

ON THE CASES OF EXCEPTION IN JACOBI'S THEOREM CONCERNING DOUBLE RESIDUES

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In a former paper* I gave a proof of Jacobi's double residue theorem, and shewed how the chief general propositions of the theory of point groups in a plane could be deduced from it, all the points being taken as isolated. In the present paper the same methods are applied to the case when the points coalesce in any way.

Some theorems are first proved concerning matrices whose constituents are polynomials in one letter. These theorems are known when the polynomials are of the first degree; they are here applied to the discussion of two ternary quantics and their eliminant, and they lead to Jacobi's theorem both in its original form and also in the modified form which is the main object of the paper.

With a view to applications the form of the result is examined when the numerator polynomial breaks up into factors, and this discussion leads to the modified theorems of Cayley and Bacharach, Riemann and Roch, the latter being so extended as to include the former.

Jacobi's theorem may be proved by evaluating over a suitable closed region the double integral

$$\iint \frac{\omega}{\phi\psi} dt du,$$

or the single integral

$$\int \frac{\omega}{\phi \frac{\partial \psi}{\partial u}} dt,$$

the notation being that of § 4 of this paper. See Baker's *Multiply-Periodic Functions*, p. 178, or *Camb. Phil. Proc.*, Vol. xiv, pp. 387–390. The treatment in the present paper is purely algebraic.

* *Camb. Phil. Proc.*, Vol. xv, pp 472–481.

Properties of a Matrix whose Constituents are Polynomials.

1. Let x, y, a, b denote rows of quantities, there being n in each row. Let M denote a matrix of n rows and columns, whose coefficients are polynomials in a variable t , and N the associated matrix formed from M by the interchange of rows and columns. Let D be the value of the determinant of M and N , which is itself a polynomial in t and which will be supposed not to vanish identically.

Then if we solve the equations $Mx = a$, or $Ny = b$, where a, b are rows of polynomials in t , we have the values of x, y as rows of polynomials divided by the common denominator D . When D has been resolved into factors the values of x, y can be resolved into partial fractions, and these partial fractions will now be investigated.

2. If D has the factor $t - t'$, then when $t = t'$, the equations $Mx = a$, $Ny = b$ cannot be solved for all values of a, b , but $Mx = 0$, $Ny = 0$ can be satisfied by rows x, y which do not wholly vanish. That is, a row of polynomials ξ can be found, not all containing the factor $t - t'$, but such that $t - t'$ is a factor of the row $M\xi$. There may be several rows such as ξ ; let ξ_1 be one of them such that $M\xi_1$ contains the highest possible power of $t - t'$, say $(t - t')^{r_1}$, so that

$$M\xi_1 = (t - t')^{r_1} a_1,$$

where a_1 does not vanish when $t = t'$.

Let ξ_2 be a row, not a multiple of ξ_1 , such that $M\xi_2$ contains $(t - t')^{r_2}$, the highest power of $t - t'$ that is possible when multiples of ξ_1 are excluded, and so on until all the linearly independent solutions of $Mx = 0$ when $t = t'$ are exhausted.

It is convenient to complete the tale of n rows ξ by means of rows linearly independent of each other, and of the original rows ξ , so that for these Mx does not vanish when $t = t'$; that is, there are n indices r_1, r_2, \dots in descending order of magnitude, some at the end being zero. If, then, D is multiplied by the determinant of the n rows ξ , which does not vanish with $t - t'$, the product may be written as the determinant of the n rows $M\xi$, and its rows will contain the factor $t - t'$ to the powers r_1, r_2, \dots . Hence D contains $t - t'$ to the power $r_1 + r_2 + \dots$.

D does not contain any higher power, for if it did we could by a combination of the rows ξ replace at least one of them by another for which $M\xi$ would contain a higher power of $t - t'$, and this is excluded by the method of construction.

Also $t-t'$ is contained in all the first minors of D to the power $r_2+r_3+\dots$, in the second minors to the power $r_3+r_4+\dots$, and so on.

Similarly, the equations $Ny = 0$ are satisfied when $t = t'$ by rows η to which the same considerations apply, and by noting the degree of the factor $t-t'$ in D and its minors we are led to the conclusion that the series of indices r_1, r_2, \dots is the same for η as for ξ .

3. Since the determinant of the rows η does not vanish when $t = t'$, the set of equations $Mx = a$ is equivalent to the set

$$\eta Mx = \eta a,$$

which may also be written $x N\eta = \eta a$.

Here the left sides contain the factor $t-t'$ to the powers r_1, r_2, \dots , and thus it is necessary that the same should be true of the right sides if x is to be limited when $t \rightarrow t'$. No further condition is necessary, since the determinant of the coefficients on the left does not vanish when the factors in question have been taken out.

Hence, if $\eta_1 a, \eta_2 a, \dots$ contain $t-t'$ to the powers r_1, r_2, \dots , the fractions with powers of $t-t'$ in the denominator are absent from the expressions for x , and, in general, the numerators of these fractions are linear in the coefficients of the powers

$$0, 1, \dots, r_1-1 \text{ of } t-t' \text{ in } \eta_1 a,$$

$$0, 1, \dots, r_2-1 \text{ of } t-t' \text{ in } \eta_2 a,$$

and so on.

The number of these coefficients, that is, of the conditions that x should be limited when $t \rightarrow t'$, is $r_1+r_2+\dots$, the degree of the factor $t-t'$ in D , and all these conditions are independent, for (1) in any particular expression of the type ηa the coefficient of any power of $t-t'$ involves coefficients from a that have not occurred with lower powers, and these are multiplied by the absolute terms in η , while (2) the absolute terms in the rows η_1, η_2, \dots are linearly independent: the absolute terms here referred to are those of the expansions in powers of $t-t'$, and the polynomials a are supposed quite general and of as high a degree as may be necessary.

Application to the Theory of the Eliminant.

4. Let ϕ, ψ be two polynomials in t, u of degrees m, n in u , which need not be irreducible, and let x denote collectively the $m+n$ coefficients

of powers of u in two polynomials λ, μ in t, u of degrees $n-1, m-1$ in u , and a those in a polynomial ω in t, u of degree $m+n-1$ in u .

$$\text{The identity} \quad \lambda\phi + \mu\psi \equiv \omega \quad (1)$$

gives a system of $m+n$ equations for x of the type $Mx = a$ which has just been discussed.

The determinant D is the u -eliminant of ϕ, ψ in the form given by Sylvester's dialytic method, but with rows and columns interchanged.

The conclusion from the matrix theory in the present case is that if D contains a factor $(t-t')^q$, and not $(t-t')^{q+1}$, then the number of conditions arising therefrom to be satisfied by the coefficients of ω is exactly q if the identity is to be a possible one, and that otherwise the expressions for x derived from the equations

$$Mx = a$$

contain q rows of partial fractions, each having a power of $t-t'$ as denominator, and each independent of a save for a coefficient whose vanishing is one of the conditions referred to.

