ART. VI.—*The Theory of the Wind Vane*; by GEORGE E.
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THE common weather-cock, or wind-vane, is no doubt the oldest, as well as the simplest, of meteorological instruments. In its earliest use, as a popular indicator of the weather, erected at the summit of a flag staff or a church tower, or, in New England, forming the usual apex of the town liberty pole, the vane assumed the form of a cock, a fish, a trumpeter, or other ornamental device; but, with its application to accurate

meteorological observations, these forms were replaced by simple plane plates, discs, or arrows, designed solely with regard to the mechanical action of the wind upon them.

The form in general use at present is that of an arrow with a double or spread tail. The first reference I find to the construction of a vane of this design is by the elder Parrot, (*Voigt's Magazin*, i, 1797). In order to diminish the continual oscillations to which the vane is subject during variable winds, he made a vane of two thin plates joined at one end at an angle of about 45° . This was used by the younger Parrot, in his well known expedition to the Caucasus in 1811, with Engelhardt. Similar vanes were soon introduced in other countries, and by 1840 were in common use by English meteorologists. With respect to the advantages of their use, Jelinek, writing in 1850, makes the following statement: "The English seek to obviate the disadvantages of the complete rotations by making the vane of two plane surfaces set at an angle to each other of generally about $22\frac{1}{2}^\circ$. They say that thereby the oscillations are smaller, and the complete rotations less frequent." Stating that experience shows this to be true, Jelinek adopted the same form for a self-recording anemoscope, designed as an improvement on that constructed for the Austrian meteorological service by Dr. Kreil in the preceding year, which, consisting of a single plane circular plate, had occasioned much difficulty by its frequent complete rotations.

While the spread vane has thus grown in favor, the angle of the wings has gradually been reduced. Starting with an angle of 45° , as made by Parrot in 1800, the angle in 1850 was generally $22\frac{1}{2}^\circ$, and at present even a smaller angle is frequently used. In practice the surfaces are made with a slight curvature, so that the actual angle made by their tangents increases from zero at the vertex, to 30° , or more, at their extremities. *Voigt's Magazin*, in which Parrot's description of his vane appeared, is not accessible to me, and his reasons for adopting 45° are not given by reviewers. Similarly, I have found no explanation of the reason for adopting any of the smaller angles subsequently used. The following analysis has therefore been made, in order to find definitely what advantages the spread vane possesses, as compared with a straight vane of the same length and shape, and at what angle the wings should be set to secure the greatest efficiency.

The wind vane, its surfaces assumed to be plane, is a case of a body immersed in a fluid current, whose resultant pressure is normal to the exposed surface, and proportional to some function of the angle of incidence. If α be the angle between the surface normal and the perpendicular to the direction of the current, the old theory of the resistance of fluids made the re-

sultant normal pressure upon the surface proportional to $\sin \alpha$. This was based upon the theory that the amount of fluid impinging on a unit of surface is proportional to $\sin \alpha$, and the normal component of pressure exerted also proportional to $\sin \alpha$: or again, the velocity of the fluid resolved perpendicularly to the surface being $v \sin \alpha$, the resultant force exerted against the surface must be proportional to $v^2 \sin^2 \alpha$. But all this reasoning is now known to be worthless. Experiment shows that the resistance is much more nearly proportional to the $\sin \alpha$ than to its square. Colonel Duchemin* was one of the first to give a more accurate formula based upon the results of experiments. He found that the pressure on a plane moving obliquely in a fluid may be represented by the following empirical formula.

$$N = P \cdot \frac{2 \sin \alpha}{1 + \sin^2 \alpha}$$

in which P is the resistance to a plane moving normally. Later experiments, made for the London Aeronautical Society in 1872, indicate that for large values of α the observations would be better satisfied by making the denominator $1 + \sin \alpha$ instead of $1 + \sin^2 \alpha$. This change has also received a theoretical confirmation. By the analytical method of Kirchhoff and Helmholtz, Lord Rayleigh has derived a formula for the case of a blade-shaped surface which makes the pressure per unit of area

$\frac{\pi \sin \alpha}{4 + \pi \sin \alpha} \rho v^2$. For small values of α neither of these formulæ differ much from $\sin \alpha$, and for the purposes of this paper it will be assumed that this latter relation holds good; therefore the normal pressure on the surface of the vane tending to produce rotation will be proportional to the sine of the angle between the surface and the wind direction.

