GEOMETRY—POSITIVE VS. NEGATIVE.¹

BY A. KENNEDY, M.A.,
Inspector of Schools, Weyburn, Sask.

The teaching of mathematics, particularly of geometry, has not given satisfactory results, has not given the pupils power and confidence nor has it produced teachers emancipated from the methods which have perpetuated this weakness.

Canadian teachers admit that they are not satisfied; the general type of geometry paper and the marks assigned respectively for proposition and deduction—usually ten and five—reveal this. A prominent member of the American Federation of Teachers of the Mathematical and the Natural Sciences writes: "There is very great danger of losing geometry as a required subject from the high school course." The Mathematical Association of Great Britain was organized in 1871 because of the recognized need for changes, particularly in geometry. The report of the Committee of the Mathematical Association, as well as the reports of the University Committees, recommended "a practical course in experimental geometry."

The unsatisfactory results, it was assumed, were due to the lack of inductive reasoning and to the inaccuracy of illustration. The author of one of the recent texts has stated: "The present little book is an attempt to make the processes of elementary geometry both inductive and deductive. . . . I have sought to show how a practical course in measurement, use of simple instruments and accurate construction may afford a training of service in anticipating the truths afterwards reached by deductive geometry. . . . It too often happens that in the class room triangles are alleged to be equal which are ridiculously unlike and lines are proved to be equal which the eye tells us differ in length by several inches." A second author writes: . . . "unless the teachers of geometry are willing to accept this modification in presenting the subject—the application of the theorems to things which personally interest them . . ." and still a third: "The aim of the authors has been to make geometry an attractive subject." But the practical course with its greater accuracy of illustration, its interesting and attractive features has not secured more satisfactory results. The teaching of geometry with "a practical course in experimental geometry" has failed not because of the

¹An abstract from an address before the Mathematical and Physical Section of the Ontario Educational Association, 1910.

practical course nor because of the interest and attractiveness but because of the spirit of Euclidean treatment.

Euclid's treatment of geometry focuses the attention on the truth which he wishes one to reach regardless of the fact that often the argument may lead to other equally important truths. Euclid asks us to prove some one truth instead of asking us to prove all we can. Further, he dictates the course to be followed regardless of the course which another mind might take toward the same end. In Proposition 5, Book I, it is stated that "The angles at the base of an isosceles triangle are equal" or the pupil is asked to "prove that the angles at the base of an isosceles triangle are equal." Even if stated: "If a triangle is isosceles prove that the angles at the base are equal," the attention is focused on the equality of the pair of internal basal angles whereas the equality of the other pairs of basal angles is also involved. The full, free consideration of an isosceles triangle would yield other truths.

Again, the course of argument is arbitrary; I have no doubt that Euclid himself investigated freely but his argument gives no hint of the various other lines of thought he may have followed nor does it allow the pupil the same freedom nor encourage investigation. "In BD take any point F, and from AE the greater cut off AG equal to AF the less. Join FC and GB, Because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, the remainder BF is equal to the remainder CG." The spirit of this argument aptly expressed by "the result will justify the course and the truth selected" develops an arbitrary and dogmatic type of mind.

Proposition 47, Book I: "In any right-angled triangle the square on the" or "If one angle of a triangle is a right angle show that the square on the" Euclid actually passed over a truth of infinitely greater value than the end aimed at and reached; the real truth is "therefore the parallelogram BL is equal to the square GB." By continuing to "therefore the square described on the side BC is equal to the squares described on the sides, BA, AC," he missed the opportunity of opening up a whole field of truth.

"Through A draw AL parallel to BD or CE, and join AD, FC. Because the angle BAC is a right angle, and the angle BAG is also a right angle, therefore CA is in the same straight line with AG" The difficulty experienced with the argument of this proposition is surely convincing that a choice of course

would be more effective. In at least one instance many of the candidates for a first-class certificate were unable to do more with Proposition 47 than to put the illustration on the paper.

The negative spirit of Euclidean method may be further seen in the deductions. Among the exercises usually following Proposition 4, Book I, we find: "If two straight lines, AB and CD, bisect each other at right angles in O, any point in either of them, AB, is equidistant from the ends, C and D, of the other," or "Two straight lines AB and CD bisect one another at E. Prove that the triangles AEC, BED are equal in all respects." The attention is focused on one truth; I shall show later the wealth of truth which a full, free consideration of the "two straight lines bisecting one another at right angles" will give.

Euclid's treatment of the indirect argument, as in Proposition 6, Book I, and his method of solving problems, as in Proposition 2, Book II, are seldom grasped and made to increase the power of the pupil. These and other propositions are in many cases memorized for examinations.

Only a method the spirit of which is distinctly open-minded, positive, and exhaustive, will satisfy the demand and develop in the pupil power and confidence. Such a method has been used and has shown itself power-giving. Evidence in support of this statement is difficult to present save by quoting from the notes of a pupil of the Junior Form, Part I.