5. When $t = t'$, D vanishes and thus x can be determined so that

$$\lambda\phi + \mu\psi = 0.$$

Since λ, μ are of lower degrees than ψ, ϕ , this implies that ϕ, ψ have a common factor, say $u-u'$. If the identity (1) is to be possible, it is clear that ω must have the same factor when $t = t'$, but without making this assumption we may find the corresponding row of partial fractions as follows when the factor $t-t'$ is simple.

By separating these partial fractions from the rest we divide λ, μ , each into two parts, say

$$\frac{\alpha}{t-t'} + \lambda_1, \quad \frac{\beta}{t-t'} + \mu_1,$$

where $t-t'$ does not occur as a denominator in λ_1, μ_1 , and α, β are polynomials in u , of degrees $n-1, m-1$ respectively, with constant coefficients.

We have then

$$\omega(t-t') = \{a + \lambda_1(t-t')\} \phi + \{\beta + \mu_1(t-t')\} \psi,$$

and when $t = t'$, $a\phi(t', u) + \beta\psi(t', u) = 0$.

Thus* $\alpha = C\psi(t', u)/(u-u')$ and $\beta = -C\phi(t', u)/(u-u')$,

where C is a constant, and the expression for ω contains the term

$$\frac{C}{(t-t')(u-u')} [\phi(t, u)\psi(t', u) - \psi(t, u)\phi(t', u)],$$

$$\text{or } C \left[\frac{\phi(t, u) - \phi(t', u)}{t - t'} \frac{\psi(t', u) - \psi(t', u')}{u - u'} - \frac{\psi(t, u) - \psi(t', u)}{t - t'} \frac{\phi(t', u) - \phi(t', u')}{u - u'} \right].$$

The expression in brackets is only fractional in appearance: when the divisions are carried out and we put $t = t'$, $u = u'$, it becomes $CJ(t', u')$, where J is the Jacobian of ϕ, ψ . Thus

$$\omega(t', u') = CJ(t', u'),^\dagger$$

for $\lambda_1\phi + \mu_1\psi = 0$, when $t = t'$, $u = u'$, since ϕ, ψ vanish then.

If, then, all the factors of D are simple, we have

$$\omega(t, u) = \Sigma \frac{\omega(t', u')}{J(t', u')} \times \frac{\phi(t, u)\psi(t', u) - \psi(t, u)\phi(t', u)}{(t-t')(u-u')} + \eta\phi + \theta\psi, \quad (2)$$

where η, θ are polynomials in t and u .

6. The condition to be satisfied by ω in order that the fractions with $t-t'$ as denominator may be absent is

$$\omega(t', u') = 0,$$

that is, the minors of a column in D are proportional to

$$1, u', u'^2, \dots, u'^{m+n-1},$$

when $t = t'$, in accordance with Sylvester's theory.

7. Suppose now that $m, n, m+n-1$ are the degrees of ϕ, ψ, ω in t and u , and that the axes and line at infinity have no special position with respect to the curves $\phi = 0, \psi = 0$, so that in particular no common point lies at infinity and no two common points are on the same line $t = \text{const.}$

* The common factor in $\phi(t', u), \psi(t', u)$ can only be a simple one, such as $u-u'$, for otherwise α, β would contain two arbitrary coefficients, all the first minors of D would vanish when $t = t'$, and $t-t'$ would be a repeated factor in D .

† This method can be carried out when $t-t'$ is not repeated as a factor in D , and it follows that in that case $J(t', u')$ cannot vanish.

Then the terms of fractional form in the expression for ω that has been found (2) are really polynomials in t, u of degree $m+n-2$ only, so that the terms of degree $m+n-1$ in ω are identical with those in $\eta\phi + \theta\psi$. In particular, if ω is of the degree $m+n-2$ only, these terms must be absent in $\eta\phi + \theta\psi$, and therefore η, θ must be of degrees $n-2, m-2$, since the assemblages of highest terms in ϕ, ψ have no common factor.

Now compare the terms of degree $m+n-2$ on the two sides of (2). In all the fractions on the right these are the same, being independent of t', u' , except for the factors $\frac{\omega(t', u')}{J(t', u')}$, whose sum therefore appears as the coefficient of a polynomial which for the moment we will call U . Distinguishing the terms of highest degree by a bar, we have then

$$\bar{\omega} = U \Sigma \frac{\omega(t', u')}{J(t', u')} + \bar{\eta}\bar{\phi} + \bar{\theta}\bar{\psi}.$$

There are $m+n-1$ independent coefficients in $\bar{\omega}$, and hence there must be the same number on the right, that is, the $n-1$ coefficients in $\bar{\eta}$, the $m-1$ in $\bar{\theta}$ and $\Sigma \frac{\omega(t', u')}{J(t', u')}$ must be linearly independent and must all vanish if all the coefficients in $\bar{\omega}$ do so. That is, if ω is of the degree $m+n-3$ only,

$$\Sigma \frac{\omega(t', u')}{J(t', u')} = 0,$$

which is *Jacobi's theorem* in its ordinary form.

8. Jacobi's theorem leads to a rule, which corresponds to Newton's method in the theory of equations, and which might be useful in practice, for finding the sum of the values of such a product as $t^a u^b$ at the different intersections. To find $\Sigma \omega(t', u')$ express $\omega(t, u)J(t, u)$ in the form

$$\lambda\phi + \mu\psi + cJ + \chi,$$

where λ, μ, χ are polynomials, χ of degree $m+n-3$ at most, and c is a constant. This can be done by successive reduction of the highest terms, and then

$$\Sigma \omega(t', u') = mnc.$$

9. In discussing the extension of Jacobi's theorem to cases when the intersections are not all isolated, it will be convenient to speak of ω as reconcilable with ϕ, ψ, \dots at (t', u') when ω can be expressed in the form $\lambda\phi + \mu\psi + \dots$, where λ, μ are not necessarily finite polynomials, but may be infinite series of ascending powers and products of $t-t', u-u'$, say *regular* series. The identity $\omega = \lambda\phi + \mu\psi + \dots$ is to be considered as merely

formal, no question of convergency being material. A fundamental result, now to be proved, is that *the number of linearly independent polynomials not reconcilable with ϕ, ψ at a point (t', u') , where ϕ, ψ vanish, is exactly the degree to which the factor $t-t'$ enters in D .*

If ϕ, ψ do not both vanish at (t', u') , suppose ϕ not to; then, in the identity

$$\omega = \lambda\phi + \mu\psi,$$

μ may be taken arbitrarily and λ found by long division, term by term: thus any polynomial is reconcilable with ϕ, ψ at (t', u') in this case.

