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XXIV. *On Tracing Rays through an Optical System. (Third Paper).* By T. SMITH, B.A. (*From the National Physical Laboratory.*)

THE great majority of the rays that are traced by optical computers through a train of lenses lie in a plane containing the axis. The calculations for such rays are very simple compared with those for rays not remaining in the same plane during their passage through the system, but are distinctly more laborious than for those rays which lie close to the optical axis. The equation connecting conjugate axial points when refraction takes place at a single surface may be modified* in a variety of ways to meet the case of rays reaching the surface at large angles and at considerable distances from the axis. It has been pointed out that in some special cases these modified equations may provide a more rapid means of tracing rays than the usual trigonometrical methods. The object of the present note is to illustrate how such equations may be fitted into a scheme for ray tracing of quite general applicability.

The usual equation for the refraction of paraxial rays is

$$\frac{\mu'}{x'} - \frac{\mu}{x} = \frac{\mu' - \mu}{r} = (\mu' - \mu)R = \kappa,$$

or if m is the transverse magnification

$$x\kappa = -\mu(1 - 1/m), \quad x'\kappa = \mu'(1 - m).$$

One of the generalised forms is

$$\frac{\mu'}{x' - p} - \frac{\mu}{x - p} = \frac{\mu' - \mu}{r - p} = j, \quad . \quad . \quad . \quad (1)$$

or if $\mathcal{M}$ is the sine ratio $\mu \sin \psi / \mu' \sin \psi'$, where ψ and ψ' are the inclinations of the ray to the axis before and after refraction,

$$(x - p)j = -\mu(1 - 1/\mathcal{M}), \quad (x' - p)j = \mu'(1 - \mathcal{M}) \quad . \quad . \quad (2)$$

Such forms may be justified immediately, for the sine ratio depends only on the relative distances of the object and image intersection points from the centre of curvature, and is, therefore, undisturbed by replacing the actual refraction by paraxial refraction at a concentric surface; moreover, by changing the radius of curvature the two conjugate points may be made

* Trans. Opt. Soc., Vol. 20., p. 23.

conjugate for paraxial refraction. The satisfaction of these conditions evidently defines the refraction completely.

It has been shown* previously that whenever refraction in an axial plane takes place without aberration, the perpendiculars to the ray from the vertex of the refracting surface before and after refraction are equal. If, then, refraction according to the paraxial law is to account satisfactorily for this non-paraxial ray, the vertex must be so chosen that these two perpendiculars are equal. This result also follows at once from equations (2) if they are multiplied by $\sin \psi$ and $\sin \psi'$ respectively. If, as before, the perpendiculars from the actual vertex are denoted by h and h' ,

$$h + p \sin \psi = h' + p \sin \psi', \quad (3)$$

and since the refraction follows paraxial laws exactly

$$\mu' \sin \psi' - \mu \sin \psi = -j(h + p \sin \psi). \quad (4)$$

Equations (3) and (4) are equivalent to (2), but are in a more desirable form for ray tracing since h and h' are always finite, while x and x' may attain any real values.

If the angular aperture at which refraction takes place is -2α , the deviation is 2δ , and the angle made with the axis by the join of the intersection of the rays with the aplanatic surfaces is 2γ , the inclinations ψ , ψ' to the axis and the angles ϕ , ϕ' with the normal are given by

$$\psi = \gamma - \alpha - \delta, \quad \psi' = \gamma - \alpha + \delta, \quad \phi = \gamma + \alpha - \delta, \quad \phi' = \gamma + \alpha + \delta. \quad (5)$$

Since the equivalent vertex is the point where the external bisector of the angle made by the rays meets the axis, it is easily seen that

$$\frac{r-p}{r} = \frac{\cos(\gamma+\alpha)}{\cos(\gamma-\alpha)},$$

or

$$p = \frac{2er}{1+e}, \quad (6)$$

where

$$e = \tan \alpha \tan \gamma. \quad (7)$$

This is the equation given previously for finding p . Evidently p will ordinarily be a very small quantity. In certain cases its value may be written down with sufficient accuracy without calculation, but in general it must be determined from the known values of h and $\sin \psi$. The equation expressing it in terms of these variables is a bi-quadratic. This would be expected, since there will be solutions corresponding to both

* Proc. Phys. Soc., Vol. XXX., p. 227.

the internal and the external bisectors of the angle between the rays, and two further solutions corresponding to refraction at a surface of opposite curvature, which may evidently satisfy the conditions. The high order of this equation occasions no difficulty, as the small root required is given very approximately by the two lowest terms of the equation, the remaining terms of which may be regarded as corrections to this approximate value. The value of p so found may be applied to (4) to find $\sin \psi'$, and the value of h for the next surface may be determined by applying the usual equation of transference, reckoning the separation from the equivalent vertex.

The object of this method of calculation is to avoid the use of the equation $\psi' = \psi - \phi + \phi'$ applied to the angles themselves, this, the essential characteristic of the trigonometrical method, necessarily involving four references to tables, and usually an interpolation for each. The value of any alternative procedure must lie in its furnishing the value of $\sin \psi'$ to a corresponding degree of accuracy in less time.

