Researches on Satellites. By *Albert Marth*, Esq.

First paper: Preliminary investigations.

I propose to make the motions of the satellites of Saturn, Uranus and Neptune the subject of investigation. I shall endeavour to elucidate their theories as far as the imperfect data at my disposal may seem fairly to warrant. In the present paper I give the developement of the general formulae employed in my further researches. In so doing I cannot of course be too sensible to the consideration, that I am engaged on a problem, which has already been treated by the master-hand of *Bessel*.*) Nevertheless I venture to submit this essay to the notice of astronomers, as I trust, that some parts of it will be found acceptable. At least I believe I am correct in stating, that the new formulae, which I am about to propose, have saved me much trouble in my later calculations.

In order that a series of observations of a satellite may be made a contribution of full value towards an investigation of its motions, it is essential to adopt such a method in observing, that the results of reduction may give positions of the satellite referred to the centre of the planet. Such an arrangement is however sometimes, or indeed usually, a matter of difficulty, since the great faintness of most satellites throws very serious obstacles in the way of obtaining results unimpeachably trustworthy. The excellent observations, taken by *Bessel* with the Koenigsberg Heliometer, furnish a good illustration of the disadvantageous effects which these difficulties produce in a greater or less degree. Jupiter's satellites are bright enough to be brought directly in contact with the limbs of the planet. By referring therefore their places to four symmetrical points of the limbs, *Bessel* derived their positions in reference to the centre, merely assuming the form of the planet to be symmetrical. Titan, the Huyghenian satellite of Saturn, does not in the usual state of the atmosphere allow direct comparisons with

the borders of the ball. Accordingly *Bessel* referred the places of this satellite to two symmetrical but otherwise arbitrary points in the apparent major axis of the ring, which points he chose with regard either to the ball or to the ring, as circumstances might permit. The positions of Titan therefore, deduced from his observations, rest on an assumption, the strict correctness of which is at least not beyond all doubt, namely, that the centre of the figure of the ring coincides with the centre of the planet. The faintness of Japetus and Rhea, the two other satellites of Saturn observed by *Bessel*, thwarted however even this arrangement; and he was in consequence obliged to disregard the planet and ring entirely and to resort to Titan as a fiducial object, so that the investigation of the orbits of the two other satellites from his observations is only possible after the motions of Titan have been duly determined.

Modifications of the formulae, which have their origin in considerations of this kind, are properly allowed for wherever they occur in special calculations. In the present preliminary researches I shall disregard them and suppose, that the observations furnish positions of the satellite in reference to the centre of the planet.

§ 1.

Let p denote the angle of position of the satellite at the centre of the planet and s its apparent angular distance from it, corresponding to the time t . Since observed angles of position, reckoned in the usual manner, refer directly to the circle of declination and the geocentric places of planets in the ephemerides are usually given in reference to the equator, I shall adopt the (moveable) equator as the fundamental-plane and refer the plane of the satellites orbit to it by means of the longitude of the ascending node N and of the inclination J , which last I shall reckon from 0° to 180° , so that there may be no need to distinguish between direct and retrograde motion. In the solid angle, which is formed at the planets centre by the prolonged line X from the observer to the planet, by the line O to the Northpole of the equator and by the line T to the (positive) pole of the satellite's orbit, let H be the angle included between the

*) The chief papers of *Bessel* on the subject of satellites are these:

Bestimmung der Bahn des *Hugen'schen* Saturns-Satelliten. — *Astronomische Nachrichten*. Vol. IX.

Bestimmung der Masse des Jupiter. — *Astr. Untersuch.* Vol. II.

and the unfinished posthumous essay:

Theorie des Saturn-Systems. — *Astr. Nachr.* Vol. XXVIII.

two planes, which meet in the line T , K the supplement of the angle included between the planes meeting in X and h the angle between the lines T and X . If then A and D denote the apparent right ascension and declination of the planet, as seen from the observers place, H , K , h are found by the formulae

$$\left. \begin{aligned} \sin \frac{1}{2} h \sin \frac{1}{2} (H + K) &= + \sin \frac{1}{2} (90^\circ - N + A) \sin \frac{1}{2} (90^\circ - D + J) \\ \sin \frac{1}{2} h \cos \frac{1}{2} (H + K) &= - \cos \frac{1}{2} (90^\circ - N + A) \sin \frac{1}{2} (90^\circ - D + J) \\ \cos \frac{1}{2} h \sin \frac{1}{2} (H - K) &= - \sin \frac{1}{2} (90^\circ - N + A) \cos \frac{1}{2} (90^\circ - D + J) \\ \cos \frac{1}{2} h \cos \frac{1}{2} (H - K) &= + \cos \frac{1}{2} (90^\circ - N + A) \cos \frac{1}{2} (90^\circ - D + J) \end{aligned} \right\} \dots \dots \dots (2)$$

According to these equations the angles H and $180^\circ - K$ are reckoned from the plane TX in the direction of the satellites motion. The angle h is of course always to be taken between 0° and 180° .

