If in the result for t we make $ED=0$, $t=0$. If we make $CD=0$, $t=\infty$.

If we multiply through by CD and make $CD=\infty$, then $ED=\infty$. Similar conclusions would be drawn from the figure.

If AE and EB , Fig. 1, be taken unequal, we get a similar expression for t .

MECHANICS, PHYSICS, AND CHEMISTRY.

The Macrograph. By WM. B. MORGAN, M. A., Prof. of Math. and
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To the Editor of the Journal of the Franklin Institute.

The following article is a copy of a thesis which was prepared by Mr. Morgan previous to his taking the degree of Civil Engineer at the University of Michigan, in June, 1863.

I offer it to you for publication on account of its originality, and the hope that it will induce further investigation. It may result in reducing the principles involved to some useful application.

D. V. WOOD, Prof. Civ. Eng.

This instrument, being as yet without a name, the author of the following article has concluded to call it *The Macrograph*, from its property of describing long curves. (Gr. μακρος, *long*, and γραφω, *to write*.)

The Macrograph is the invention of Oliver Butler, Attorney at Law, Richmond, Indiana.

I. The instrument is represented by Fig. 1, Plate I. It has two cog-wheels, A and B—the latter moving the former—the *number* of teeth in the two being different.

Each wheel carries an arm, MM' , which may be shortened or lengthened at pleasure by means of adjusting screws. At the extremity of the arm, M , is a rotating pivot, P , with a binding screw and a mortise through which the rod R passes. The rod R is attached to the extremity of M' , and also passes through pivot P' , which may be placed any where upon it at pleasure, and fixed there by means of the binding screw. The pivot P' rotates in the slider S . The latter may be placed any where upon the rod R . This latter rod has holes in it at H, H' , &c., into which a pencil may be inserted.

Upon turning the crank of wheel B, while the instrument is in the position indicated, it will describe a curve or rather a series of curves, of which Fig. 2 is a specimen. By placing the pencil in other parts of the same, four figures may be made which differ so much from this specimen, that one would hardly suppose that they were governed by the same law.

It is evident from the construction of the instrument, that if the wheels contained the same number of cogs in each, with any particular arrangement of the pencil, rods, pivots, &c., the pencil must follow in the same path at successive revolutions of the wheels. This

path *might* be a straight line, a loop, or some other shaped figure. While the number of cogs is unequal, it is evident that the path must vary continually until the same relative position of the cogs is reached which existed at the commencement of the curve.

II. The rod R' may be removed, and the pivot at L inserted into slide s . The mortises in slide s and pivot P have slits in them equal to the thickness of the pencil through which the latter may work. Since the wheels are unequal in size, it is evident that, in order to manipulate the instrument, the binding screw of either the slider or the pivot must be loosened. By this arrangement we may describe two systems of curves differing somewhat in principle of construction.

Fig. 3 is a sample of the curves described when the binding screw of the slider is loosened.

If the binding screw of the pivot be loosened, we may describe curves of which Fig. 4 is a sample.

III. Lastly, the wheel A may be removed, and a pivot in the lower part of the slider s' inserted into the slider s . The binding screws of s' and s being tightened, and that of pivot P loosened, a pencil inserted into any of the holes will describe an arc of a circle. If the slider s' is loosened and pivot P tightened, the pencil immediately under s' will describe a straight line; at every other point an oval, with its extremity nearer to s' , more or less sharp, according to its proximity to s' .

Figs. 5 and 6 are examples.

Again s' may be fastened at any point of rod R' , the pivot P being tightened, and the slider s loosened. Fig. 7 shows three such curves described, with the pencil at three different points.

It is this class of curves that I have partially investigated. The discussion of the curves described by the first two arrangements of the instrument, would probably be more curious than practically useful. The only useful application of the first two combinations which suggests itself to the writer, belongs to the engraver's art. A graver's tool, properly constructed upon the same principle, might produce curves which it would be impossible to counterfeit. By the use of cams, eccentrics, and elliptic wheels, not exhibited in the drawing, the figures may be infinitely more diversified.

It is possible that the *single* curves described by the *third* arrangement may be useful to the engineer. At any rate, it is his province to investigate every manner of curve, and ascertain whether in his extensive department for applied mathematics, each may not find some place for conducing to strength and symmetry.