10. LEMMA I.—If D vanishes identically, then ϕ, ψ must have a common polynomial factor.

For now polynomials λ, μ in t, u exist of degrees in u lower than those of ψ, ϕ , such that

$$\lambda\phi + \mu\psi = 0$$

identically. Suppose any common factors of λ, μ to have been taken out. Then by the regular H.C.F. process applied to λ, μ as polynomials in u , other polynomials η, θ in t, u are found such that

$$\eta\lambda + \theta\mu = \kappa,$$

a polynomial in t only, and thus

$$\kappa\phi = \eta\lambda\phi + \theta\mu\phi = \mu(\theta\phi - \eta\psi),$$

$$\kappa\psi = -\lambda(\theta\phi - \eta\psi).$$

Since λ, μ have no common factor they cannot both vanish identically for any particular value of t , and therefore all the factors of κ must divide $\theta\phi - \eta\psi$, and since μ is of lower degree in u than ϕ ,

$$\phi = \mu\chi, \quad \psi = -\lambda\chi,$$

where χ is a polynomial in t, u , as was to be proved.* In the present discussion the case of ϕ, ψ having a common factor is excluded, so that D cannot vanish identically.

LEMMA II.—If there is a formal identity

$$\eta\phi + \xi\psi = 0,$$

where η, ξ are regular series in $t-t', u-u'$, then these series must be expressible in the forms $\theta\psi, -\theta\phi$, where θ is a regular series in $t-t', u-u'$.

* The proof is on the same lines as that of the similar theorem for polynomials in u with constant coefficients, which has been used in § 5.

For there is an identity

$$D = \rho\phi + \sigma\psi,$$

where ρ, σ are polynomials, and thus

$$D\eta = (\sigma\eta - \rho\xi)\psi, \quad D\xi = -(\sigma\eta - \rho\xi)\phi.$$

Any power of $t-t'$ which is a factor in D must occur in each term of the series $\sigma\eta - \rho\xi$, since $t-t'$ cannot be a common factor of ϕ, ψ . The division by the other factor of D can be carried out in an ascending series.

Thus $\eta = \theta\psi$, $\xi = -\theta\phi$, where $\theta = (\sigma\eta - \rho\xi)/D$, which is a regular series in $t-t', u-u'$.

11. Since any fraction whose denominator is a power of $t-t''$ can be expanded in positive powers of $t-t'$, when $t' \neq t''$, it follows that if the $t-t'$ fractions are absent from the expressions for λ, μ deduced from (1), then ω is reconcilable with ϕ, ψ at (t', u') . What we have to prove is the converse statement.

Suppose then that ω , a polynomial in t, u of degree $m+n-1$ at most, is reconcilable with ϕ, ψ at (t', u') , that is,

$$\omega = \alpha\phi + \beta\psi,$$

where α, β are regular series in $t-t', u-u'$.

We have $D\omega = \rho\phi + \sigma\psi$, where ρ, σ are polynomials in t, u of degrees $n-1, m-1$ in u .

$$\text{Thus} \quad \phi(\rho - D\alpha) + \psi(\sigma - D\beta) = 0,$$

and, by Lemma II, $\rho = D\alpha + \theta\psi$, $\sigma = D\beta - \theta\phi$,

where θ is a regular series in $t-t', u-u'$.

Here put $t = t'$, then $\rho(t', u) = \theta(t', u)\psi(t', u)$ for D vanishes; and therefore $\rho(t', u)$ contains the factor $u-u'$ to at least as high a degree as $\psi(t', u)$.

Also, when $t = t'$, $\rho\phi + \sigma\psi = 0$, and therefore $\rho(t', u)$ contains all the factors of $\psi(t', u)$ that are not factors of $\phi(t', u)$, that is, all the factors of $\psi(t', u)$ but $u-u'$, since the axes of t, u have no special position. Hence $\rho(t', u)$ contains all the factors of $\psi(t', u)$ and must vanish identically, since it is only of degree $n-1$. So also $\theta(t', u)$ and $\sigma(t', u) = 0$ identically.

The same argument applies to the terms in ρ, σ, θ of lowest degree in

$t-t'$, so long as that degree is less than q , the power of $t-t'$ which is a factor of D .

Now $\lambda = \rho/D$, $\mu = \sigma/D$, and ρ , σ both contain as high a power of $t-t'$ as D , so that the $t-t'$ series of partial fractions is absent in λ and μ when ω is reconcilable with ϕ , ψ at (t', u') and there is no other intersection for which $t = t'$. It has been supposed that the degree of ω is not greater than $m+n-1$; any polynomial of higher degree can have its degree reduced by subtracting an expression of the form $\eta\phi + \xi\psi$, where η , ξ are polynomials.

It is therefore proved that q , the degree to which the factor $t-t'$ occurs in D , is the same as the number of linearly independent polynomials which are not reconcilable with ϕ , ψ at (t', u') .

Geometrically q is the number of intersections for which (t', u') counts.

If ω is reconcilable with ϕ , ψ at every intersection, then

$$\omega = \lambda\phi + \mu\psi,$$

where λ , μ are polynomials, all the partial fractions disappearing.

12. An important part is now played by Kronecker's interpolation function $\Delta(t, u, \tau, v)$ which may be defined as

$$\begin{vmatrix} \phi(t, u) & \phi(\tau, u) & \phi(\tau, v) \\ \psi(t, u) & \psi(\tau, u) & \psi(\tau, v) \\ 1 & 1 & 1 \end{vmatrix} \div (t-\tau)(u-v).^*$$

Subtracting in turn the third and first columns from the second in the determinant, we see that

$$(t-t')\Delta(t, u, t', u') \quad \text{and} \quad (u-u')\Delta(t, u, t', u')$$

are everywhere reconcilable with ϕ , ψ , if

$$\phi(t', u') = \psi(t', u') = 0,$$

but an important fact now to be proved is that $\Delta(t, u, t', u')$ is not reconcilable with ϕ , ψ at (t', u') .

For suppose that $\Delta(t, u, t', u') = \lambda\phi + \mu\psi$,

where λ , μ are regular series in $t-t'$, $u-u'$. Then

$$(t-t')(u-u')(\lambda\phi + \mu\psi) = \phi(t, u)\psi(t', u) - \phi(t', u)\psi(t, u),$$

* For expressions of this nature compare A. L. Dixon, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp: 51-69, 476-492; Vol. 8, pp. 341-352.

and, by Lemma II,

$$\psi(t', u) = (t-t')(u-u')\lambda + \theta\psi(t, u),$$

$$\phi(t', u) = -(t-t')(u-u')\mu + \theta\phi(t, u),$$

where θ is a regular series in $t-t'$, $u-u'$.

Putting $u = u'$ we have $\theta(t, u') = 0$, since $u-u'$ is not a factor of both $\phi(t, u)$ and $\psi(t, u)$. Thus there is no absolute term in the series θ .

Putting $t = t'$, we have $\theta(t', u) = 1$, so that θ must have an absolute term 1, which is absurd.