If u denotes the argument of latitude or the angle between the node and the place of the satellite, reckoned in the direction of the motion, and σ the supplement of the angle contained between the line X and a line S , drawn from the planet to the satellite, then we have in the solid angle between the lines XTS the equations

$$\left. \begin{aligned} \sin \sigma \sin (p - K) &= \cos (H + u) \\ \sin \sigma \cos (p - K) &= \sin (H + u) \cos h \\ \cos \sigma &= \sin (H + u) \sin h \end{aligned} \right\} \dots \dots (3)$$

which give therefore the angle of position p and the angle σ , which last of course is to be taken between 0° and 180° .

Further, denoting by r the radius vector of the satellite in its orbit round the planet and by Δ and Δ_1 the linear distances of the observer from the planet and from the satellite, we obtain the apparent angular distance s and also Δ_1 by means of the formulae

$$\left. \begin{aligned} \Delta_1 \sin s &= r \sin \sigma \\ \Delta_1 \cos s &= r \cos \sigma + \Delta \end{aligned} \right\} \dots \dots \dots (4)$$

so that Δ_1 , s , p may be found from

$$\left. \begin{aligned} \Delta_1 \sin s \sin (p - K) &= r \cos (H + u) \\ \Delta_1 \sin s \cos (p - K) &= r \sin (H + u) \cos h \\ \Delta_1 \cos s &= r \sin (H + u) \sin h + \Delta \end{aligned} \right\} \dots \dots \dots (5)$$

or, in case Δ_1 is not wanted, by putting

$$\left. \begin{aligned} \frac{\Delta}{r} + \sin h \sin (H + u) &= \frac{1}{\tau}, \text{ from } \\ \tau \sin (p - K) &= \cos (H + u) \\ \tau \cos (p - K) &= \sin (H + u) \cos h \end{aligned} \right\} \dots \dots \dots (6)$$

In most cases however it will be more advantageous to retain the angle σ and to calculate s indirectly by the equation

$$\left. \begin{aligned} \sin s &= \frac{r}{\Delta} \sin (\sigma - s) \\ \text{and, in case } \Delta_1 \text{ is wanted, } \Delta_1 &= \frac{\sin \sigma}{\sin (\sigma - s)} \end{aligned} \right\} \dots \dots \dots (7)$$

$$\left. \begin{aligned} \sin h \sin H &= \cos (A - N) \cos D \\ \sin h \cos H &= \sin (A - N) \cos D \cos J + \sin D \sin J \\ \cos h &= - \sin (A - N) \cos D \sin J + \sin D \cos J \\ \sin h \sin K &= \cos (A - N) \sin J \\ \sin h \cos K &= - \sin (A - N) \sin J \sin D - \cos J \cos D \end{aligned} \right\} \dots (1)$$

instead of which the trigonometrical formulae of Gauss may be used, so that

Although the calculation of these rigorous formulae for s is extremely simple, yet in some cases it is superfluous to use them, since the approximation $s'' = \frac{r \sin \sigma}{\Delta \arc 1''}$ will be

found sufficient, so that equations (5) become

$$\left. \begin{aligned} s'' \sin (p - K) &= \frac{r}{\Delta} \cdot \omega'' \cos (H + u) \\ s'' \cos (p - K) &= \frac{r}{\Delta} \cdot \omega'' \sin (H + u) \cos h \\ \text{where } \omega'' &= \frac{1}{\arc 1''} = 206264'' \end{aligned} \right\} \dots \dots (8)$$

For, equations (4) give

$$\begin{aligned} \tau g s &= \frac{r \sin \sigma}{\Delta + r \cos \sigma} \\ &= \frac{r \sin \sigma}{\Delta} \left\{ 1 - \frac{r}{\Delta} \cos \sigma + \frac{r^2}{\Delta^2} \cos^2 \sigma - \dots \right\}, \end{aligned}$$

therefore

$$s'' = \frac{r \sin \sigma}{\Delta \arc 1''} \left\{ 1 - \frac{r}{\Delta} \cos \sigma + \frac{r^2}{\Delta^2} \cos^2 \sigma (1 - \frac{1}{3} \tau g^2 \sigma) - \dots \right\}$$