"Two $\left\{ \begin{array}{l} \text{positions determine a} \\ \text{points} \end{array} \right. \left\{ \begin{array}{l} \text{direction} \\ \text{straight line and} \end{array} \right.$
also a distance, $\left\{ \begin{array}{l} \text{the distance between the two points} \\ \text{the length of the segment of the line.} \end{array} \right.$

Between these two points there is a point
 $\left\{ \begin{array}{l} \text{equally distant from the original points} \\ \text{equidistant from the two points} \\ \text{midway between them;} \end{array} \right.$
this point $\left\{ \begin{array}{l} \text{bisects the segment determined by the two points} \\ \text{is a bisector of the segment} \\ \text{is a mid-point of the segment.} \end{array} \right.$

Through the mid-point of the line-segment may be a line
 $\left\{ \begin{array}{l} \text{determining with the original line} \\ \text{perpendicular to the original line.} \end{array} \right. \left\{ \begin{array}{l} \text{a right angle} \\ \text{and} \\ \text{a right angle} \end{array} \right.$

The straight line through the mid-point of a line-segment and perpendicular to the line, of which the segment is a part, is known as the RIGHT BISECTOR of the segment.

Consider a point on the right bisector of a line-segment. The end-points, A and B, of a line-segment, the mid-point, M, of the segment, and any point, P, on the right bisector of the segment determine three triangles, ABP, AMP, BMP.

Comparing two of these, AMP and BMP, we know that the segments of two sides and the angle which they determine are respectively

$$\text{equal, } \begin{cases} AM = BM \\ MP = MP \\ \text{the angle } AMP = \text{the angle } BMP, \end{cases}$$

so that the two triangles must be congruent, including

$$\begin{cases} \text{the equality of the segments of the third sides} \\ \text{and} \\ \text{the equality of the corresponding basal angles.} \end{cases}$$

$$\begin{cases} AP = BP \\ \text{the angle } MAP = \text{the angle } MBP \\ \text{the angle } APM = \text{the angle } BPM. \end{cases}$$

I. $AP = BP$.

The segments determined by two points and the moving point on the right bisector of the segment determined by the two points are equal.

The distances of two points from any point on the right bisector of the line-segment determined by the two points are equal.

Any point on the right bisector of a line-segment is equidistant from the end-points of the segment.

Two points are equidistant from any point on the right bisector of the segment which they determine.

Two points and any point on the right bisector of the segment which they determine, determine a triangle with the segments of two sides equal.

Consider a point, R, equidistant from two points, M and N.

Is it possible that this point, R, may not be on the right bisector of the segment determined by the two points, M and N? If so, the line determined by R and M would intersect the right bisector of the segment MN in a point, S, and the point S would be

equidistant from the two original points, or SM would be equal to SN . But RM is equal to RN .

Then $RM - SM$ would be equal to RS , $RN - SN$ or $RS + SN$ would be equal to RN .

But the distance along a crooked line, from R to N , is greater than the distance along the straight line so that the preceding statement cannot be true; hence the point equidistant from the two original points must be *on* the right bisector of the segment determined by these two points.

Or, consider a point equidistant from two points.

Suppose this point, P , is not on the right bisector of the segment determined by the two points, A and B ; then the line PA would intersect the right bisector of the segment AB in a point, R , and RA would be equal to RB .

But $\begin{cases} PR + RA \text{ is equal to } PA. \\ PA \text{ is equal to } PB. \end{cases}$

Then $PR + RB$ would be equal to PB ; but this cannot be true. Therefore the point, P , equidistant from two points must be *on* the right bisector of the segment determined by the two points, A and B .

A point equidistant from two points must be *on* the right bisector of the segment which those points determine.

A triangle with the segments of two sides equal must have its vertex *on* the right bisector of the segment of the base.

A triangle with the segments of two sides equal is known as an isosceles triangle.

A triangle with the segments of three sides equal is known as an equilateral triangle.

An equilateral triangle then has each vertex *on* the right bisector of the segment of the corresponding base.

The hypothesis—"Any point on the right bisector of a line-segment," gave us by a direct argument a conclusion of three parts, each important. The first part we have expressed—"Any point on the right bisector of a line-segment must be equidistant from the points which determine the segment."

We then asked ourselves regarding "any point equidistant from two points" and answered the question by means of an indirect argument obtaining the theorem, "An isosceles triangle must have its vertex *on* the right bisector of the segment of the base."

This theorem is the *converse* of the preceding theorem.

Again from the hypothesis, "A point on the right bisector of a line-segment," we obtained as a second part of the conclusion, "The angle MAP is equal to the angle MBP."

The angle determined by a straight line and the line through one of the end-points determining a segment of the original line and a point on the right bisector of the line-segment is equal to the angle determined by the original line and the line through the other end-point on the right bisector of the segment.

The angles determined by a straight line and the lines through the end-points of a segment of the original line and any point on the right bisector of the segment are equal.