Hence $\Delta(t, u, t', u')$ is not reconcilable with ϕ, ψ at (t', u') .

At any other point of intersection (t'', u'') , $\Delta(t, u, t', u')$ is reconcilable with ϕ, ψ for $(t-t')\Delta$ and $(u-u')\Delta$ are so, and the reciprocal of $t-t'$ or $u-u'$ is a regular series in $t-t''$ or $u-u''$.

13. Now let U be such an operation that $U\omega$ denotes collectively the linear combinations of the coefficients in a polynomial ω which must vanish if ω is to be reconcilable with ϕ, ψ at (t', u') , the variables in ω, ϕ, ψ being t, u , and let Y be the similar operation when the variables are τ, v , Y_1 any particular linear combination of the members of the row Y with constant coefficients.

We have

$$(t-\tau)\Delta(t, u, \tau, v) = \begin{vmatrix} \phi(t, u), & \frac{\phi(\tau, u) - \phi(\tau, v)}{u-v}, & \phi(\tau, v) \\ \psi(t, u), & \frac{\psi(\tau, u) - \psi(\tau, v)}{u-v}, & \psi(\tau, v) \\ 1, & 0, & 1 \end{vmatrix},$$

where the second column is fractional only in appearance, so that each term in the expanded determinant is destroyed by U or by Y .

$$\text{Thus} \quad UY_1 \{ (t-\tau)\Delta(t, u, \tau, v) \} = 0,$$

$$\text{or} \quad U \{ (t-t')Y_1\Delta \} = UY_1 \{ (\tau-t')\Delta \}.$$

From this it may be proved that $Y_1\Delta$ cannot be reconcilable with ϕ, ψ at (t', u') . If it were, then $(t-t')Y_1\Delta$ would be so, that is, if

$$UY_1\Delta = 0,$$

$$\text{then} \quad U \{ (t-t')Y_1\Delta \} = 0,$$

$$\text{and} \quad UY_1 \{ (\tau-t')\Delta \} = 0.$$

Thus $Y_1 \{(\tau-t')\Delta\}$ is reconcilable and similarly so is $Y_1 \{(\nu-u')\Delta\}$, and, in general, $Y_1 \{(\tau-t')^\alpha (\nu-u')^\beta \Delta\}$, where α, β are any positive integers.

Now, for at least one pair of indices α, β ,

$$Y_1 \{(\tau-t')^\alpha (\nu-u')^\beta \Delta\}$$

is a mere multiple of the absolute term in Δ as expanded in a series regular in $\tau-t', \nu-u'$, and therefore this absolute term, namely $\Delta(t, u, t', u')$, is reconcilable with ϕ, ψ at (t', u') , which has been proved impossible.

Hence $Y_1 \Delta(t, u, \tau, \nu)$ is not reconcilable with ϕ, ψ at (t', u') , whatever linear combination Y_1 may be of the row Y , but *the row*

$$Y\Delta(t, u, \tau, \nu)$$

furnishes the complete set of q polynomials in t, u that are not reconcilable with ϕ, ψ at (t', u') .

A like argument shews that at (t'', u'') , any other intersection, these functions are all reconcilable with ϕ, ψ .

14. Now any polynomial ω may be made reconcilable at (t', u') with ϕ and ψ by subtracting from it constant multiples of such a complete set as $Y\Delta(t, u, \tau, \nu)$ has been proved to be: let the row of multipliers be C , so that

$$\omega - CY\Delta(t, u, \tau, \nu)$$

is reconcilable with ϕ, ψ at (t', u') .

The row $Y\Delta(t, u, \tau, \nu)$ includes $\Delta(t, u, t', u')$ itself, a polynomial of degree $m+n-2$ in t, u : all the $q-1$ other members of the row may be taken as of lower degree than this, since any term in $\Delta(t, u, \tau, \nu)$ which actually contains τ or ν must be of degree $m+n-3$ at most in t, u .

By carrying out the same process at each point of intersection we arrive at a polynomial

$$\omega - \Sigma CY\Delta(t, u, \tau, \nu),$$

in which Σ refers to the different intersections, and for this polynomial all the partial fractions in λ, μ (§ 4) are wanting, that is,

$$\omega = \Sigma CY\Delta(t, u, \tau, \nu) + \eta\phi + \xi\psi,$$

where η, ξ are polynomials.

The reasoning by which Jacobi's theorem was proved (§ 7) still applies, since the terms of degree $m+n-2$ in $\Delta(t, u, t', u')$ are still independent of (t', u') . Thus, if ω is of the degree $m+n-3$ only, the sum of the co-

efficients of such functions as $\Delta(t, u, t', u')$, say of the *last** coefficients in ω , at the different intersections, is zero: this is *Jacobi's theorem in its generalized form*, and it clearly reduces to the ordinary form when all the intersections are simple.† Some care is needed in forming the last coefficient; an illustration is given below in Example III, § 25.

It follows from the theorem that, *if ω satisfies the conditions of reconcilability at all the intersections with the exception of the last at one intersection, this last coefficient must vanish also, and $\omega = \lambda\phi + \mu\psi$, where λ, μ are polynomials.*

The use of the phrase "last coefficient" accords with the theory of priority to be developed in § 15. It may be worth while to recapitulate the method of its formation at a point (t', u') .

Form the expression $\Delta(t, u, \tau, v) - \Delta(t, u, t', u')$, and write down the conditions for this, as a function of τ, v , to be reconcilable with $\phi(\tau, v)$ and $\psi(\tau, v)$ at (t', u') . These conditions are that $q-1$ polynomials in t, u should vanish, say the polynomials

$$\omega_1, \omega_2, \dots, \omega_{q-1}.$$

Any polynomial ω can be expressed in the form

$$a_1\omega_1 + a_2\omega_2 + \dots + a_{q-1}\omega_{q-1} + a_q\omega_q + \varpi,$$

where ϖ is reconcilable with ϕ, ψ at (t', u') and

$$\omega_q = \Delta(t, u, t', u').$$

Then a_q is the "last coefficient" of ω at (t', u') .

Priority among the Conditions of Reconcilability.

15. Among the conditions that ω should be reconcilable with ϕ, ψ at (t', u') a certain order is to be observed. The conditions include those for the reconcilability of $(t-t')^\alpha(u-u')^\beta\omega$, where α, β are any positive integers, and indeed of $\omega\chi$ where χ is any polynomial. When two condi-

* From the above discussion of $\tau\Delta$ (§ 13) it appears that $\Delta(t, u, \tau, v)$ can be divided into three parts—

- (1) $\Delta(t, u, t', u')$ with coefficient unity.
- (2) Other members of the row $\tau\Delta(t, u, \tau, v)$, multiplied by powers or products of $\tau - t'$ and $v - u'$, which are not reconcilable at (t', u') with $\phi(\tau, v), \psi(\tau, v)$.
- (3) A part reconcilable at (t', u') with $\phi(\tau, v)$ and $\psi(\tau, v)$, and involving t, u also.