Let $(1+n)P$ be the total effective pressure on a *straight vane* when at right angles to the wind, where n takes account of the diminution of pressure on the sheltered side. The value of n varies with the angle of obliquity in some ratio not yet fully known, but, for its first approximation may be assumed, like the pressure on the exposed surface, to be proportional to the sine. On these assumptions the gyratory force tending to restore the equilibrium of the vane will be, for an angle θ between the vane and the wind direction, proportional to $(1+n)P \sin \theta$.

For a *spread vane*, let ϵ be half the angle between the two wings and θ the angle between the medial line of the vane and the wind direction. If θ be less than ϵ , the wind will act on

* Duchemin: Recherches experimentales sur les lois de la resistance des fluides, Paris, 1842.

both wings in opposite directions, but when θ is equal to or greater than ϵ , the wind will strike only one wing as in the case of a straight vane. Therefore, to obtain a measure of relative sensitiveness, we must compare the gyrotory force acting in each of these two cases with the corresponding force acting on a straight vane. For simplicity, suppose the air between the wings to be unaffected by the wind, as would be the case if the space were entirely enclosed.

1. When $\theta < \epsilon$. The gyrotory force is proportional to $P[\sin(\epsilon + \theta) - \sin(\epsilon - \theta)] = P \sin \theta \cdot 2 \cos \epsilon$, in which, for all ordinary angles of the wings, $2 \cos \epsilon$ may be put equal to 1.9. To compare this expression with $P \sin \theta (1 + n)$, the expression above given for the corresponding gyrotory force acting upon a straight vane, we must know the approximate value of n , which varies with the size and shape of the plate. Experiments on normal planes of small size give values of n ranging from 0.2 to 0.86; for surfaces as large as ordinary wind vanes, the upper limit of n can hardly be as great as 0.5 and probably is less. With this value, the ratio of the gyrotory force acting on the straight vane to that acting on the spread vane, when $\theta < \epsilon$, is

$$1.5:1.9$$

2. When $\theta =$ or $> \epsilon$. The wind strikes the exposed surface at an angle $\theta + \epsilon$ and passes the sheltered surface at an angle $\theta - \epsilon$.

The total gyrotory force, therefore, will be

$$P \sin(\theta + \epsilon) + nP \sin(\theta - \epsilon).$$

Comparing this expression with $P \sin \theta + nP \sin \theta$, the corresponding gyrotory force upon a straight vane, the first term is seen to be larger, and the second term smaller, than the respective terms of the latter formula.

If 0.5, the maximum value of n , be substituted in these formulæ, the difference between the two forces will be

$$u = \frac{1}{2} \sin \epsilon \cos \theta - \frac{3}{2} (1 - \cos \epsilon) \sin \theta,$$

which is the excess of the force upon the spread vane over that upon the straight vane. For all values of ϵ less than $41^\circ.4$ this excess is positive and has its greatest* value when $\theta = \epsilon$; with the increase of θ , the excess diminishes and becomes 0 at different points depending on the value of ϵ , thus

for $\epsilon = 10^\circ$	the excess vanishes for	$\theta = 75^\circ.3$
$\epsilon = 20^\circ$	“	$\theta = 62^\circ.1$
$\epsilon = 30^\circ$	“	$\theta = 51^\circ.2$
$\epsilon = 41^\circ.4$	“	$\theta = 41^\circ.4$

* By an analytical investigation of the nature of u considered as a function of the two independent variables, θ and ϵ , it appears that this function has no critical value for ranges of ϵ and θ between 0° and 90° ; but the greatest positive value of u (not a critical value) will occur when $\theta = \epsilon$, the limiting value of θ under the present case.

For a smaller value of n , the excess would vanish at larger values of θ .

Consequently, for all moderate values of ϵ and θ when $\theta < \epsilon$, as well as when $\theta < \epsilon$, the gyratory force acting on the spread vane is greater than that on the straight vane.

If a perfectly sensitive vane be defined as one that instantly responds to the slightest change in the direction of the wind, that vane upon which the winds exert the greatest gyratory force will be the most sensitive, supposing the vanes all have the same friction. The increased sensitiveness accruing to the spread vane from its greater gyratory force is diminished and, for large values of ϵ , may be quite overcome by its greater lateral friction, due to that component of the wind pressure which passes through the axis transversely. This component is proportional to $\sin^2 \epsilon$, nearly, and so its effect, which is small and negligible for small values of ϵ , increases rapidly and becomes an important factor for values of ϵ greater than 20° or 25° . This lateral friction, therefore, constitutes a condition requiring the angle between the wings to be small. A further indication as to the angle at which the wings should be set for maximum sensitiveness is found by an analysis of the expressions given above for the gyratory force.