The problem which I have partially solved may be thus stated:

The end of a straight line is made to follow the circumference of a circle, while the line passes through any fixed point; what is the equation and some of the properties of the curve described by any point of the given line?

Let E , Fig. 8, be the centre of the given circle, AB the given line, and p the given point.

Let R = radius of the given circle.

a = length of $AB = A'B'$.

$r = PB = P'B'$ = radius vector of curve.

$d = PE$.

t = angle BPK .

We have,

$$r = a - AP \text{ (or } A'P') = a \pm AF - PF, \quad (A)$$

But $PF = d \cos t$,

and from triangle PEA , $d : R :: \sin A : \sin t$;

$$\therefore R \sin A = d \sin t, \quad (a)$$

Subtract the square of both members of a from R^2 , and we find

$$R \cos A = \pm \sqrt{R^2 - d^2 \sin^2 t} = AF \text{ or } A'F, \quad (b)$$

These in (A) give

$$r = a \pm \sqrt{R^2 - d^2 \sin^2 t} - d \cos t,$$

which is the polar equation of the curve.

From this equation we see that for every value of t there are two values for r , the difference of which is $2 \sqrt{R^2 - d^2 \sin^2 t} = 2 R \cos A$ (see (b)). Whence it appears, as is also evident from the construction, that the chord $BB' =$ the chord AA' .

From (a) we have $\sin t = \frac{R \sin A}{d}$, from which we see that the superior limit of t is

$$\sin^{-1} \frac{R}{d}, \quad (c)$$

$$\text{or, } \cos^{-1} \sqrt{\frac{d^2 - R^2}{d^2}}$$

$\therefore$ if $d = R$ the limit is 90° .

On condition that $d = R$, or that the point through which the given line passes is situated on the circumference of the given circle, the equation of the curve becomes $r = a$ or $r = a - 2 R \cos t$.

With the former the point describes a semicircle, and with the latter a curve meeting the extremities of this semicircle, modified by the length of a .

Fig. 9 shows two such curves. The first differential co-efficient of the latter equation is $\frac{dr}{dt} = 2 R \sin t$; $\therefore \frac{dt}{dr} = \frac{1}{2 R \sin t}$, which is the value of the subnormal. Hence the value of the subnormal is independent of the radius vector. Then, for the same value of t in a system of such curves described with different radii, if normals be drawn to the curves at the extremity of the radii, as at L, L' , &c., these normals will all meet in a single point c , Pc being the subnormal.

For every radius vector drawn from P through the curves, there would be a particular point, c , where the normals would meet. The

locus of these points would not be the straight line pc , but would form a curve.

[If several lines of intrenchments be made, one behind the other, and of the form of these curves, then batteries placed in the same radius vector, as at $L, L', \&c.$, might concentrate their fire at the same point, c , when their line of sight is normal to the curve.]

From the theory of curves we have

$$\text{the subtangent} = \frac{r^2 dt}{dr} = \frac{(a-2R \cos t)^2}{2R \sin t} \quad (D)$$

$$\text{Tangent} = r \sqrt{1 + \frac{r^2 dt^2}{dr^2}} = \frac{a-2R \cos t}{2R \sin t} \sqrt{4R^2 + a^2 - 4aR \cos t} \quad (E)$$

$$\text{Normal} = \sqrt{r^2 + \frac{dr^2}{dt^2}} = \sqrt{(a-2R \cos t)^2 + \frac{1}{4R \sin^2 t}} \quad (F)$$

If in these we suppose $a=2R$, corresponding to the curve MMP we have

$$\text{Subtangent} = 2R \frac{(1-\cos t)^2}{\sin t}, \text{ for } t=90^\circ = 2R \quad (d)$$

$$\text{Tangent} = R\sqrt{8} \frac{(1-\cos t)^3}{\sin t}, \text{ for } t=90^\circ = \sqrt{8}R \quad (e)$$

$$\text{Normal} = \sqrt{\frac{1}{4R^2 \sin^2 t} + 4R^2 (1-\cos t)^2}, \text{ for } t=90^\circ = \sqrt{\frac{1+16R^4}{4R^2}} \quad (f)$$

and if $t=0$ we have subtang = $\frac{0}{0}$, tan = $\frac{0}{0}$, normal = ∞ , which indicates a cusp.