Hence in $\Delta(t, u, \tau, v)$ the "last" coefficient at (t', u') with respect to t, u is a function of τ, v differing from 1 by an expression reconcilable at (t', u') with ϕ and ψ .

† In the other extreme case when $\phi = t^m, \psi = u^n$, so that

$$t' = u' = 0 \quad \text{and} \quad \Delta(t, u, t', u') = t^{m-1}u^{n-1},$$

the theorem states that when ω is of degree $m+n-3$ it cannot contain a term in $t^{m-1}u^{n-1}$, which is evident.

tions (1), (2) are so related that ω must satisfy (1) as a condition necessary that (2) may be satisfied by $\omega\chi$, where χ is an arbitrary polynomial, then (1) may be called *earlier* than (2), and (2) *later* than (1). The vanishing of ω itself, that is, of the absolute term in the expansion at (t', u') , is the earliest condition to be satisfied at that point, for α, β may always be so chosen that this is the only coefficient affected by the condition (2) as applied to $(t-t')^\alpha (u-u')^\beta \omega$.

The whole system of conditions at (t', u') may be called a complete one in the sense that if satisfied by ω it is also satisfied by $\omega\chi$, where χ is any polynomial. It includes what may be called minor complete systems which have the property of completeness in the same sense but form only part of the whole system. Any minor complete system must contain all conditions which are earlier than any one of its members, and it does not cease to be complete if it is reduced by the cancelling of a condition when it contains no later condition than the one cancelled. Thus any minor complete system may be taken to belong to a sequence of complete systems, in which each contains the system before it and an additional condition, the first of the sequence consisting of one condition and the last of q .

16. Again, if $(t-t')\omega$ and $(u-u')\omega$ are both reconcilable at (t', u') , then ω can be made reconcilable there by subtracting a constant multiple of $\Delta(t, u, t', u')$.

For let $E(t)$ be D divided by the highest power of $t-t'$ that is a factor in D . Then, since the $t-t'$ fractions are absent from the expressions of § 4 for $(t-t')\omega$ and $(u-u')\omega$, we have

$$E(t)(t-t')\omega = \alpha\phi + \beta\psi,$$

$$E(t)(u-u')\omega = \gamma\phi + \delta\psi,$$

where $\alpha, \beta, \gamma, \delta$ are polynomials.

$$\text{Thus } \phi[a(u-u') - \gamma(t-t')] = \psi[\delta(t-t') - \beta(u-u')]$$

and

$$a(u-u') - \gamma(t-t') = \rho\psi,$$

$$\delta(t-t') - \beta(u-u') = \rho\phi,$$

where ρ is a polynomial.

Hence

$$\begin{aligned} a(u-u') - \rho\{\psi(t', u) - \psi(t', u')\} &= \gamma(t-t') + \rho\{\psi(t, u) - \psi(t', u)\} \\ &= \sigma(t-t')(u-u'), \text{ say,} \end{aligned}$$

where σ is a polynomial; for the first member of the equality has the factor $u-u'$, and the second has $t-t'$.

Similarly,

$$\begin{aligned}\beta(u-u') + \rho \{ \phi(t', u) - \phi(t', u') \} &= \delta(t-t') - \rho \{ \phi(t, u) - \phi(t', u) \} \\ &= \theta(t-t')(u-u'),\end{aligned}$$

where θ is also a polynomial.

Hence

$$\begin{aligned}(t-t')(u-u')(\sigma\phi + \theta\psi) \\ &= (t-t')(\gamma\phi + \delta\psi) + \rho\phi(t, u) \{ \psi(t, u) - \psi(t', u) \} - \rho\psi(t, u) \{ \phi(t, u) - \phi(t', u) \} \\ &= E(t)(t-t')(u-u')\omega - \rho(t-t')(u-u')\Delta(t, u, t', u'),\end{aligned}$$

or
$$E(t)\omega = \rho\Delta(t, u, t', u') + \sigma\phi + \theta\psi,$$

which proves the theorem since ρ, σ, θ are polynomials, and $E(t')$ is not zero, for the reciprocal of $E(t)$ is a regular series in $t-t'$, and the only irreconcilable term on the right is that which contains the absolute term of ρ .

17. Let $\omega_1, \omega_2, \dots, \omega_q$ be a complete set of the polynomials not reconcilable with ϕ, ψ at (t', u') , so that any polynomial ω whatever can be made reconcilable by subtracting such an expression as

$$a_1\omega_1 + a_2\omega_2 + \dots + a_q\omega_q,$$

where a_1, a_2, \dots, a_q are constants. The whole system of conditions for ω to be reconcilable is

$$a_1 = a_2 = \dots = a_q = 0,$$

and we shall suppose $\omega_1, \omega_2, \dots, \omega_q$, such that the minor system

$$a_1 = a_2 = \dots = a_p = 0$$

is complete for all values of p less than q .

18. It follows that ω_q must be substantially identical with $\Delta(t, u, t', u')$. For the conditions $a_1 = a_2 = \dots = a_{q-1} = 0$ form a minor complete system restricting ω to the form $a_q\omega_q$, and therefore $\omega_q(t-t')$ and $\omega_q(u-u')$ satisfy the same system, and constants a, b exist, such that $\omega_q(t-t') + a\omega_q$ and $\omega_q(u-u') + b\omega_q$ are reconcilable. If a, b are not both zero, it follows that ω_q is itself reconcilable, which is contrary to hypothesis. Hence $\omega_q(t-t')$ and $\omega_q(u-u')$ are both reconcilable, that

is, ω_q only differs by reconcilable expressions from a constant multiple of $\Delta(t, u, t', u')$ according to the theorem just now proved (§ 16).

There is then no loss of generality in putting

$$\omega_q = \Delta(t, u, t', u').$$

By subtracting suitable multiples of ω_q from $\omega_1, \omega_2, \dots, \omega_{q-1}$, we secure that the "last" coefficient in each of these is zero, and then the "last" coefficient in ω will be a_q : this step does not affect a_1, a_2, \dots, a_{q-1} .

19. Putting
$$a_1 \omega_1 + a_2 \omega_2 + \dots + a_q \omega_q = \omega$$

and
$$b_1 \omega_1 + b_2 \omega_2 + \dots + b_q \omega_q = \chi,$$

we have $\omega\chi$ made reconcilable by subtracting a new expression of the same form, say

$$c_1 \omega_1 + c_2 \omega_2 + \dots + c_q \omega_q;$$

and c_1, c_2, \dots, c_q are all linear in the a 's and also in the b 's. Let the "last" coefficient in $\omega\chi$, that is

$$\begin{aligned} c_q &= a_1 B_1 + a_2 B_2 + \dots + a_q B_q \\ &= b_1 A_1 + b_2 A_2 + \dots + b_q A_q, \end{aligned}$$

where A_1, \dots, A_q are linear and homogeneous in a_1, \dots, a_q with constant coefficients, and B_1, \dots, B_q are the same functions of b_1, \dots, b_q .