If we omit in this expression all terms but the first, we are neglecting in s'' the quantity

$$- \frac{r^2}{\Delta^2} \cdot \frac{\sin 2 \sigma}{2 \arc 1''} + \dots$$

This quantity reaches its maximum for $\sigma = 45^\circ$ or 135° . In order therefore to avoid in all parts of the apparent orbit an error $= \varepsilon$ in s , using the approximative form, the proportion $\frac{r}{\Delta}$ must not exceed the limit $\sqrt{2 \varepsilon \arc 1''}$. This limit or its reciprocal is

for $\varepsilon = 0''005$	$\frac{\Delta}{r} = 4542$	or greatest apparent elongation $45''$
$\varepsilon = 0''01$	3212	$64''$
$\varepsilon = 0''05$	1437	$143''$
$\varepsilon = 0''1$	1016	$203''$

For the satellites of Uranus and Neptune and also for the inner ones of Saturn the error of the substitution

$$s'' = \frac{r}{\Delta} \cdot \omega'' \sin \sigma \text{ is accordingly always below } \frac{1}{200}''.$$

§ 2.

I shall now develop the alterations which the calculated values of p and s undergo by alterations of the elements upon which they are based.

By differentiating the equations (3) we have

$$\begin{aligned} -\sin^2 \sigma (dp - dK) &= \cosh(dH + du) - \cos(H + u) \cos \sigma dh \\ -\sin \sigma d\sigma &= \cos(H + u) \sinh(dH + du) + \sin(H + u) \cosh dh \end{aligned}$$

But the differentiation of (1) gives, A and D being considered constants,

$$\begin{aligned} -\sin h dH &= \cos h \sin H dJ + \cos K \cos D dN \\ -\sin h dK &= \sin H dJ + \cos H \sin J dN \\ dh &= \cos H dJ - \sin H \sin J dN \\ &= \cos H dJ - \sin K \cos D dN \end{aligned}$$

Hence we derive after some reductions

$$\left. \begin{aligned} \sin^2 \sigma dp &= -\cosh du + \sin h \cos(H + u) \sin u dJ \\ &\quad + \{\sin h \sin(H + u) \sin u \sin J - \sin D\} dN \\ \sin \sigma d\sigma &= -\sin h \cos(H + u) du - \cosh \sin u dJ \\ &\quad + \{\cos K \cos(H + u) + \sin K \sin(H + u) \cos h\} \cos D dN \end{aligned} \right\} \dots \dots \dots (9)$$

This last equation follows more shortly by differentiation of the third equation (3). Let it also be observed, that the coefficient of $\cos D dN$ is $= \sin \sigma \sin p$.

By differentiating the equation $\sin s = \frac{r}{\Delta} \sin(\sigma - s)$ we obtain

$$\frac{ds}{\sin s} = \frac{\cos(\sigma - s)}{\sin \sigma} d\sigma + \frac{\sin(\sigma - s)}{\sin \sigma} \cdot \frac{dr}{r}$$

or by substituting for $d\sigma$ its value from (9)

$$\left. \begin{aligned} \frac{\sin \sigma}{\sin s} \cdot ds &= -\frac{\cos(\sigma - s)}{\sin \sigma} \{\sin h \cos(H + u) du + \cosh \sin u dJ - \sin \sigma \sin p \cos D dN\} \\ &\quad + \sin(\sigma - s) \frac{dr}{r} \end{aligned} \right\} \dots \dots \dots (10)$$

As unit of the angular quantities dp , $d\sigma$, du etc. I will assume the minute of arc; the linear quantities are therefore still to be multiplied by $w' = \frac{1}{\text{arc } 1'} = 3437.7$, $\lg w' = 3.53627$.