If in (D), (E), and (F), we make $t=90^\circ$, we shall have

$$\text{subtan} = \frac{a^2}{R^2}, \tan = \frac{a}{2R} \sqrt{\frac{4a^2 R^2 + a^4}{4R^2}}, \text{ normal} = \sqrt{\frac{1+4a^2 R^2}{4R^2}} \text{ and if } t=0,$$

$$\text{subtan} = \frac{(a-2R)^2}{0} = \infty, \tan = \infty, \text{ normal} = \infty.$$

To find the area, use the equation $dA = \frac{1}{2} r^2 dt$, in which substitute the value of $r = a - 2R \cos t$, and integrate, and we readily find

$$A = \frac{1}{2} a^2 t - 2aR \sin t + R^2 t + \frac{1}{2} R^2 \sin 2t + (c=0).$$

For $t=90^\circ$ we have

$$A = \frac{1}{2} \pi a^2 - 2aR + \frac{1}{2} \pi R^2$$

Hence the whole area $NNN = \pi (\frac{1}{2} a^2 + R^2) - 4aR$.

If $a=2R$ we find for the area $MMPMM = (3\pi - 8)R^2$.

Radius of Curvature.

In the general expression

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{dt} \right)^2 \right]^{\frac{3}{2}}}{r^2 + 2 \left(\frac{dr}{dt} \right)^2 - r \frac{d^2 r}{dt^2}}$$

Substitute $r=a-2R \cos t$, $\frac{dr}{dt}=2R \sin t$, $\frac{d^2r}{dt^2}=2R \cos t$ and reduce, and we find

$$\rho = \frac{[a^2 - 4aR \cos t + 4R^2]^{\frac{3}{2}}}{a^2 - 6aR \cos t + 8R^2} \quad (G)$$

$$\text{If } t=0, \text{ then } \rho=\rho_0 = \frac{(a-2R)}{a^2 - 6aR + 8R^2} \quad (G')$$

$$\text{If } t=90^\circ \text{ then } \rho=\rho_1 = \frac{(a^2 + 4R^2)^{\frac{3}{2}}}{a^2 + 8R^2} \quad (G'')$$

In these make $a=2R$ and we find

$$\rho = \frac{2^{\frac{5}{2}}}{3} R (1 - \cos t)^{\frac{1}{2}} \quad (H)$$

$$\rho_0 = 0 \quad (H')$$

$$\rho_1 = \frac{2^{\frac{5}{2}}}{3} R \quad (H'')$$

For T, T' we have $TT' = r \cos t = a \cos t - 2R \cos^2 t$,

$$\text{which is a maximum for } \cos t = \frac{a}{4R} \quad (I)$$

For $a=2R$, $\cos t = \frac{1}{2} \therefore t=60^\circ$

Application to the Arch.

Suppose we have an arch constructed with the curvature $NNNN$, having a pier in the centre. This would make a graceful arch.

Let θ = the angle which the tangent line to the curve makes with NPN , or with the axis of x , and

v = the angle which the tangent line makes with the radius vector at the point of tangency.

$$\text{Then } \tan v = r \frac{dt}{dr} = \frac{a - 2R \cos t}{2R \sin t}$$

$$\theta = v - (90 - t) = v + t - 90^\circ$$

$$\therefore \tan \theta = -\cot(v+t) = \frac{\tan v \tan t - 1}{\tan v + \tan t} = \frac{a \tan t - 4R \sin t}{a - 2R (\cos t - \sin t \tan t)}$$

Substitute $\frac{\sin t}{\cos t}$ for $\tan t$ and reduce, and we have

$$\therefore \sec^3 \theta = \left[1 + \left(\frac{a \sin t - 2R \sin 2t}{a \cos t - 2R \cos 2t} \right)^2 \right]^{\frac{3}{2}} \quad (K)$$

Now if z = depth of loading at any point,

m = depth of loading at the crown, N ,

ρ_0 = radius of curvature at the crown,

ρ = radius of curvature at the point corresponding to z , we have (See Professor Wood's Lectures)

$$z = \frac{m\rho_0}{\rho} \sec^3 \theta \quad (L)$$

Fig. 5.

Fig. 6.

Fig. 7.

Fig. 9.

Fig. 1.

Fig. 3.

Fig. 8.

Fig. 4.

Fig. 2.

Equation (L) is thus deduced—

If the arch is a curve of equilibrium, the resultant of the forces at any point will be in the direction of the tangent.