The determinant of the coefficients in these functions, that is,

$$\frac{\partial(A_1, \dots, A_q)}{\partial(a_1, \dots, a_q)}$$

does not vanish.

For to any irreconcilable polynomial ω there corresponds an index p , such that $\omega \{a(t-t') + b(u-u')\}^p$ is reconcilable for all values of the constants a, b , while $\omega \{a(t-t') + b(u-u')\}^{p-1}$ is not. That is, for certain indices α, β , $\omega(t-t')^\alpha (u-u')^\beta$ is not reconcilable while $\omega(t-t')^{\alpha+1} (u-u')^\beta$ and $\omega(t-t')^\alpha (u-u')^{\beta+1}$ are so. Hence, if b_1, b_2, \dots, b_q are so chosen that $\chi - (t-t')^\alpha (u-u')^\beta$ is reconcilable, c_q does not vanish: that is, A_1, \dots, A_q cannot all vanish unless a_1, \dots, a_q do so, which is the thing to be proved.

Another way of stating this result is to say that the conditions for ω to be reconcilable with ϕ, ψ at (t', u') are the conditions that c_q , the "last" coefficient in $\omega\chi$, may vanish for an arbitrary polynomial χ .

20. Now let ω be restricted, but only by the minor complete system

$$a_1 = a_2 = \dots = a_p = 0,$$

and consider the conditions to be satisfied by χ if $\omega\chi$ is to be reconcilable. Since

$$c_q = a_{p+1}B_{p+1} + \dots + a_qB_q,$$

it is necessary that $B_{p+1} = B_{p+2} = \dots = B_q = 0$.

The other conditions are all of the form $c'_q = 0$, where c'_q is the coefficient of ω_q in $(t-t')^a(u-u')^b\omega\chi$, and may therefore be written

$$a'_1B_1 + a'_2B_2 + \dots + a'_qB_q = 0,$$

where a'_1, a'_2, \dots, a'_q are for $(t-t')^a(u-u')^b\omega$ what a_1, a_2, \dots, a_q are for ω . Now, by hypothesis, a'_1, a'_2, \dots, a'_p vanish, because a_1, a_2, \dots, a_p vanish, and thus the conditions

$$B_{p+1} = B_{p+2} = \dots = B_q = 0$$

are sufficient as well as necessary. These conditions form a minor complete system, for clearly if $\omega\chi$ is reconcilable, so is $\omega \cdot \chi(t-t')^a(u-u')^b$. The two systems

$$a_1 = a_2 = \dots = a_p = 0$$

and

$$A_{p+1} = A_{p+2} = \dots = A_q = 0,$$

will be called *associated* systems.

21. The extended residue theorem as applied to $\omega\chi$ is that the sum of all such coefficients as c_q with respect to the different intersections is zero. It has now been found that c_q is the sum of q terms, each of which is a product of the form aB , the first factor being a linear function of the coefficients of ω , and the second a linear function of the coefficients of χ ; when $q > 1$, the reduction of c_q to this form is not always unique.

We may, by a convention, speak of the curves $\phi = 0$, $\psi = 0$ as having q coincident or consecutive intersections at (t', u') , and we may distribute these, assigning p to ω and $q-p$ to χ , the meaning being that ω is to satisfy a minor complete system of p conditions, and χ the associated complete system of $q-p$ conditions. The total number of intersections is the degree of the eliminant, mn , and this is also the total number of terms of form aB in the residue theorem.

Thus, if χ passes through all the points assigned to it, and ω through all but one of those assigned to it, all these terms vanish but one: hence that term must also vanish, and ω must pass through the excepted point, unless χ passes through an additional point in the conventional sense.

Thus we have the modified *theorem of Cayley and Bacharach*:—If ω is a polynomial of degree r , less than $m+n-2$, restricted by "complete" systems of conditions at the different points where ϕ, ψ vanish, the whole

number of conditions being $mn - \frac{1}{2}(m+n-r-1)(m+n-r-2)$, then ω is of the form $\lambda\phi + \mu\psi$, where λ, μ are polynomials, unless there is a polynomial χ , of degree $m+n-r-3$, which satisfies all the

$$\frac{1}{2}(m+n-r-1)(m+n-r-2)$$

conditions which are included in the "associated" complete systems. (Compare *Camb. Phil. Proc.*, Vol. x., p. 474.)

22. *Modified Theorem of Riemann and Roch*:—Of the mn intersections of ϕ, ψ in the conventional sense take a number h , that is, take minor, or other, complete systems (A) of conditions at their common points, amounting to h conditions in all. Let B denote the systems associated with A , amounting to $mn-h$ in all.

Let k be the number of arbitrary coefficients in an r -ic satisfying B , where $r < m+n-2$, and l the number of arbitrary coefficients in an $(m+n-r-3)$ -ic satisfying A . Then shall

$$k = \frac{1}{2}(r+1)(r+2) - mn + h + l.$$

For, if ω, χ are of degrees $r, m+n-r-3$, and we denote by $A\chi, B\omega$ the rows of linear functions in their coefficients that have to vanish on account of the conditions specified, Jacobi's theorem applied to $\omega\chi$ gives a result that may be written

$$A\chi \cdot B'\omega + A'\chi \cdot B\omega = 0.$$

Suppose then that $B\omega = 0$, and this equation includes k relations involving $A\chi$, but of these some are illusory, namely, those in which $B'\omega = 0$. The number of these, if $r > m-3$, and also $> n-3$, is

$$\frac{1}{2}(r-m+1)(r-m+2) + \frac{1}{2}(r-n+1)(r-n+2),$$

for ω may be $\sigma\psi$ or $\rho\phi$, where σ, ρ are of degrees $r-n, r-m$.

Hence the conditions $A\chi = 0$ are in number only

$$h - k + \frac{1}{2}(r-m+1)(r-m+2) + \frac{1}{2}(r-n+1)(r-n+2),$$

and $l \geq \frac{1}{2}(m+n-r-1)(m+n-r-2) - h + k - \frac{1}{2}(r-m+1)(r-m+2)$

$$- \frac{1}{2}(r-n+1)(r-n+2),$$

or $l \geq mn - \frac{1}{2}(r+1)(r+2) + k - h. \quad (3)$

Again, suppose $A\chi = 0$, and it follows that

$$A'\chi B\omega = 0,$$

so that the expressions $B\omega$ are connected by l linear relations of which none can be illusory, since χ is of lower degree than either ϕ or ψ , and cannot therefore satisfy all the relations $A\chi = A'\chi = 0$.

