{2}}$$

Therefore, the general series solution is given as follows:

$$u(x)=\sum_{i=0}^{\infty}u_i(x)=2\sqrt{x}-\pi x+\frac{4}{3}\pi x^{\frac{3}{2}}-\frac{1}{2}\pi^2x^2+\frac{8}{15}\pi^2x^{\frac{5}{2}}-\cdots$$
$$=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\pi x)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}+1\right)}=1-E_{\frac{1}{2}}\left(-\sqrt{\pi x}\right)=1-e^{\pi x}erfc\left(\sqrt{\pi x}\right), \tag{55}$$

where $E_\alpha(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+1)},(\alpha>0)$ is the Mittag-Leffler function in one parameter and $u(x)$ is the exact solution.

Then, the series solution till fourth term is given as:

$$u(x)=\sum_{i=0}^{4}u_i(x)=2\sqrt{x}-\pi x+\frac{4}{3}\pi x^{\frac{3}{2}}-\frac{1}{2}\pi^2x^2+\frac{8}{15}\pi^2x^{\frac{5}{2}}. \tag{56}$$

Case 3: Optimal Homotopy Asymptotic method

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The OHAM formulation of the above example is:

The linear section is considered as:

$$L(u(x;p)) = u(x) - 2\sqrt{x} \qquad (57)$$

and the non-linear section is defined as:

$$N(u(x,p)) = \int_0^x \frac{u(t)}{\sqrt{x-t}}dt \qquad (58)$$

and the leading term $f(x)$ is:

$$f(x) = 2\sqrt{x} \qquad (59)$$

which satisfies in the homotopy function as follows:

$$\begin{gathered} H(u,p) = (1-p)\left(u(x) - 2\sqrt{x}\right) = \\ \left(pc_1 + p^2c_2 + p^3c_3 + \cdots\right)\left\{u(x) - 2\sqrt{x} + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt\right\} \end{gathered} \qquad (60)$$

The same as process which was done in example 1, we find optimal parameters $c_i, i = 1,2,3,4$ among a set of complex and real roots as follows:

$$\begin{aligned} c_1 &= -3.1590023647687975, \quad c_2 = -2.28938783997985, \\ c_3 &= 0.006314235025025266, \quad c_4 = -2.5452866240998273 \end{aligned}$$

So, the series solution is given as follows:

$$u(x) = \sum_{i=0}^{4} u_i(x) = 5.26379x^{\frac{5}{2}} + 10.2671x^{\frac{3}{2}} - 12.2235x^2 - 4.55633x + 2\sqrt{x}. \qquad (61)$$

The graph of approximate and exact solutions till fourth iteration is given in Fig.2.

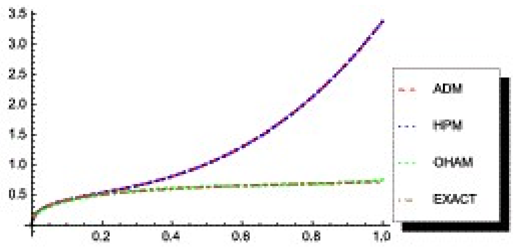

Fig. 2 - Graph of approximate and exact solutions of Example 2

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6.3. Example 3. Consider the linear generalized Abel's integral equation

$$u(x)=x^2+\frac{27}{40}x^{\frac{8}{3}}-\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{3}}}dt \qquad (62)$$

The exact solution is $y(x)=x^2$.

Case 1: Adomian decomposition method:

After computations, the components $u_i(x), i=0,1,\cdots,$ are given as follows:

$$u_0(x)=x^2+\frac{27}{40}x^{\frac{8}{3}}$$

$$u_1(x)=-\frac{27}{40}x^{\frac{8}{3}}-2\frac{\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{13}{3}\right)}x^{\frac{10}{3}}$$

$$u_2(x)=\frac{864.2^{\frac{2}{3}}\sqrt{3}\pi}{21505}x^{\frac{23}{6}}+\frac{27\sqrt{\pi}\Gamma\left(\frac{11}{3}\right)}{40\Gamma\left(\frac{25}{6}\right)}x^{\frac{19}{6}}$$

$$u_3(x)=-\frac{27\pi x^{\frac{11}{3}}}{172040}\left(\frac{256.2^{\frac{2}{3}}\sqrt{3}\pi x^{\frac{2}{3}}\Gamma\left(\frac{29}{6}\right)}{\Gamma\left(\frac{16}{3}\right)}+1173\right)$$

$$u_4(x)=\frac{5184.2^{\frac{2}{3}}\sqrt{3}\pi^2 x^{\frac{29}{6}}}{623645}+\frac{81\pi^{\frac{3}{2}}x^{\frac{25}{6}}\Gamma\left(\frac{14}{3}\right)}{440\Gamma\left(\frac{31}{6}\right)}.$$

Then the series solution till fourth term is given by:

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$$u(x)=\sum_{i=0}^{4}u_i(x)=\frac{5184.2^{\frac{2}{3}}\sqrt{3}\pi^2}{623645}x^{\frac{29}{6}}+\frac{864.2^{\frac{2}{3}}\sqrt{3}\pi}{21505}x^{\frac{23}{6}}+\frac{81\pi^{\frac{3}{2}}\Gamma\left(\frac{14}{3}\right)}{440\Gamma\left(\frac{31}{6}\right)}x^{\frac{25}{6}}+\frac{27\sqrt{\pi}\Gamma\left(\frac{11}{3}\right)}{40\Gamma\left(\frac{25}{6}\right)}$$

$$-\frac{27\pi\left(\frac{256.2^{\frac{2}{3}}\sqrt{3}\pi\Gamma\left(\frac{29}{6}\right)}{\Gamma\left(\frac{16}{3}\right)}+1173\right)}{172040}x^{\frac{11}{3}}-\frac{2\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{13}{3}\right)}x^{\frac{10}{3}}+\frac{27\sqrt{\pi}\Gamma\left(\frac{11}{3}\right)}{40\Gamma\left(\frac{25}{6}\right)}x^{\frac{19}{6}}+x^2$$

Case 2: Homotopy perturbation method:

A homotopy perturbation function can be constructed as follows:

$$H(u,p)=u(x)-\left(x^2+\frac{27}{40}x^{\frac{8}{3}}\right)+p\int_0^x\frac{u(t)}{(x-t)^{\frac{1}{3}}}dt \qquad (63)$$

So, the various components $u_i(x), i=0,1,\cdots$ are obtained as:

$$u_0(x)=\frac{27}{40}x^{\frac{8}{3}}+x^2,$$

$$u_1(x)=-\frac{27}{40}x^{\frac{8}{3}}-\frac{2x^{\frac{10}{3}}\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{13}{3}\right)}x^{\frac{10}{3}}$$

$$u_2(x)=\frac{27\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{11}{3}\right)}{40\Gamma\left(\frac{13}{3}\right)}+\frac{1}{12}x^4\Gamma\left(\frac{2}{3}\right)^3,$$

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$$u_3(x) = -\frac{\left(729x^{\frac{2}{3}} + 1540\right)\Gamma\left(\frac{2}{3}\right)^3}{18480}x^4,$$

$$u_4(x) = -\frac{2\Gamma\left(\frac{2}{3}\right)^5}{\Gamma\left(\frac{19}{3}\right)}x^{\frac{16}{3}} + \frac{243\Gamma\left(\frac{2}{3}\right)^3}{6160}x^{\frac{14}{3}}$$

So, the series solution till fourth term is given by:

$$u(x) = \frac{2\Gamma\left(\frac{2}{3}\right)^5}{\Gamma\left(\frac{19}{3}\right)}x^{\frac{16}{3}} + \frac{243\Gamma\left(\frac{2}{3}\right)^3}{6160}x^{\frac{14}{3}} + \frac{27\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{11}{3}\right)}{40\Gamma\left(\frac{13}{3}\right)}x^{\frac{10}{3}} \tag{64}$$

$$-\frac{2\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{13}{3}\right)}x^{\frac{10}{3}} + \frac{1}{12}x^4\Gamma\left(\frac{2}{3}\right)^3 + x^2 - \frac{\left(729x^{\frac{2}{3}} + 1540\right)\Gamma\left(\frac{2}{3}\right)^3}{18480}x^4.$$