The straight lines determined by any point on the right bisector of a line-segment and the end-points of the segment determine with the line of which the segment is a part, angles which are equal.

A point on the right bisector of a line-segment determines with the end-points of the segment two lines which determine with the original line angles which are equal.

A triangle with the segments of two sides equal has its basal angles equal.

An isosceles triangle must have its basal angles equal.

This theorem refers not only to the equality of the internal basal angles but also to the equality of the external basal angles.

What must be true of a triangle which has two of its angles equal?

Or, consider a triangle, ABC, with two of its angles equal.

Can { this triangle not be isosceles?
 this triangle not have the segments of two sides equal?
 the vertex not be equidistant from the basal vertices?
 the vertex not be on the right bisector of the segment of the base?

If so, either side of the triangle would intersect the right bisector of the segment of the base in a point, R, other than the vertex. This point would determine with the basal vertices an isosceles triangle and the basal angles of this triangle would be equal; the angle RBC would be equal to the angle RCB.

But the angle ABC is equal to the angle ACB . It is impossible that an angle may be equal to two different angles. Then the vertex of the original triangle must be on the right bisector of the segment of the base.

Or, the vertex must be equidistant from the basal vertices.

Or, the triangle must be isosceles.

A triangle with two of its angles equal must be isosceles.

As a special case—A triangle with its three angles equal must be equilateral.

From the hypothesis, "A point on the right bisector of a line-segment," we obtained as a third part of the conclusion, "The angle APM is equal to the angle BPM ."

The angle determined by the right bisector of a line-segment and the line through any point on the right bisector and one of the end-points of the segment is equal to the angle determined by the right bisector and the line through the point on the right bisector and the other end-points of the segment.

The angles determined by the right bisector of a line-segment and the lines through any point on the right bisector and the end-points of the segment are equal.

The right bisector of a line-segment determines with the lines through any point on the right bisector and the end-points of the segment angles which are equal.

The right bisector of the segment of the base of an isosceles triangle bisects the vertical angle.

The right bisector of the segment of the base of an isosceles triangle is $\left\{ \begin{array}{l} \text{the bisector of the internal vertical angle.} \\ \text{the internal bisector of the vertical angle.} \end{array} \right.$

The external bisector of the vertical angle of an isosceles triangle is then parallel to the base.

Since the segment of the base of an isosceles triangle has *but one* right bisector, and the vertical angle has *but one* internal bisector we may be satisfied that the internal bisector of the vertical angle of an isosceles triangle must be the right bisector of the segment of the base.

As a special case, the internal bisectors of the angles of an equilateral triangle are the right bisectors of the segments of the corresponding bases.

Having defined a circle, a secant, chord, and so on, the theorem, "Any point on the right bisector of a line-segment is equidistant

from the points which determine the segment" may be otherwise expressed as follows: "The right bisector of any chord must be a center line."

As a special case—The line through the point of contact of a tangent, perpendicular to the tangent, must be a center line.

Since a segment has *but one* right bisector and a circle has *but one* center, a center line which bisects a chord must be perpendicular to the secant.

As a special case—A center line which passes through the point of contact of a tangent must be perpendicular to the tangent.

Also a center line which is perpendicular to a secant must bisect the chord.

As a special case—A center line which is perpendicular to a tangent must pass through its point of contact."

Permit me to call attention to the expression in this treatment. The pupils of the Junior Form, Part I, expressed the truths in that variety of ways which is here quoted and this power of expression is to that extent an indication of their power of thought.

These notes also give a satisfactory answer to a question frequently asked as to the value of such a method to pupils approaching geometry for the first time.

GOLD FROM COPPER ORES.

The production of gold from copper ores in 1909 was valued at \$5,631,885, against an output of \$4,812,993 in 1908. Material increases of gold from these ores are noted for Colorado, Nevada, California, and Utah, in order named, due to increased copper output. The copper ores are mainly sulphides, with a greater tonnage containing chalcocite than chalcopyrite; and the great disseminated deposits of Utah, Arizona, and Nevada are of constantly increasing importance. These ores are mainly concentrated before smelting. Oxidized ores of decreasing relative importance still come from the Bisbee and Tintic districts; these are commonly smelted crude and are sought for as fluxing ores. Gold in copper ores is mainly recovered in the electrolytic copper refineries and is, therefore, lost and not reported for the relatively small proportion of copper not so refined.

EFFECT OF WATER IN WATER-PROOFING CONCRETE has long been recognized, according to C. M. Chapman in a paper read before the National Association of Cement Users, that the quantity of water used in mixing concrete has a large influence upon the impermeability of the resulting mass. A concrete may be too wet or too dry to produce the most impermeable product, and one extreme is as bad as the other. The theory is generally held that, other conditions being the same, a wet mix makes a more waterproof concrete than a dry mix, as the latter produces a more porous mass, unless very tightly compacted by much hard tamping or enormous pressure. A slight difference in the percentage of water used will make a large difference in permeability.