Thus k , the number of arbitrary coefficients in ω when $B\omega = 0$, is

$$\geq \frac{1}{2}(r+1)(r+2) - mn + h + l. \quad (4)$$

By comparing (3) and (4) we have the result stated, that is, the modified and extended theorem of Riemann and Roch, which includes, substantially, that of Cayley and Bacharach.

23. The following is a statement of the *theorem* which appears to be the *converse of Jacobi's*.

Take a number of pairs of polynomials in t, u , say ϕ_1 and ψ_1 both vanishing at (t_1, u_1) , ϕ_2 and ψ_2 both vanishing at (t_2, u_2) , and so on. Let B denote the conditions that a polynomial may be reconcilable with ϕ_1 and ψ_1 at (t_1, u_1) with ϕ_2 and ψ_2 at (t_2, u_2) , and so on, and let these conditions be altogether mn in number. Any given polynomial will have a "last" coefficient at each of the points (t_1, u_1) , (t_2, u_2) , ... with respect to the pair of polynomials thus associated with that point. If the "last" coefficients at the different points in all polynomials of degree $m+n-3$ satisfy a certain homogeneous linear relation with constant coefficients, none of which is zero, then the number of m -ics and $(n-3)$ -ics which satisfy B is together at least

$$\frac{1}{2}(m-n+1)(m-n+2)+1,$$

and the number of n -ics and $(m-3)$ -ics which satisfy B is at least

$$\frac{1}{2}(n-m+1)(n-m+2)+1,$$

For the proof see *Camb. Phil. Proc.*, Vol. xv, pp. 479, 480.

The Jacobian.

24. When (t, u) and (τ, v) coincide, $\Delta(t, u, \tau, v)$ becomes the Jacobian $\partial(\phi, \psi)/\partial(t, u)$. An interesting property of this Jacobian is that the "last" coefficient in it at (t', u') is q : this may be proved as follows.

Since

$$(t-\tau)\Delta(t, u, \tau, v) = \lambda\{\phi(t, u) - \phi(\tau, v)\} + \mu\{\psi(t, u) - \psi(\tau, v)\},$$

where λ, μ are polynomials in t, u, τ, v , it follows that the "last" coefficient with respect to t, u in $(t-t')^{a-1}(u-u')^b(t-\tau)\Delta(t, u, \tau, v)$ is the same as in $-(t-t')^{a-1}(u-u')^b\{\lambda\phi(\tau, v) + \mu\psi(\tau, v)\}$, and therefore, with respect to τ, v , is reconcilable with ϕ, ψ .

Thus, as a function of τ, v , the last coefficient with respect to t, u in

$$(t-t')^a(u-u')^b\Delta(t, u, \tau, v)$$

differs only by a reconcilable polynomial from that in

$$(\tau - t')(t - t')^{\alpha-1} (u - u')^{\beta} \Delta(t, u, \tau, \nu),$$

or similarly, that in

$$(\nu - u')(t - t')^{\alpha} (u - u')^{\beta-1} \Delta(t, u, \tau, \nu),$$

or, by successive steps, that in

$$(\tau - t')^{\alpha} (\nu - u')^{\beta} \Delta(t, u, \tau, \nu),$$

which is $(\tau - t')^{\alpha} (\nu - u')^{\beta}$ (§ 14, note).

Since any polynomial ω is the sum of such terms as $A(t - t')^{\alpha} (u - u')^{\beta}$, it follows that the last coefficient at (t', u') with respect to t, u in $\omega(t, u) \Delta(t, u, \tau, \nu)$ differs from $\omega(\tau, \nu)$ by a polynomial reconcilable with ϕ, ψ at (t', u') .

Thus, if we write

$$\Delta(t, u, \tau, \nu) \equiv \omega_1 \bar{\Omega}_1 + \omega_2 \bar{\Omega}_2 + \dots + \omega_q \bar{\Omega}_q \pmod{\phi, \psi},$$

where $\omega_1, \omega_2, \dots, \omega_q$ are the polynomials of § 18 in t, u , and the bar indicates that τ, ν are to be substituted for t, u in polynomials $\Omega_1, \dots, \Omega_q$, we have

$$\begin{aligned} \bar{\omega} &\equiv \text{last coefficient in } (a_1 \omega_1 + \dots + a_q \omega_q)(\omega_1 \bar{\Omega}_1 + \dots + \omega_q \bar{\Omega}_q) \pmod{\bar{\phi}, \bar{\psi}} \\ &\equiv A_1 \bar{\Omega}_1 + A_2 \bar{\Omega}_2 + \dots + A_q \bar{\Omega}_q \quad (\S 19). \end{aligned}$$

Thus the substitutions Ω for ω and A for a are contragredient.

Now the last coefficient in $\omega\chi$ is

$$a_1 B_1 + \dots + a_q B_q,$$

and has the value 1, if

$$a_p = B_p = 1,$$

$$a_r = B_r = 0 \quad (r \neq p),$$

that is, if

$$\omega = \omega_p, \quad \chi = \Omega_p.$$

Any other coefficient in $\omega\chi$ is of the form

$$a'_1 B_1 + \dots + a'_q B_q,$$

where a'_1, \dots, a'_p must vanish, since a_1, \dots, a_{p-1} vanish. Thus

$$\omega_p \Omega_p - \Delta(t, u, t', u')$$

is reconcilable with ϕ, ψ at (t', u') .

$$\begin{aligned} \text{Now, at } (t', u'), \quad \Delta(t, u, t, u) &\equiv \omega_1 \Omega_1 + \dots + \omega_q \Omega_q \pmod{\phi, \psi} \\ &\equiv q \Delta(t, u, t', u'), \end{aligned}$$

that is, the "last" coefficient in the Jacobian at any common point is q , the number of intersections for which that point counts.

The sum of all the numbers such as q is mn , and in accordance with this the sum of the highest terms in the Jacobian, that is, those of degree $m+n-2$, differs from mn times the sum of the highest terms in $\Delta(t, u, \tau, \nu)$ by a polynomial which can be expressed in the form

$$\lambda \bar{\phi} + \mu \bar{\psi},$$

where $\bar{\phi}$ is the sum of the highest terms in ϕ ,

and $\bar{\psi}$ " " " " ψ ,

and λ, μ are polynomials.

Illustrations.

25. A few special illustrations of the general theory may be given.

I. Suppose the curves $\phi = 0, \psi = 0$ to have contact, of order α only, at the origin, but no other special feature, that is, suppose ϕ, ψ to be reconcilable at the origin with

$$u - p_1 t - p_2 t^2 - \dots - p_\alpha t^\alpha, t^{\alpha+1},$$

where $p_1, p_2, \dots, p_\alpha$ are constants.