From the known values of $\sin \psi$, h and R , the sines of the angles of incidence and emergence are found from the usual formula

$$\mu' \sin \phi' = \mu \sin \phi = \mu (h R + \sin \psi). \quad . \quad . \quad (8)$$

It is then evident from equations (5) that to a first approximation

$$2\alpha = \sin \phi - \sin \psi, \quad 2\gamma = \sin \phi' + \sin \psi,$$

and therefore

$$4e = h R (\sin \phi' + \sin \psi), \quad 2p = h (\sin \phi' + \sin \psi). \quad . \quad (9)$$

This is an underestimate for p , the exact relation being

$$2p = \frac{h (\sin \phi' + \sin \psi)}{\cos (\gamma - \delta) \cos (\gamma - \alpha) \cos (\alpha + \delta)},$$

which suggests as a corrected form

$$2p = \frac{h (\sin \phi' + \sin \psi)}{1 - \frac{1}{8} \{ (\sin \phi + \sin \psi)^2 + (\sin \phi' - \sin \psi)^2 + (\sin \phi' - \sin \phi + 2 \sin \psi)^2 \}} \quad (10)$$

and this is sufficiently accurate for all ordinary purposes. Other forms may be found without difficulty by constructing equations satisfied by e . Thus from equations (5)

$$\begin{aligned} \sin \phi - \sin \psi &= 2 \sin \alpha \cos (\gamma - \delta), \quad \sin \phi + \sin \psi = 2 \cos \alpha \sin (\gamma - \delta) \\ \sin \phi' + \sin \psi &= 2 \sin \gamma \cos (\alpha + \delta), \quad \sin \phi' - \sin \psi = 2 \cos \gamma \sin (\alpha + \delta). \end{aligned}$$

Combining these equations with e

$$\begin{aligned}\sin \varphi - \sin \psi + e(\sin \varphi + \sin \psi) &= 2 \sin \alpha \cos \delta / \cos \gamma, \\ \sin \varphi' + \sin \psi + e(\sin \varphi' - \sin \psi) &= 2 \sin \gamma \cos \delta / \cos \alpha, \\ \sin \varphi - \sin \psi - e(\sin \varphi' - \sin \psi) &= 2 \tan \alpha \cos (\alpha + \gamma) \cos \delta, \\ \sin \varphi' + \sin \psi - e(\sin \varphi + \sin \psi) &= 2 \tan \gamma \cos (\alpha + \gamma) \cos \delta, \\ (\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi) - e(\sin \varphi + \sin \psi)(\sin \varphi' - \sin \psi) \\ &= 4 \sin \alpha \sin \gamma \cos (\alpha + \gamma), \\ (\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi) - e^2(\sin \varphi + \sin \psi)(\sin \varphi' - \sin \psi) \\ &= 4e \cos (\alpha + \gamma) \cos \delta \cos \psi,\end{aligned}$$

from which $\cos (\alpha + \gamma)$ and $\cos \delta$ have to be eliminated. Thus

$$\begin{aligned}& 4e[\sin \varphi - \sin \psi - e(\sin \varphi' - \sin \psi)][\sin \varphi' + \sin \psi - e(\sin \varphi + \sin \psi)] \\ &= (1 - e)[\sin \varphi - \sin \psi + e(\sin \varphi + \sin \psi)][\sin \varphi' + \sin \psi + e(\sin \varphi' - \sin \psi)] \\ &\quad \times [(\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi) - e(\sin \varphi + \sin \psi)(\sin \varphi' - \sin \psi)] \\ &= [(\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi) - e^2(\sin \varphi + \sin \psi)(\sin \varphi' - \sin \psi)]^2 \sec^2 \psi.\end{aligned}$$

The second of these equations is ordinarily to be preferred. Let the correcting factor $\mathfrak{E}$ be taken as independent variable, where

$$4e\mathfrak{E} = (\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi). \quad (11)$$

The second equation then becomes

$$\begin{aligned}\mathfrak{E}[\mathfrak{E}^2 - \tfrac{1}{4}\mathfrak{E}(\sin^2 \varphi + \sin^2 \varphi' - 2 \sin^2 \psi) \\ + \tfrac{1}{16}(\sin^2 \varphi - \sin^2 \psi)(\sin^2 \varphi' - \sin^2 \psi)] \cos^2 \psi \\ = [\mathfrak{E}^2 - \tfrac{1}{16}(\sin^2 \varphi - \sin^2 \psi)(\sin^2 \varphi' - \sin^2 \psi)]^2. \quad (12)\end{aligned}$$

with the solution

$$\begin{aligned}\mathfrak{E} = 1 - \tfrac{1}{4}(\sin^2 \varphi + \sin^2 \varphi' + 2 \sin^2 \psi) - \tfrac{1}{16}(\sin^4 \varphi + \sin^4 \varphi' + \sin^4 \psi \\ - \sin^2 \varphi \sin^2 \varphi' - \sin^2 \varphi \sin^2 \psi - \sin^2 \varphi' \sin^2 \psi) - \dots\end{aligned}$$

The terms of the fourth order are obviously small in many cases, and since $\mathfrak{E}$ is a correcting factor to a small quantity, they may often be neglected. The value of p is to be obtained from

$$2p = \frac{h(\sin \varphi' + \sin \psi)}{\mathfrak{E} + \tfrac{1}{4}hR(\sin \varphi' + \sin \psi)}, \quad (13)$$

which agrees with (10) when terms of the fourth order in $\mathfrak{E}$ are neglected.