Case 3: Optimal Homotopy Asymptotic method:

The OHAM formulation of the above example is:

The linear section is considered as:

$$L\left(u\left(x;p\right)\right) = u(x) - x^2 - \frac{27}{40}x^{\frac{8}{3}}, \tag{65}$$

and the non-linear section is defined as:

$$N(u(x,p)) = \int_0^x \frac{u(t)}{\sqrt{x-t}}dt \tag{66}$$

and the leading term $f(x)$ is:

$$f(x) = x^2 + \frac{27}{40}x^{\frac{8}{3}} \tag{67}$$

which satisfies in the homotopy function as follows:

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$$H(u,p)=(1-p)\left\{u(x)-x^2-\frac{27}{40}x^{\frac{8}{3}}\right\}=$$
$$\left(pc_1+p^2c_2+p^3c_3+\cdots\right)\left\{u(x)-x^2-\frac{27}{40}x^{\frac{8}{3}}+\int_0^x\frac{u(t)}{\sqrt{x-t}}dt\right\} \tag{68}$$

After computations, we find optimal real parameters $c_i, i=1,2,3,4$ among a set of complex and real roots as follows:

$$c_1=0.6793587312516111,\ c_2=-1.9291411777146121,$$
$$c_3=3.127385017446066,\ \ c_4=-7.440541189887519$$

Then, the series solution is given by:

$$u(x)=\sum_{i=0}^{4}u_i(x)=x^2(0.0133399x^{\frac{10}{3}}-0.0402786x^{\frac{8}{3}}-0.0214071x^{\frac{4}{3}}$$
$$+0.00373266x^{\frac{2}{3}}+0.0446198x^2+1). \tag{69}$$

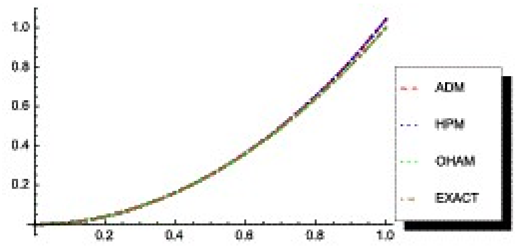

Fig. 3 - Graph of approximate and exact solutions of Example 3

The graph of approximate and exact solutions till fourth iteration is given in Fig.3. In this example, we observe that the curves of approximate and exact solutions roughly coincidence on each other. Also, we can conclude that the power of OHAM in 4-th iteration is more than ADM and HPM. It should be noted that the best reason to justify this phenomena can be asymptotic behavior of singular Volterra integral equations when $\alpha \neq \frac{1}{2}$ that has been comprehensively studied in [26]. Also, its noticed that the computations must be done till fourth iteration to obtain suitable approximate solutions close to exact solution for all three compared methods.

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6.4. Example 4. Consider the singular Volterra integral equation

$$u(x)=\frac{1}{\sqrt{x}}+\pi-\int_0^x \frac{u(t)}{\sqrt{x-t}}dt, 0\le x\le 1, \qquad (70)$$

which has $u(x)=\frac{1}{\sqrt{x}}$ as the exact solution.