1. When $\epsilon = \theta$ or $\epsilon > \theta$. Place $\epsilon = \theta + \alpha$; then

$$2 \sin \theta \cos \epsilon = 2 \sin \theta \cos (\theta + \alpha).$$

Noting that $(\theta + \alpha)$ must always lie in the first quadrant, this expression has its greatest value when $\alpha = 0$, i. e. when the half angle of the wings is not larger than the angle of the wind's deviation.

2. When $\epsilon < \theta$. Examining the expression for gyratory force, $\sin (\theta + \epsilon) + n \sin (\theta - \epsilon)$, to find a critical value, it appears that the function is a maximum when $\tan \epsilon = \frac{1-n}{1+n} \cot \theta$.

Making $n = \frac{1}{2}$ as before, $\tan \epsilon = \frac{1}{3} \cot \theta$. For all values of θ up to 30° , this makes the corresponding value of ϵ to be greater than θ , which for the present case is an impossible value; but for this range of θ , from θ° to 30° , the *greatest* value of the function (not a *critical* value) will occur when $\epsilon = \theta$. For values of θ above 30° , the value of ϵ giving a maximum diminishes downward from 30° , and for $\theta = 90$ becomes 0. Consequently the best values for ϵ for deviation of the wind below 30° will hold good also for deviations greater than 30° . The conclusion, therefore, is that, for maximum sensitiveness, the half angle of the wings should be the average angle through which the wind makes sudden deviations, this average being taken from 0 to 30° .

The question of the relative stability of a spread and straight vane remains to be discussed. That the continual oscillations of the wind vane are diminished, was the advantage claimed

by Parrot for his spread vane, and has been the justified claim of his followers. Given two vanes of the same moment of inertia, for slowly shifting winds, the oscillations of the spread vane will be smaller than those of a straight vane, for, being more sensitive, it responds more quickly to any gradual change in wind direction, and, therefore, has a less amplitude of oscillation, and will sooner come to rest. When the wind suddenly changes direction through a considerable angle, the straight and spread vanes start out with the same angle of deviation from the new direction. The following analysis shows that, in this case also, the spread vane comes sooner to rest. If v be the velocity of the wind, r a constant depending on the length of the vane, θ the angle between the direction of the wind and the axis of a straight vane, the resultant relative velocity, R , of the wind and the vane will be the third side of a triangle of which two sides are the velocities of the wind and vane, and

$90^\circ - \theta$, their included angle.

Whence $R^2 = v^2 + 2vr \frac{d\theta}{dt} \sin \theta + \left(r \frac{d\theta}{dt}\right)^2$, $\frac{d\theta}{dt}$ being negative or positive, according as the motion of the vane is with or against the wind. This resultant makes an angle φ with the vane such that

$$\begin{aligned} R \sin \varphi &= v \sin \theta + r \frac{d\theta}{dt}, \\ R \cos \varphi &= v \cos \theta; \end{aligned}$$

The effective force tending to turn the vane, which is proportional to $v^2 \sin \theta$ when the vane is at rest, becomes $R^2 \sin \varphi$ when the vane is in motion. Considering the motion when θ and $\frac{d\theta}{dt}$ are so small that we may neglect the square of $\frac{d\theta}{dt}$ and the product, $\sin \theta \frac{d\theta}{dt}$,

$$R^2 \text{ becomes } v^2, \text{ and } \sin \varphi = \sin \theta + \frac{r}{v} \cdot \frac{d\theta}{dt},$$

$$\text{Whence } R^2 \sin \varphi = v^2 \left(\sin \theta + \frac{r}{v} \cdot \frac{d\theta}{dt} \right).$$