One of the advantages to be expected from tracing rays by calculating the p 's of successive surfaces is that under similar conditions of refraction the values of p are closely proportional to the thicknesses of the lenses if these are determined by the lens aperture, and are independent of the focal length. It will

thus often happen that a computer who employs this system will be able from his past experience to assume sufficiently accurate values for p by basing them on the lens thicknesses. Nevertheless it is at times convenient to disregard p entirely, and to eliminate it from equations (3) and (4). It will be shown that this leads to a correcting factor quite symmetrical in ϕ , ϕ' and ψ .

Since there is no aberration when $h=0$, it is permissible to assume for p the form

$$p = \frac{hP}{1-P\sin\psi}, \quad \dots \dots \dots (14)$$

which gives

$$h + p \sin \psi = \frac{h}{1-P\sin\psi}$$

and

$$\frac{r-p}{r} = \frac{1-P\sin\phi}{1-P\sin\psi}$$

giving for (4)

$$\mu' \sin \psi' - \mu \sin \psi = -\kappa h / (1-P\sin\phi). \quad \dots (15)$$

In this form the equation is not inconvenient for use, but an alternative form free from refractive indices

$$\sin \psi' - \sin \psi = (\sin \phi' - \sin \phi) \frac{1-P\sin\psi}{1-P\sin\phi}. \quad \dots (16)$$

is obtained by adding $(\mu - \mu') \sin \psi$ to both sides. By substituting in (3) this value of $\sin \psi'$ and by the original assumption the two equations

$$\frac{h'}{h} = \frac{1-P\sin\psi'}{1-P\sin\psi} = \frac{1-P\sin\phi'}{1-P\sin\phi} \quad \dots \dots (17)$$

for h' are found. It remains to find a formula for P . Since P vanishes with $\sin \phi' + \sin \psi$, put

$$2PE = \sin \phi' + \sin \psi \quad \dots \dots \dots (18)$$

Comparing (13) and (14)

$$E = \mathcal{E} + \frac{1}{4}(\sin \phi + \sin \psi)(\sin \phi' + \sin \psi) \quad \dots \dots (19)$$

and the elimination of $\mathcal{E}$ between (12) and (19) gives

$$E^4 - E^3(1+s_2) + \frac{1}{4}E^2(s_2^2 + s_1s_3 + s_1^2 + s_2) - \frac{1}{8}E(s_1 + s_3)(s_1s_2 - s_3) + \frac{1}{64}(s_1s_2 - s_3)^2 = 0 \quad (20)$$

where

$$s_1 = \sin \phi + \sin \phi' + \sin \psi, \quad s_2 = \sin \phi \sin \phi' + \sin \phi \sin \psi + \sin \phi' \sin \psi, \\ s_3 = \sin \phi \sin \phi' \sin \psi,$$

showing that E is symmetrical in φ , φ' , ψ . This is also seen by noting that

$$E = \cos \alpha \cos \delta \cos(\alpha + \delta) \\ = \frac{1}{4} \{ 1 + \cos(\varphi - \psi) + \cos(\varphi' - \psi) + \cos(\varphi' - \varphi) \} \quad (21)$$

From either (20) or (21) it is easy to derive the solution

$$E = 1 + \frac{1}{8}(S_1^2 - 3S_2) + \frac{1}{32}(S_2^2 - 3S_4) + \frac{1}{64}(S_2S_4 - 3S_6) + \dots \quad (22) \\ = \frac{1}{8}[-1 + S_1^2 + (3 - \frac{1}{2}S_2 - \frac{1}{8}S_4 - \frac{1}{16}S_6 \dots)^2]$$

where

$$S_n = \sin^n \varphi + \sin^n \varphi' + \sin^n \psi,$$

an exact solution when required being formed from the sum of the three cosines. In numerical calculations equations (16) and (17) will be used in the forms

$$\sin \varphi' - \sin \psi = (\sin \varphi' - \sin \varphi) \left[1 + \frac{(\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi)}{2E - \sin \varphi(\sin \varphi' + \sin \psi)} \right] \\ \eta' = \eta \left[1 + \frac{(\sin \varphi - \sin \varphi')(\sin \varphi' + \sin \psi)}{2E - \sin \varphi(\sin \varphi' + \sin \psi)} \right],$$

the denominator being frequently represented with sufficient accuracy by

$$2 - \frac{1}{4}[(\sin \varphi + \sin \varphi')^2 + (\sin \varphi + \sin \psi)^2 + (\sin \varphi' - \sin \psi)^2]$$

where the sines in each term are connected by the opposite sign to the one in the terms of the numerators. It should be noted that the correcting term in the first of these equations is obtained by multiplying that for the second equation by $-R$, as is evident from the initial equation for η' in terms of φ' and ψ' .