Case 1: Adomian decomposition method

After computations, the components $u_i(x), i=0,1,\cdots$ are given as follows:

$$u_0(x)=\frac{1}{\sqrt{x}}+\pi,$$
$$u_1(x)=-\pi-2\pi\sqrt{x},$$
$$u_2(x)=2\pi\left(\sqrt{x}+\frac{\pi x}{2}\right),$$
$$u_3(x)=\frac{1}{6}\pi^2\left(8+3\pi\sqrt{x}\right)x^{\frac{3}{2}},$$
$$u_4(x)=-\frac{1}{30}\pi^3\left(15+16\sqrt{x}\right)x^2$$

Then, the series solution is given by:

$$u(x)=\sum_{i=0}^{4}u_i(x)=\frac{1}{6}\pi^2\left[\left(3\pi\sqrt{x}+8\right)x^{\frac{3}{2}}-\pi^2\left(8\sqrt{x}+6\right)x\right]+\pi^2 x+\frac{1}{\sqrt{x}} \qquad (71)$$

Case 2: Homotopy perturbation method :

A homotopy perturbation function can be constructed as follows:

$$H(u,p)=u(x)-\frac{1}{\sqrt{x}}-\pi+p\int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt \qquad (72)$$

So, the sequence of solutions $u_i(x)$, $i=1,2,\cdots$ are given as follows:

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$$\begin{aligned}
u_0(x) &= \frac{1}{\sqrt{x}} + \pi, \\
u_1(x) &= -\pi - 2\sqrt{x}, \\
u_2(x) &= \pi^2 x + 2\pi\sqrt{x}, \\
u_3(x) &= -\pi^2 x - \frac{4}{3}\pi^2 x^{\frac{3}{2}} \\
u_4(x) &= -\frac{1}{6}\pi^2\left(3\pi\sqrt{x}+8\right)x^{\frac{3}{2}}
\end{aligned}$$

Then, we have series solution as follows:

$$u(x) = \sum_{i=0}^{4} u_i(x) = -\frac{1}{6}\pi^2\left(3\pi\sqrt{x}+8\right)x^{\frac{3}{2}} - \frac{4}{3}\pi^2 x^{\frac{3}{2}} + \frac{1}{\sqrt{x}}. \qquad (73)$$

Case 3: Optimal Homotopy Asymptotic method

The OHAM formulation of the above example is:

The linear section is considered as:

$$L\big(u(x;p)\big) = u(x) - \frac{1}{\sqrt{x}} - \pi, \qquad (74)$$

and the non-linear section is defined as:

$$N\big(u(x;p)\big) = \int_0^x \frac{u(t)}{\sqrt{x-t}}dt \qquad (75)$$

and the leading term $f(x)$ is:

$$f(x) = \frac{1}{\sqrt{x}} + \pi \qquad (76)$$

which satisfies in the optimal homotopy function as follows:

$$\begin{aligned}
&H(u,p) = (1-p)\left\{u(x) - \frac{1}{\sqrt{x}} - \pi\right\} = \\
&\left(pc_1 + p^2c_2 + p^3c_3 + \cdots\right)\left\{u(x) - x^2 - \frac{27}{40}x^{\frac{8}{3}} + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt\right\}
\end{aligned} \qquad (77)$$

After computations, we find optimal real parameters $c_i, i=1,2,3,4$ among a set of complex and real roots as follows:

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$$c_1 = 0.11452179930429117, c_2 = -0.6018928997443851,$$
$$c_3 = 0.5453895225845069, c_4 = -0.2562489807524551,$$

Then, after computations, the series solution is given by:

$$u(x) = \sum_{i=0}^{4} u_i(x) =$$
$$1.14823\left\{1 - \frac{0.223526}{1.14823}x^{\frac{3}{2}} + \frac{0.00266669}{1.14823}x^2 + \frac{0.990074}{1.14823}x - \frac{1.63856}{1.14823}\sqrt{x} + \frac{0.743751}{1.14823\sqrt{x}}\right\} \quad (78)$$

The graph of approximate and exact solutions till fourth iteration is given in Fig. 4.