Let I be the moment of inertia of the vane around its axis, ρ the air density, A the surface of the vane, $f_0(P)$ the pressure of a wind of unit velocity on a unit surface under a standard density of the air ρ_0 , and r the distance of the point of applica-

tion of P from the axis. Then the differential equation of motion for a straight vane will be

$$\frac{d^2\theta}{dt^2} + Kv^2 \left(\sin \theta + \frac{r}{v} \frac{d\theta}{dt} \right) = 0$$

$$\text{where } K = \frac{\rho A f_0(P) \cdot r}{I}.$$

When θ is small, $\sin \theta$ may be replaced by θ , and simplifying the constants by making $Kv^2 = \omega^2$, and $Krv = 2k$ the equation becomes

$$\frac{d^2\theta}{dt^2} + 2k \frac{d\theta}{dt} + \omega^2 \theta = 0.$$

This is the equation of a circular pendulum in a resisting medium where the resistance varies as the velocity. The general integral of this equation takes three forms according as $k^2 - \omega^2$ is positive, negative, or vanishes; a numerical evaluation shows that it is always negative, for which case, the integral becomes

$$\theta = \alpha e^{-kt} \left\{ \cos h t + \frac{k}{h} \sin h t \right\}$$

where α is the greatest value of θ and $h^2 = \omega^2 - k^2$.

For a *spread vane*, where the half angle of the wings is greater than the deviation of the wind, the wind acts on both wings at once in opposite directions, and the resultant pressure will be

$$\sin(\epsilon + \theta) + \frac{r}{v} \frac{d\theta}{dt} - \left[\sin(\epsilon - \theta) + \frac{r}{v} \frac{d\theta}{dt} \right]$$

in which $\frac{d\theta}{dt}$ is to be taken with opposite signs in the two terms, and the expression reduces to

$$2 \sin \theta \cos \epsilon + 2 \frac{r}{v} \frac{d\theta}{dt}.$$

Using the same notation as in the previous case, and making $2 \cos \epsilon = 1.9$, which is approximately true for all ordinary cases, we have the equation

$$\frac{d^2\theta}{dt^2} + 4k \frac{d\theta}{dt} + 1.9\omega^2\theta = 0. \quad \text{Making } h_1^2 = 1.9\omega^2 - 4k^2,$$

$$\theta = \alpha e^{-2kt} \left\{ \cos h_1 t + \frac{2k}{h_1} \sin h_1 t \right\}$$

$$\frac{d\theta}{dt} = -\alpha \cdot \frac{h_1^2 + 4k^2}{h_1} \cdot e^{-2kt} \sin h_1 t.$$

$$\frac{d\theta}{dt} = 0, \text{ when } t=0, t = \frac{\pi}{h_1}, t = \frac{2\pi}{h_1}, \text{ etc.}$$

Substituting these values of t in the preceding equation for θ , the amplitudes of vibration become successively

$$\alpha, \alpha l \frac{-2k}{h_i} \pi, \alpha e \frac{-4k}{h_i} \pi, \dots$$

Similarly, the amplitudes of vibration of a straight vane are, successively,

$$\alpha, \alpha e \frac{-k}{h} \cdot \pi, \alpha e \frac{-2k}{h} \cdot \pi, \dots$$

To determine the relative magnitude of the oscillation in these two cases, let us substitute for $\frac{2k}{h_i}$ and $\frac{k}{h}$ their original values,

$$\frac{2k}{h_i} = \frac{Kr}{\sqrt{1.9K - K^2 r^2}}$$

$$\frac{k}{h} = \frac{Kr}{\sqrt{4K - K^2 r^2}}$$

If it be desired to take account of the diminution of pressure on the sheltered side of the straight vane, the value of K is not the same for both vanes, but will become so by multiplying K in the equations of the straight vane by the factor $1+n$.

Making this correction, $\frac{k}{h} = \frac{Kr}{\sqrt{\frac{4}{1+n} \cdot K - K^2 r^2}}$. For all possi-

ble values of n , $\frac{2k}{h_i} > \frac{k}{h}$, and, consequently, at any time t , the amplitude of oscillation of the spread vane will be less than that of the straight vane, and so the former will sooner come to rest.

This difference in amplitude of oscillation is least when $K^2 r^2$ is negligible in comparison with $1.9K$, in which case the difference between the amplitudes of oscillation is determined by the ratio $\sqrt{\frac{4}{1+n}} : \sqrt{1.9}$ as exponential factors in the expression for the amplitude. This difference is increased by the term $K^2 r^2$, r and K being constants depending on the length and surface of the vane.

The formulæ, therefore, show 1st, that the oscillations of both vanes are smaller as the vanes are longer and larger; 2d, that the spread vane is always more stable than the straight vane; and 3d, that this advantage in stability is greater for long vanes than for short vanes, and is independent of the wind velocity.

The above analysis obtains for a frictionless bearing. From the discussion of relative sensitiveness, we have found that with equal friction, a spread vane is more sensitive than a similar straight vane; consequently, for two vanes of equal sensitiveness, the spread vane will have the greater friction and will come to rest more quickly.