When the cosines of the angles are to be found, the denominator will be most simply calculated from the formula

$$\frac{1}{4} \{ (\sin \psi - \sin \varphi + \sin \varphi')^2 + (\cos \psi + \cos \varphi + \cos \varphi')^2 - 1 \}$$

and it is in this form that the equations have been found most convenient in practice. It will be noted that since the difference between any cosine and unity enters into the equations as a correction to the aberration alone and does not affect the normal refraction according to Gaussian laws, a knowledge of the cosine to a comparatively small number of figures yields an accuracy corresponding to the use of a greater number of figures in the usual trigonometrical calculation. It thus becomes possible to obtain a very high degree of accuracy

by constructing a short four-figure table giving the cosines of angles in terms of their sines, such as that given below. The interpolations involved in the use of this table only affect the last figure and can always be made mentally. They thus differ fundamentally from those required with the tables ordinarily used, and the time usually spent in turning over the pages of tables to reach the angles required is entirely avoided. Calculations made by this method on photographic lenses of complex form with angles of incidence reaching 60° show that the final results are accurate to about $3''$ in direction, with a corresponding degree of accuracy in the position of the ray. Unlike the methods described previously the perpendicular after refraction is obtained before the new direction of the ray is found. The order of operations is indicated by the equations

$$\begin{aligned}\sin \phi_\lambda &= \sin \psi_{\lambda-1} + h_\lambda R_\lambda \\ \mu_\lambda \sin \phi'_\lambda &= \mu_{\lambda-1} \sin \phi_\lambda \\ h'_\lambda &= h_\lambda + \\ &\quad \frac{4 h_\lambda (\sin \phi_\lambda - \sin \phi'_\lambda) (\sin \phi'_\lambda + \sin \psi_{\lambda-1})}{(\sin \psi_{\lambda-1} - \sin \phi_\lambda + \sin \phi'_\lambda)^2 + (\cos \psi_{\lambda-1} + \cos \phi_\lambda + \cos \phi'_\lambda)^2 - 1} \quad (23) \\ \sin \psi_\lambda &= \sin \phi'_\lambda - h'_\lambda R_\lambda \\ h_{\lambda+1} &= h'_\lambda + t_\lambda \sin \psi_\lambda.\end{aligned}$$

The formulæ as they stand are applicable to every case that can arise, and it does not appear desirable to depart from them in any case with the possible exception of a flat surface, where the value of $\sin \psi_\lambda$ is obtained from that of $\sin \psi_{\lambda-1}$ by the law of refraction, and h'_λ may be obtained from the relation

$$\frac{h'_\lambda}{\cos \psi_\lambda} = \frac{h_\lambda}{\cos \psi_{\lambda-1}}.$$

If this equation is used the cosines must be calculated directly, the table being insufficiently accurate. If the table is to be utilised the equation should be expressed in the form

$$h'_\lambda = h_\lambda \left[1 + \frac{(\sin \psi_{\lambda-1} - \sin \psi_\lambda)(\sin \psi_{\lambda-1} + \sin \psi_\lambda)}{\cos \psi_{\lambda-1}(\cos \psi_{\lambda-1} + \cos \psi_\lambda)} \right].$$

Perhaps the simplest method of deriving the new equation for h'_λ from those given previously is to start from the relations

$$\frac{h'}{h} = \frac{\sin \phi' - \sin \psi'}{\sin \phi - \sin \psi} = \frac{\cos \phi' + \cos \psi'}{\cos \phi + \cos \psi},$$

COSINES IN TERMS OF SINES.