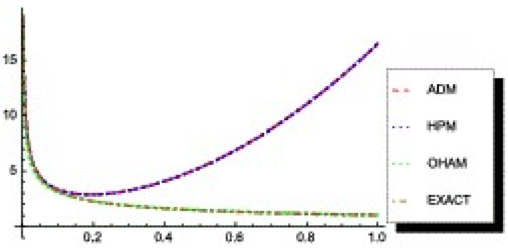

Fig. 4 - Graph of approximate and exact solutions of Example 4

7 RESULTS AND DISCUSSION

The purpose of the present paper is to propose the power of OHAM in order to find the solutions of weakly singular Volterra integral equations of Abel type rather to ADM and HPM methods. We implemented all computations in a laptop by processor 2.53 GHz and we could handle them in OHAM till 4-th iteration. Because, after 4-th iteration, we confront with high cost of computations that occupy a vast space of our memory and its natural that the process of computations takes a long time especially to compute integral components directly. If we have more powerful processors, we can enhance accuracy of the OHAM approximations. In fact, we have presented a powerful performance of OHAM compared to classic analytic-approximation techniques such as ADM and HPM. We can conclude that the obtained results of OHAM method can be applied to solve weakly singular Volterra integral equations of Abel type in just few iterations and make favorite approximations close to exact solutions of mentioned equations, whereas our experience shows that ADM and HPM are not

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capable to solve equations in few iterations. It means that by using ADM and HPM, we can find a suitable series solution close to exact solution of expressed examples provided that we continue to computations in more iterations. It can be said that in spite of the volume of computations , the OHAM method is very powerful, reliable, efficient and accurate compared to many competitive numerical methods that sometimes equations need to be changed before solving the problem by some approaches such as change of variables, Laplace transform and so on. Therefore, we can perform OHAM directly on any favorite problem without any concern. Programming the OHAM method is very easy and we can modify the method for solving stiff types of singular integral equations by approximating the integral components that take a long time while computing by many computers that have different type of processors.

8 CONCLUSION

In this research paper , we have implemented ADM , HPM and OHAM on singular integral equations of Abel type. Obtained results demonstrate efficiency of OHAM rather to ADM and HPM. We can conclude that the power of OHAM is enough to obtain approximate solutions with best accuracy, Whereas ADM and HPM methods need to use many iterations to reach the best approximate solution.

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10 REFERENCES

[1] G. Adomian, On product nonlinearities in stochastic differential equations, Appl.Math. Comput. ,1981, 8,79-82.

[2] G. Adomian, R. Rach, Polynomial nonlinearities in differential equations, J. Math. Anal. Appl ,1985,109 , 90-95.

VOLUME 1, NÚMERO 3, DEZEMBRO DE 2018
ISSN: 2595-8402
DOI: 10.5281/zenodo.2529459

[3] G. Adomian and R. Rach, Nonlinear plasma response, J. Math. Anal. Appl.1985, 111,,114-118.

[4] G. Adomian and R. Rach, Anharmonic oscillator systems, J.Math.Anal.Appl ,1983,91 229-236.

[5] G.Adomian and R.Rach,Light scattering in crystals, J.Appl.Phys ,1984,56, 2592-2594.

[6] G. Adomian and R. Rach, On composite nonlinearities and the decomposition method, J. Math. Anal.Appl,1986, 113, 504-509.

[7] G. Adomian and R. Rach, Nonlinear differential equations with negative power nonlinearities, J. Math.Anal. Appl ,1985,112, 497-501.

[8] G. Adomian, R. Rach and D. Sarafyan, On the solution of equations containing radicals by the decompositionmethod, J. Math. Anal. Appl ,1985,111, 423-426.

[9] G. Adomian and R. Rach, Solving nonlinear differential equations with decimal power nonlinearities,J. Math.Anal. Appl,1986,114 ,423-425.

[10] P. Baratella, A Nystrom interpolant for some weakly singular linear Volterra integral equations, J.Comput. Appl. Math ,2009, 231 ,725-734.

[11] Z. Chen, W. Jiang, The exact solution of a class of Volterra integral equation with weakly singularkernel, Appl. Math. Comput,2011, 217,7515-7519.

[12] M.Dehghan, and F.Shakeri, Approximate solution of a differential equation arising in astrophysicsusing the variational iteration method, New Astronomy,2008,13 ,53-59.

[13] M.Dehghan and F.Shakeri ,Solution of an Integro-Differential equation arising in oscillating Magneticfields using He Homotopy Perturbation Method, PIER,2008,78, 361-376.