Then it is easy to see that there are $\alpha+1$ polynomials not reconcilable at the origin with ϕ, ψ , and that these may be taken to be

$$1, t, t^2, \dots, t^\alpha,$$

or, more generally, to be expressions beginning with these terms when arranged in ascending powers.

$$\text{If } \omega = a_0 + a_1 t + a_2 t^2 + \dots + a_\alpha t^\alpha,$$

$$\chi = b_0 + b_1 t + \dots + b_\alpha t^\alpha,$$

then the coefficient of t^α in $\omega\chi$ is

$$a_0 b_\alpha + a_1 b_{\alpha-1} + \dots + a_\alpha b_0,$$

and the "last" coefficient is a constant multiple of this, at least when all the earlier coefficients vanish.

The statement that ω passes through β consecutive points at the origin on the curve ϕ or ψ means that

$$a_0, a_1, \dots, a_{\beta-1},$$

all vanish, and the "associated" system of conditions for χ is that $b_0, b_1, \dots, b_{\alpha-\beta}$ all vanish, or that χ pass through $\alpha-\beta+1$ consecutive points of ϕ or ψ at the origin.

II. Let the curves have a common node at the origin, and let the axes be so chosen that

$$\phi \equiv p_1 t^2 + q_1 u^2 + \dots,$$

$$\psi \equiv p_2 t^2 + q_2 u^2 + \dots,$$

the term in tu being absent, and

$$p_1 q_2 - p_2 q_1 \neq 0.$$

Then four irreconcilable polynomials at the origin are 1, t , u , tu , and so are

$$1 + at + bu + ctu, \quad a't + b'u + c'tu, \quad a''t + b''u + c''tu, \quad tu,$$

unless

$$a'b'' = a''b'.$$

Either of these sets may be taken for $\omega_1, \omega_2, \omega_3, \omega_4$ in § 17, but the further condition of § 18 is not generally satisfied by them.

Now, if

$$\omega = a_1 + a_2 t + a_3 u + a_4 tu,$$

$$\chi = b_1 + b_2 t + b_3 u + b_4 tu,$$

the coefficient of tu in $\omega\chi$ is $a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$, and this is proportional to the last coefficient if the three earlier ones vanish.

If, then, $\omega\chi$ is to be reconcilable with ϕ and ψ at the origin,

- (1) when ω must pass through the origin, but is not otherwise restricted, χ must have a double point at the origin;
- (2) if ω touches $u = \mu t$ at the origin, χ must touch $u = -\mu t$ at the origin;
- (3) if a double point at the origin is assigned to ω , then χ must pass through the origin.

Hence many theorems may be deduced such as the following:—If two given quartics have a common node at the origin, no quartic can pass through the origin, touching a given line there, and also pass through eleven of the other twelve common points without having a node at the origin and passing through the twelfth other common point, unless the given tangent at the origin, and the line from the origin to the twelfth point, are conjugate in the involution to which belong the tangents at the origin to the given quartics.

III. Take $\phi = t^2 + au^3, \quad \psi = u^2 + bt^3,$

$$\omega = c + et + fu + gt^2 + htu + ku^2 + lt^3 + mt^2u + nu^2 + pu^3,$$

and let us verify Jacobi's theorem.

We have

$$\Delta(t, u, \tau, v) = (t + \tau)(u + v) - ab(t^2 + t\tau + \tau^2)(u^2 + uv + v^2),$$

and in order to find the "last" coefficient in ω at $(0, 0)$ we must pick out the coefficients of $1, \tau, v, \tau v$ in $\Delta(t, u, \tau, v)$; they are

$$tu - abt^2u^2, \quad u - abtu^2, \quad t - abt^2u, \quad 1 - abtu,$$

and on account of the second term in the fourth of these, the last coefficient in ω is not h but $h + abc$: the other second terms are reconcilable with ϕ, ψ at the origin.

The unreconcilable part of ω may, in fact, be written

$$c(1 - abtu) + e.t + f.u + (h + abc)tu.$$

The intersections other than $(0, 0)$ are given by

$$t^5 = -a^{-2}b^{-3}, \quad tu = a^{-1}b^{-1},$$

and

$$J = 4tu - 9abt^2u^2,$$

which has the value $-5a^{-1}b^{-1}$ at each of the five points.

The sum of the values of ω at the five points is

$$5c + 5ha^{-1}b^{-1},$$

and the sum of the values of ω/J is therefore $-h - abc$, that is, equal and opposite to the last coefficient at the origin, as it should be.

This example illustrates the method of finding the last coefficient, and shews that it is not safe to neglect all but the lowest terms in ϕ, ψ for the purpose.

IV. For the ordinary theory of adjoint polynomials of a curve whose only singularities are ordinary multiple points, take ϕ to have a multiple point of order i at the origin, and ψ to have one of order $i-1$ at the origin, but not to touch any branch of ϕ there. Let ϕ_0, ψ_0, \dots denote the terms of lowest degree in ϕ, ψ, \dots .

Suppose ω to be restricted by the condition of having a multiple point of order $i-1$ at the origin: to find the associated system of conditions, suppose $\omega\chi$ to be reconcilable at the origin with ϕ and ψ , and χ_0 to be of degree j . Since ω_0 may be $t^{i-1}, t^{i-2}u, \dots$, we must have

$$t^{i-1}\chi_0 = \lambda\phi_0 + \mu\psi_0,$$

$$t^{i-2}u\chi_0 = \rho\phi_0 + \sigma\psi_0, \dots,$$

where $\lambda, \rho, \mu, \sigma$ are homogeneous polynomials of degrees $j-1, j-1, j, j$. Thus

$$(u\lambda - t\rho)\phi_0 + (u\mu - t\sigma)\psi_0 = 0,$$

and $u\lambda = t\rho$, $u\mu = t\sigma$, if $j < i-1$. In this way it follows that the coefficients of ϕ_0 , such as λ , ρ , in the expressions for

$$t^{i-1}\chi_0, t^{i-2}u\chi_0, t^{i-3}u^2\chi_0, \dots, u^{i-1}\chi_0,$$

form a geometrical progression, and are, in fact, proportional to

$$t^{i-1}, t^{i-2}u, t^{i-3}u^2, \dots, u^{i-1},$$

and the same is true of μ , σ , ... the coefficients of ψ_0 . This is impossible if $j < i-1$, and hence χ_0 must be of degree $i-1$, and χ must also have a multiple point of order $i-1$ at the origin, the associated system of conditions being the same as the original.

Thus, if ϕ has only ordinary multiple points, and ψ is adjoint to ϕ , that is, has an $(i-1)$ -ple point wherever ϕ has an i -ple point, and if k is the number of arbitrary coefficients in a polynomial ω of degree r ($< m+n-2$ but $> m-3$ or $n-3$), which is also adjoint to ϕ and passes through all but h of the other intersections of ϕ and ψ , and l is the number of arbitrary coefficients in a polynomial χ of degree $m+n-r-3$, which is also adjoint to ϕ and passes through the h excepted intersections, then

$$k = \frac{1}{2}(r+1)(r+2) - mn + h + l.$$