	0	1	2	3	4	5	6	7	8	9	D	D ⁻¹
·70	·7141	·7132	·7122	·7112	·7102	·7092	·7082	·7072	·7062	·7052	—10	—10
1	·7042	·7032	·7022	·7012	·7001	·6991	·6981	·6971	·6960	·6950	—10	—10
2	·6940	·6929	·6919	·6908	·6898	·6887	·6877	·6866	·6856	·6845	—11	—9
3	·6834	·6824	·6813	·6802	·6791	·6781	·6770	·6759	·6748	·6737	—11	—9
4	·6726	·6715	·6704	·6693	·6682	·6671	·6659	·6648	·6637	·6626	—11	—9
5	·6614	·6603	·6592	·6580	·6569	·6557	·6546	·6534	·6523	·6511	—11	—9
6	·6499	·6488	·6476	·6464	·6452	·6440	·6428	·6416	·6404	·6392	—12	—9
7	·6380	·6368	·6356	·6344	·6332	·6320	·6307	·6295	·6283	·6270	—12	—8
8	·6258	·6245	·6233	·6220	·6208	·6195	·6182	·6170	·6157	·6144	—13	—8
9	·6131	·6118	·6105	·6092	·6079	·6066	·6053	·6040	·6027	·6013	—13	—8
·80	·6000	·5987	·5973	·5960	·5946	·5933	·5919	·5906	·5892	·5878	—14	—7
1	·5864	·5851	·5837	·5823	·5809	·5795	·5781	·5766	·5752	·5738	—14	—7
2	·5724	·5709	·5695	·5680	·5666	·5651	·5637	·5622	·5607	·5592	—15	—7
3	·5578	·5563	·5548	·5533	·5518	·5502	·5487	·5472	·5457	·5441	—15	—7
4	·5426	·5410	·5395	·5379	·5363	·5348	·5332	·5316	·5300	·5284	—16	—6
5	·5268	·5252	·5235	·5219	·5203	·5186	·5170	·5153	·5136	·5120	—16	—6
6	·5103	·5086	·5069	·5052	·5035	·5018	·5000	·4983	·4966	·4948	—17	—6
7	·4931	·4913	·4895	·4877	·4859	·4841	·4823	·4805	·4787	·4768	—18	—6
8	·4750	·4731	·4712	·4694	·4675	·4656	·4637	·4618	·4598	·4579	—19	—5
9	·4560	·4540	·4520	·4501	·4481	·4461	·4441	·4420	·4400	·4379	—20	—5
·90	·4359	·4338	·4317	·4296	·4275	·4254	·4233	·4211	·4190	·4168	—21	—5
1	·4146	·4124	·4102	·4080	·4057	·4035	·4012	·3989	·3966	·3943	—23	—4
2	·3919	·3896	·3872	·3848	·3824	·3800	·3775	·3751	·3726	·3701	—24	—4
3	·3676	·3650	·3625	·3599	·3573	·3546	·3520	·3493	·3466	·3439	—26	—4
4	·3412	·3384	·3356	·3328	·3299	·3271	·3242	·3212	·3183	·3153	—29	—3
5	·3122	·3092	·3061	·3030	·2998	·2966	·2934	·2901	·2868	·2834	—32	—3
6	·2800	·2765	·2730	·2695	·2659	·2622	·2585	·2548	·2510	·2471	—37	—3
7	·2431	·2391	·2350	·2308	·2265	·2222	·2178	·2132	·2086	·2039	—44	—2
8	·1990	·1940	·1889	·1836	·1782	·1726	·1667	·1607	·1545	·1479	—53	—2
9	·1411	·1339	·1262	·1181	·1094	·0999	·0894	·0774	·0632	·0447	...	...
·990	·1411	·1404	·1397	·1389	·1382	·1375	·1368	·1361	·1353	·1346	...	...
1	·1339	·1331	·1324	·1316	·1309	·1301	·1293	·1286	·1278	·1270	...	...
2	·1262	·1254	·1247	·1239	·1231	·1222	·1214	·1206	·1198	·1190	...	...
3	·1181	·1173	·1164	·1156	·1147	·1138	·1130	·1121	·1112	·1103	...	...
4	·1094	·1085	·1075	·1066	·1057	·1047	·1038	·1028	·1018	·1009	...	...
5	·0999	·0989	·0979	·0968	·0958	·0948	·0937	·0926	·0916	·0905	...	...
6	·0894	·0882	·0871	·0859	·0848	·0836	·0824	·0812	·0799	·0787	...	...
7	·0774	·0761	·0748	·0734	·0721	·0707	·0692	·0678	·0663	·0648	...	...
8	·0632	·0616	·0600	·0583	·0565	·0548	·0529	·0510	·0490	·0469	...	...
9	·0447	·0424	·0400	·0374	·0346	·0316	·0283	·0245	·0200	·0141	...	...

[To face p. 258.]

whence

$$\begin{aligned}
 \frac{h'}{h} &= \frac{(\cos \varphi' + \cos \psi')(\cos \varphi + \cos \varphi') + (\sin \varphi' - \sin \psi')(\sin \varphi - \sin \varphi')}{(\cos \varphi + \cos \psi)(\cos \varphi + \cos \varphi') + (\sin \varphi - \sin \psi)(\sin \varphi - \sin \varphi')} \\
 &= \frac{1 + \cos(\varphi' - \varphi) + \cos(\varphi' - \psi') + \cos(\varphi + \psi') - 2\sin^2 \varphi'}{(\cos \varphi + \cos \psi)(\cos \varphi + \cos \varphi') + (\sin \varphi - \sin \psi)(\sin \varphi - \sin \varphi')} \\
 &= \frac{1 + \cos(\varphi' - \varphi) + \cos(\varphi - \psi) + \cos(\varphi' + \psi) - 2\sin^2 \varphi'}{(\cos \varphi + \cos \psi)(\cos \varphi + \cos \varphi') + (\sin \varphi - \sin \psi)(\sin \varphi - \sin \varphi')} \\
 &= \frac{(\cos \varphi + \cos \psi)(\cos \varphi + \cos \varphi') + (\sin \varphi + 2\sin \varphi' + \sin \psi)(\sin \varphi - \sin \varphi')}{(\cos \varphi + \cos \psi)(\cos \varphi + \cos \varphi') + (\sin \varphi - \sin \psi)(\sin \varphi - \sin \varphi')} \\
 &= 1 + \frac{2(\sin \varphi - \sin \varphi')(\sin \varphi' + \sin \psi)}{(\cos \varphi + \cos \psi)(\cos \varphi + \cos \varphi') + (\sin \varphi - \sin \psi)(\sin \varphi - \sin \varphi')} \\
 &= 1 + \frac{2(\sin \varphi - \sin \varphi')(\sin \varphi' + \sin \psi)}{(\cos \varphi + \cos \psi)(\cos \varphi' + \cos \psi) - (\sin \varphi - \sin \psi)(\sin \varphi' + \sin \psi)}
 \end{aligned}$$