[14] D.D.Ganji and A.Rajabi, Assessment of Homotopy Perturbation and Perturbation Methods in heatradiation equations, Int.Commun. Heat Mass Transfer,2006, 33,391-400.

[15] S.Haq, M.Idrees, S.Islam, Application of optimal homotopy asymptotic method to eight order initialand boundary value problems, I. J. Appl. Math.Comput ,2010, 2,(4),73-80.

[16] J.H.He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg ,1999,178 ,257-262.

VOLUME 1, NÚMERO 3, DEZEMBRO DE 2018

ISSN: 2595-8402

DOI: 10.5281/zenodo.2529459

[17] J.H.He ,An approximate solution technique depending upon an artificial parameter, Commun.Nonlinear Sci. Simul ,1998, 3(2), 92-97.

[18] J.H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems,Int. J. Nonlinear Mech,2000,35, 37-43.

[19] J.H.He, Application of homotopy perturbation method to non linear wave equations, Chaos Solitons&Fractls, 2005,26 ,695-700.

[20] J.H.He, Homotopy perturbation method for bifurcation of nonlinear problems, Int. J. Nonlinear Sci.Numer. Simul ,2005, 6 , 207-208.

[21] Q.Hu, Superconvergence of numerical solutions to Volterra integral equations with singularities, SIAMJ. Numer.Anal,34,1997,1698-1707.

[22] N.Herisanu, V. Marinca, T. Dordea, and G. Madescu, A new analytical approach to nonlinear vibrationof an electricalmachine, Proceedings of the Romanian Academy , Series A,2008,vol.9,no.3,,229-236.

[23] S.Iqbal , M.Idrees , AM.Siddiqui , AR.Ansari , Some solutions of the linear and nonlinear Klein-Gordonequations using the optimal homotopy asymptotic method. Appl. Math. Comput ,2010, 216(10),2898-2909.

[24] S.Iqbal ,A.Javed ,Application of optimal homotopy asymptotic method for the analytic solution ofsingular Lane-Emden type equation , Appl. Math. Comput ,2011,217(19), ,7753-7761.

[25] N. Khan, T. Mahmood, M.S. Hashmi, OHAM solution for thin film flow of a third order fluid throughporous medium over an inclined plane, Heat Transfer Res,2013,44, (8), 1-13.

[26] A.A.Kilbas,M.Saigo ,On Asymptotic Solutions of Nonlinear and Linear Abel-Volterra Integral Equations, KURENAI,,1994, 881,91-111.

[27] S.J.Liao, An approximate solution technique not depending on small parameters: A special examples,Int. J. Nonlinear Mech ,1995, 30(3), 371-380.

[28] S.J.Liao . Boundary element method for general nonlinear differential operators, Eng. Anal. BoundaryElements ,1997 , 20, 91-98.

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MULTIDISCIPLINAR MAGAZINE
VOLUME 1, NÚMERO 3, DEZEMBRO DE 2018
ISSN: 2595-8402
DOI: 10.5281/zenodo.2529459

[29] S.J.Liao, An optimal homotopy-analysis approach for strongly nonlinear differential equations. Communicationin Nonlinear Science and Numerical Simulation 2009; doi:10.1016/j.cnsns.2009.09.002, Online.

[30] P.Lima,T.Diogo, Numerical solution of a nonuniquely solvable Volterra integral equation using extrapolationmethod, J.Comput.Appl.Math,2002,140,537-557.

[31] J.T.Ma,Y.J.Jiang, On a graded mesh method for a class of weakly singular Volterra integral equation,J.Comput.Appl.Math ,2009,23,1807-1814.

[32] V.Marinca and N.Herisanu, Application of optimal homotopy asymptotic method for solving nonlinearequations arising in heat transfer, International Communications in Heat and Mass Transfer,2008, vol.35,no.6 , 710-715.

[33] V.Marinca, N. Herisanu, I.Nemes, Optimal homotopy asymptotic method with application to thin filmflow, Cent. Eur. J. Phys,2008,6, (3) ,648-653.