Let T = the tension or compression at any point,

ΣY = the sum of all the vertical forces from the crown to the point considered,

H = the thrust at the crown,

δ = the weight of a unit of volume.

The other notation is given above.

Resolving the tension horizontally and vertically, we have

$$T \frac{dx}{ds} = \Sigma Y = \int \delta z \, dx.$$

$$T \frac{dx}{ds} = H.$$

Dividing gives

$$\frac{dy}{dx} = \frac{\int \delta z \, dx}{H} = \tan \theta.$$

Differentiating gives $\frac{d^2y}{dx^2} = \frac{\delta z}{H}$.

These in the value of the radius of curvature give

$$\rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = H \frac{\sec^3 \theta}{\delta z}$$

$$\therefore z = \frac{H \sec^3 \theta}{\delta \rho}.$$

For $\theta=0$, $z=m$, and $\rho=\rho_0$ $\therefore m = \frac{H}{\delta \rho_0}$ which combined with the former equation gives

$$z = \frac{m \rho_0 \sec^3 \theta}{\rho} \text{ as given above.}$$

To find ρ_0 substitute $\cos t = \frac{a}{4R}$. (See (I)) in (G) and we find

$$\rho_0 = \frac{16R^3}{16R^2 - a^2}. \quad (M)$$

Substitute the value of $\sec^3 \theta$ from (K), ρ from (G) and ρ_0 , in (L) and we find

$$z = m \frac{16R^3}{16R^2 - a^2} \left[\frac{a^2 - 6aR \cos t + 8R^2}{(a^2 - 4aR \cos t + 4R^2)^{\frac{3}{2}}} \right] \times \left[1 + \left(\frac{a \sin t - 2R \sin 2t}{a \cos t - 2R \cos 2t} \right)^2 \right]^{\frac{3}{2}} \quad (M')$$

If $\cos t = \frac{a}{4R}$ we have $z = m$

$$t = 90^\circ \text{ we have } z = 2m \frac{8R^2 + a^2}{16R^2 - a^2} \quad (M'')$$

$$t = 0 \text{ we have } z = m \frac{16R^3}{4R + a} \cdot \frac{(4R - a)^{\frac{1}{2}}}{(2R - a)^{\frac{3}{2}}} \quad (M''')$$

If $a = 2R$, which corresponds to the curve MMP we have

$$z = \frac{1}{2}m \frac{1}{\sin \frac{1}{2}t} \left[1 + \left(\frac{\sin t - \sin 2t}{\cos t - \cos 2t} \right)^2 \right]^{\frac{3}{2}} \quad (N)$$

$$\text{for } t = 60^\circ, z = m \quad (N')$$

$$t = 90^\circ, z = 2m \quad (N'')$$

$$t = 0, z = \infty \quad (N''')$$

From (N'') we see that the depth of loading at the extremity should be twice that at the crown.

If, then, a portion contained between $t = 60^\circ$ and $t = 90^\circ$ were taken for half the span of an arch, and the other half span constructed similarly, we should have the form of an arch which might, in some cases, be useful, for instance, in covered arches for aqueducts. The upper portion of the load may be horizontal, provided its thickness over the top of the arch is equal to the rise of the arch.

Utilization of Brine. A Practical Application of Dialysis.

From the London Chemical News, No. 225.

Mr. Whitelaw, of Glasgow, has recently described the result of a patent process of his own for utilizing the brine of salted meat. When fresh meat, he said, had been sprinkled with salt for a few days, it was found swimming in brine. Fresh meat contained more than three-fourths of its weight of water, which was retained in it as in a sponge. But flesh had not the power to retain brine to that extent, and in similar circumstances it absorbed only about half as much saturated brine as of water, so that under the action of salt flesh allowed a portion of its water to flow out. This expelled water, as might naturally be expected, was saturated with the soluble nutritive ingredients of flesh—it was, in fact, juice of flesh—soup—with all its valuable and restorative properties. In the large curing establishments of this city very considerable quantities of this brine were produced, and thrown away as useless. This was the material to which Mr. Whitelaw has applied the process of dialysis, and he thought with success, for the removal of the salts of the brine, and for the production at a cheap rate of pure fresh extract of meat. His process he stated as follows:—The brine, after being filtered to free it from any particles of flesh or other mechanical impurities it might contain, was then subjected to the operation of dialysis. The vessels or bags in which he conducted the ope-