two alternative forms for the denominator, which are suitable for use with the tabulated cosines, and may easily be resolved into the form already given in equation (23).

The cosine tables are available for all the operations which have been described in previous papers, provided the equations are arranged so that the cosines appear only in a term which may be regarded as a correction to the paraxial result. Such equations can always be found in a form convenient for calculation. The positions of the focal lines, for example, may be determined in this way. When it is desired to find errors of phase in the emergent wave front the equation given previously

$$\Sigma \frac{(h' - h)\mu \sin \varphi}{\{(1 + \cos \varphi)(1 + \cos \varphi')(1 + \cos \psi)(1 + \cos \psi')\}^{\frac{1}{2}}},$$

is probably as suitable as any other form, and fits very conveniently into the present system of calculation.

The author desires, in conclusion, to thank Miss Gillman for checking the table of cosines, and both Miss Everett and Miss Gillman for comparative calculations of rays in various optical systems which have indicated the degree of accuracy obtainable as recorded in the paper.

ABSTRACT.

Equations for tracing rays in an axial plane through an optical system have the normal refraction terms separated from those representing aberrations. By expressing the latter as a fraction with the

first order aberration as the numerator and a correcting factor, which may take various forms, as the denominator, rays may be traced exactly through the system using a short table of cosines in terms of sines in place of the extensive tables, giving sines in terms of angles generally employed. A considerable saving of time is effected in the calculations, and the estimation, without calculation, of the aberrations of other rays is facilitated.

DISCUSSION.

Mr. J. W. FRENCH communicated the following note: I have read with much interest Mr. Smith's very valuable contribution to the subject of optical computation. The principal advantage claimed for the system over, say, the Steinheil and Voit trigonometrical system, is, I understand, one of speed, as Mr. Smith's system does not involve the conversion of functions of angles to and from degrees and minutes. In order to test this point, I have had a cemented objective computed by both methods, the same operator and machine being engaged throughout. It would appear that the number of operations under Mr. Smith's system is much greater than in the case of the Steinheil system. Regarding each item that is recorded on paper by the computer when using a machine as indicating an operation, then for the complete objective the number of operations involved in Mr. Smith's method is 69, as compared with 33 in the Steinheil method. By the use of a machine having an integrating slide, it is possible that the number of paper entries involved in Mr. Smith's method could be substantially reduced, but the number of actual calculating operations would still appear to be considerably greater than in the Steinheil method. Thus, the number of additions or subtractions per surface on the Smith system is eight, as compared with four in the Steinheil system; the number of multiplications or divisions on the Smith system is eight, as compared with five on the Steinheil; the number of table references on the Smith system is three, as compared with three on the Steinheil. Now, with regard to addition, subtraction, multiplication and division, it is possible that Mr. Smith adopts short cuts which are not indicated in the Paper. If this is not the case, it would appear that the Steinheil system is nearly twice as quick as the Smith system, so far as these operations are concerned. With regard to table references, it is stated that in the trigonometrical method four references to tables are involved. I suggest that really only three table references are involved. It is true there are four angles, but in the case of an objective, at all surfaces after the first, one is known. When the object is at infinity there are only three angles involved, even at the first surface, since one of the four is zero. When the object is at a finite distance there are four angles involved at the first surface, and only three at succeeding surfaces. So far as table reference is concerned, it would appear that the only advantage that remains is that Mr. Smith, by an ingenious device, is able to obtain accurate results from four-figure tables, whereas tables of six or seven figures are required in the Steinheil method. It may be of interest to note that it is our general practice to use radian tables. In view of the small number of table references and the great difference between the two methods so far as time occupied in the other operations is concerned, and considering the facility which a trained computer acquires in the use of tables, I suggest that the Steinheil system is more rapid, unless, as already stated, Mr. Smith may employ a number of short cuts that have not been indicated. So far as accuracy is concerned, the proposed method appears to be all that Mr. Smith claims. In computing an ordinary telescope objective, the final angles obtained by the two methods corresponded to one unit in the sixth significant figure and the intercepts to three in the fifth figure.