[34] V. Marinca,N. Herisanu,C.Bota,B. Marinca, An optimal homotopy asymptotic method applied to thesteady flow of fourth-grade fluid past a porous plate, Appl. Math. Lett,2009, 22 (2) 245-251.

[35] V. Marinca, N.Herisanu, An optimal homotopy asymptotic method for solving nonlinear equationsarising in heat transfer, Int. Commun. Heat Mass Transfer ,2008 ,35 ,710-715.

[36] V.Marinca, N.Herisanu, Optimal homotopy perturbation method for strongly nonlinear differentialequations, Nonlinear Sci. Lett. A ,2010,1, (3) , 273-280.

[37] V.Marinca,N.Herisanu, Application of homotopy Asymptotic method for solving non-linear equationsarising in heat transfer, I. Comm. Heat Mass Trans,2008,35 ,710-715.

[38] V.Marinca , N. Herisanu, Determination of periodic solutions for the motion of a praticle on a rotatingparabola by means of the optimal homotopy asymptotic method, J.Sound Vib , 2010 ,329 ,1450-1459.

[39] V. Marinca, N. Herisanu, The optimal homotopy asymptotic method for solving Blasius equation, Appl.Maths. Comput ,2014,231,134-139.

[40] Z. Niu, C.Wang ,One-step optimal homotopy analysis method for nonlinear differential

VOLUME 1, NÚMERO 3, DEZEMBRO DE 2018
ISSN: 2595-8402
DOI: 10.5281/zenodo.2529459

equations. Communication in Nonlinear Science and Numerical Simulation. Online. doi:10.1016/j.cnsns.2009.08.014.

[41] R. Rach, On the Adomian decomposition method and comparisons with Picard's method, J. Math.Anal. Appl ,1987,128, 480-483.

[42] A.M.Siddiqui, R.Mahmood, Q.K.Ghori, Homotopy perturbation method for thin film flow of a fourthgrade fluid down a vertical cylinder, Phys.Lett.A , 2006, 352, 404-410.

[43] R.A.Shah, S.Islam and A.M. Siddiqui, Wire coating analysis with Oldroyd 9-constant fluid by OptimalHomotopy Asymptotic Method,Comp.Math.Appl,2012,63,695-707.

[44] A.M.Wazwaz, A comparison study between the modified decomposition method and the traditionalmethods for solving nonlinear integral equation, Appl. Math. Comput ,2006,181 ,(2),1703-1712.

[45] A.M.Wazwaz, A First Course in Integral Equations, World Scientific,New Jersey, 1997.

[46] A.M.Wazwaz, Partial Differential Equations: Methods and Applications, A.A.Balkema, Lisse, TheNetherlands, 2002.

[47] A.M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Higher Education Press, Beijing,and Springer-Verlag, Berlin, 2009.

[48] A.M.Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications, Higher EducationPress,Beijing, and Springer-Verlag, Berlin, 2011.

[49] A.M.Wazwaz and R.Rach, Comparison of the Adomian decomposition method and the variationaliteration method for solving the Lane-Emden equations of the first and second kinds, Kybernetes , 2011,40,1305-1318.

[50] A.M.Wazwaz, A reliable study for extensions of the Bratu problem with boundary conditions, Math.Methods ,Appl. Sci ,2012,35,845-856.

[51] C.Wang, Z.Niu, Reply to Comments on " A one-step optimal homotopy analysis method for nonlineardifferential equations". Communication in Nonlinear Science and Numerical Simulation. doi10.1016/j.cnsns.2009.12.026.

[52] S.A.Yousefi, Numerical solution of Abel's integral equation by using Legendre wavelets, Appl. Math.Comput ,2006,175,574-580.

VOLUME 1, NÚMERO 3, DEZEMBRO DE 2018
ISSN: 2595-8402
DOI: 10.5281/zenodo.2529459

[53] N.Zeilon , Sur quelques points de la theorie de le equation integrale de Abel,Arkiv. Mat. Astr.Fysik,1924,18,5855,1-19.