Commander T. Y. BAKER (communicated): The computing opticians' methods of ray tracing, although varying in detail as to conventions of signs, have all been practically of the same type—namely, successive calculations of the refractive equation $\mu \sin \varphi = \mu' \sin \varphi'$, and successive transference from one surface to the next usually by formulæ connecting the crossing point of the ray and the aperture of the point of its inter-section with the next surface. Such formulæ have all been purely trigonometrical, and, as Mr. Smith states, the calculation $\psi' = \psi - \varphi + \varphi'$ crops up for each surface.

Optical computation by such methods is slow. Probably, at the best, one man working alone cannot exceed a speed of 12 surfaces per hour, even with the use of a calculating machine to replace logarithmic computation. The slowness is mainly caused by the constant necessity of referring to tables of sines and the need for interpolation between tabulated values. A small improvement can be made if sines tabulated to seconds are available, although the greater bulkiness of the tables makes in itself for delay. It must, therefore, be a great pleasure to the calculating optician to find a scheme put forward, such as the present one, in which the troublesome parts of the ordinary methods are circumvented. In this scheme of work, the actual values of the angles of incidence, refraction or of inclination with the axis, never need to be found; their sines only are dealt with except in one place, and the whole set of formulæ could be expressed in an algebraic form without the use of a single trigonometrical function. Except for the tabulated values of $\sqrt{1-x^2}$, which is really the meaning of Mr. Smith's table, nothing is needed except ordinary logarithms, and not even those when the multiplications and divisions are done by calculating machine. I have had the opportunity, through Mr. Smith's kindness, of testing the speed of his scheme against that of ordinary methods, and, although nothing like full proficiency has yet been attained, we have, nevertheless, already reached the "five minutes per surface" mark, and it would appear as if the ultimate limit would be from 20 to 25 surfaces per hour. The slowest part of the scheme is the solution of equation 23, in which h_2' is determined from h_2 ; in fact, this equation takes nearly half the time. Possibly on a different type of calculating machine, where numbers could be left on the machine for subsequent use, one might be able to economise time; but, with the machine that I was using, I found that equation 23 as it stands is not in the most suitable form, and prefer to use the following, which is readily established by geometry, and is not approximate in any form:—

$$h_2' = h_2 [1 - 2 \sin \frac{1}{2}(\varphi' + \psi) \sin \frac{1}{2}(\varphi' - \varphi) \sec \frac{1}{2}(\varphi + \psi)].$$

The calculation of the expression

$$2 \sin \frac{1}{2}(\varphi' + \psi) \sin \frac{1}{2}(\varphi' - \varphi) \sec \frac{1}{2}(\varphi + \psi),$$

which is really of aberrational magnitude, can be made with sufficient accuracy by a specially constructed slide rule based on the following principles: If on a circle of unit radius three points be marked off on the circumference from a common zero, and subtending the three angles φ , φ' and $-\psi$ at the centre, the fraction above can be written

$$\frac{\text{Chord } (\varphi' \text{ to } -\psi) \times \text{chord } (\varphi' \text{ to } \varphi)}{2 \times \text{perpendicular from centre on chord } (\varphi \text{ to } -\psi)}$$

It remained, therefore, from the known values of the sines of the three angles to determine the chords and the perpendicular. This was easily done by constructing a scale of $\sin^{-1}x$. The actual distance, consequently, in this scale between the marked values of $\sin \varphi'$ and $\sin \varphi$ is the angle $\varphi' - \varphi$, and the chord corresponding to this angle can be read on the slide. A second slide read the length of the perpendicular. The accuracy obtainable depends, of course, on the range to which the values are marked and on the scale of

the slide rule. Probably, for separated lens systems, 90 per cent. of surfaces will be such that $\sin \phi$, $\sin \phi'$ or $\sin \psi$ are all below 0.3, and if the slide is made to go up to that value, the occasional cases outside the range can be dealt with by Mr. Smith's formula or directly by trigonometrical methods. The rough rule to guide one respecting the accuracy seemed to be that if the value of the above fraction had two zeros after the decimal point, it could be trusted to be right to about one in the fifth place. If there was only one zero, it was usually desirable to make the calculation independently of the slide rule. The above accuracy can, of course, be improved by increasing the scale of the slide rule, but it appears to be sufficient for separated lens systems. I have not had sufficient time to investigate to what extent the method is applicable to photographic lenses where the curves are steep and the apertures wide. In conclusion, I feel that Mr. Smith is to be congratulated on having evolved a system of ray tracing which looks to be invaluable for the optical computer, and I trust that steps will be taken to bring it prominently to the notice of instrument making firms. I have in mind several instances of delay in getting out new designs owing to congestion in the computation section, and a scheme that will enable a computer to double his output ought not to be lost sight of.

MR. F. J. W. WHIPPLE (communicated): It may be of interest to place on record a variant of Mr. Smith's formula (23), which appears to have advantages from the computer's point of view as well as the merit of simple geometrical proof. If P is the point where the ray cuts the refracting surface of which A is the vertex and C the centre, let AN , perpendicular to the incident ray, cut the bisector of the angle POA at Q . h is the sum of the projections of the equal lines PQ and QA on any perpendicular to the incident ray, so that

$$h = PQ[1 + \cos(\psi + \phi)]$$

$$\text{Similarly } h' = PQ[\cos(\phi - \phi') + \cos(\psi + \phi')]$$

$$\begin{aligned} \text{Hence } \frac{h'}{h} &= \frac{\cos(\phi - \phi') + \cos(\psi + \phi')}{1 + \cos(\psi + \phi)} \\ &= 1 - \frac{(\cos \phi' - \cos \psi)(\cos \phi' - \cos \phi) + (\sin \phi' - \sin \phi)(\sin \phi' + \sin \psi)}{2 - \frac{1}{2}(\cos \phi - \cos \psi)^2 - \frac{1}{2}(\sin \phi + \sin \psi)^2} \end{aligned}$$

Evidently small errors in the cosines will exert very little influence when the ratio of h' to h is computed from this formula. The formula (23) itself can be derived in like fashion by considering the projections of a symmetrical linkage of four equal sides. Connecting A and P , one side of this linkage, AR , is perpendicular to the incident ray, and the next (RS), which is perpendicular to the antiparallel of the refracted ray with regard to PO , terminates on the bisector of the angle POA ; the other two are images of the first two in this bisector. On taking the ratio of the projections it is found that

$$\frac{h'}{h} = \frac{\cos(\phi' - \phi) + \cos 2\phi' + \cos(\psi - \phi) + \cos(\psi + \phi')}{1 + \cos(\phi + \phi') + \cos(\psi - \phi') + \cos(\psi + \phi)}$$

The required formula can be deduced at once. The highest precision would be obtained by using it in the form

$$\frac{h'}{h} = 1 + \frac{4(\sin \phi - \sin \phi')(\sin \phi' + \sin \psi)}{8 - (\cos \phi - \cos \phi')^2 - (\cos \psi - \cos \phi')^2 - (\cos \psi - \cos \phi)^2 - (\sin \phi + \sin \phi')^2 - (\sin \phi' - \sin \psi)^2 - (\sin \psi + \sin \phi)^2}$$

The AUTHOR, in reply to Mr. French, stated that an appreciable gain in speed had been found in spite of the greater number of operations involved. These could be divided into two classes, the first concerning quantities which had to be known to the same degree of accuracy as was ultimately to be attained, the second concerning quantities where a much rougher approximation could be adopted with impunity. About two-thirds of the operations related to the latter group. The calculation of the denominator in equation (23) could frequently be shortened when a paraxial ray had already been traced by noting that in the absence of aberration

$$\sin \psi_{\lambda} - 1 - \sin \varphi_{\lambda} + \sin \varphi_{\lambda}' = \sin \psi_{\lambda}.$$

Unless $\sin \psi_{\lambda}$ were large the square of the first quantity would not be appreciably affected by neglecting the aberration. The work could be so arranged that the individual contributions to the denominator need not be recorded separately, the denominator as a whole being calculated as the sum of two products obtained automatically by not clearing the product slide between operations. In the arrangement followed by the author, 21 paper entries would be made in calculating a ray by the new system through a cemented doublet, against 26 on the Steinheil system. The number of entries in the former system can be further reduced, but, in general, this is inadvisable, as all the numbers retained involve results of value to the computer. For example, the value of the denominator gives the relative magnitudes of the aberrations of the first and of higher orders. No detailed procedure had been suggested because this would necessarily vary with the type of machine used, as well as with the amount of information ultimately required from the calculations. Although emphasis had been laid on speed, the new system offered advantages in that the operations reflected the physical events which took place at the surface, and depended on its curvature simply, while in the Steinheil and most other systems the calculation hinged around the positions of the centres of curvature; and, further, it was usually possible to judge very well without calculation how a host of intermediate rays would be affected owing to the separation of the aberration terms from those relating to the normal refraction, a point in which the ordinary methods of calculation were notably deficient. In at least one more respect the new system enjoyed an advantage over Steinheil's. The same formulæ cover every case, and are uniformly reliable. On the other hand, with Steinheil's, a number of special cases arise in which entirely distinct formulæ must be adopted if the numerical accuracy is to be maintained. With the new method there would be no great difficulty in arranging a machine to carry the calculations automatically through a number of surfaces when the optical data and the initial ray co-ordinates were given.

Mr. Whipple's geometrical proofs were very ingenious and interesting, and the rearrangement of the denominator, though ordinarily troublesome, might be advantageous when the full accuracy of the cosine table was called for. Commander Baker's equation was of a somewhat different type from those considered in the paper. Apart from slide rule methods it was evident that linkages could be contrived to give correcting factors to the first order aberration automatically, and these might well be found to give the most rapid means of tracing rays.