Supposing generally the accuracy of the measured angles of position to be directly proportional to the apparent angular distances or to their sines (a point, which of course in every case must be made a matter for especial consideration); assuming the unit of accuracy to be that, with which

the angle is measured in the greatest elongations, when the satellite is at its mean distance a from the planet and in the (arbitrary, but constant) distance Δ_0 from the observer; besides making $\frac{r}{a} = \rho$, so that ρ denotes the radius vector of the satellite expressed in parts of the semi-major axis of its orbit; and lastly, putting for shortness sake $\frac{\Delta_0}{\Delta} \cdot \frac{\sin(\sigma - s)}{\sin \sigma} \cdot \rho = \nu$; then the first equation (9) is changed into

$$\left. \begin{aligned} \nu \sin \sigma dp &= -\nu \operatorname{cosec} \sigma \cosh du + \nu \operatorname{cosec} \sigma \sin h \cos(H + u) \sin u dJ \\ &\quad + \nu \operatorname{cosec} \sigma \{\sin J \cos \sigma \sin u - \sin D\} dN \\ \text{and equation (10), supposing the accuracy of the measured distances for small and large} \\ \text{values to be the same and expressing } ds \text{ in seconds, becomes} \\ \frac{\Delta_0}{60a} ds'' &= -\nu \cdot \frac{\cos(\sigma - s)}{\sin \sigma} \sin h \cos(H + u) du - \nu \cdot \frac{\cos(\sigma - s)}{\sin \sigma} \cosh \sin u dJ \\ &\quad + \nu \cos(\sigma - s) \sin p \cos D dN + \nu \sin(\sigma - s) \cdot w' \frac{dr}{r} \end{aligned} \right\} \dots \dots \dots (11)$$

where $\frac{dr}{r} = \frac{da}{a} + \frac{d\rho}{\rho}$ and $w' = \frac{1}{\text{arc } 1'}$.

§ 3.

The variations dp and ds appear in these equations as functions of dJ , dN , da and du , $d\rho$. It is now requisite to express du and $d\rho$ by the differentials of the other elements of the orbit.

Let Q denote the angle between the line of apsides and the line of nodes, e the eccentricity of the orbit, μ , ε , v the mean, eccentric and true anomaly of the satellite at the time t , μ_0 its mean anomaly corresponding to the time t_0 , n its mean daily sidereal motion. By introducing then the angle of eccentricity ϕ , so that $\sin \phi = e$, we have for the determination of ρ , v , u , as usual, the equations

$$\left. \begin{aligned} \varepsilon - e' \sin \varepsilon &= \mu = \mu_0 + (t - t_0)n \\ \rho \sin v &= \cos \phi \sin \varepsilon \\ \rho \cos v &= \cos \varepsilon - \sin \phi \\ u &= Q + v \\ \text{and their differentials are} \\ du &= \frac{\cos \phi}{\rho^2} d\mu + dQ + \left(\sec^2 \phi + \frac{1}{\rho} \right) \sin v \cdot \omega' de \\ \omega' d\rho &= tg \phi \sin v d\mu - \cos v \cdot \omega' de \end{aligned} \right\} \quad (12)$$

It depends upon the value of J , whether Q is to be retained as an element by itself or in combination with N . At present I will consider it by itself as such and I will accordingly assume, that the inclination J of the satellite's orbit upon the equator is not inconsiderable. Let then the combination $\mu_0 + Q = u_0$ be introduced as an element instead of μ_0 , whence we have $\mu = u_0 - Q + (t - t_0)n$, and the differential-expressions for du and $d\rho$ become

$$\left. \begin{aligned} du &= \frac{\cos \phi}{\rho^2} (du_0 + (t - t_0)dn) + \left(1 - \frac{\cos \phi}{\rho^2} \right) dQ + \left(\sec^2 \phi + \frac{1}{\rho} \right) \sin v \cdot \omega' de \\ \omega' d\rho &= tg \phi \sin v (du_0 + (t - t_0)dn) - tg \phi \sin v dQ - \cos v \cdot \omega' de \end{aligned} \right\} \dots\dots\dots (13)$$

These equations are convenient and accurate, in case the eccentricity is considerable. If e however is of small amount, then the coefficients of dQ are also small, dQ itself can therefore not be found from equations of condition with sufficient accuracy and it becomes necessary to introduce instead of Q and e functions of these quantities as elements. For this purpose I will now transform equations (13). We have

$$\begin{aligned} 1 - \frac{\cos \phi}{\rho^2} &= 1 - \frac{(1 + e \cos v)^2}{\cos^3 \phi} \\ &= - \frac{(2 + e \cos v) e \cos v + 1 - \cos^3 \phi}{\cos^3 \phi} \\ &= - tg \phi \left\{ \frac{2 + e \cos v}{1 - e^2} \cos v + tg \phi \left(1 + \frac{1}{2 \cos \phi \cos^2 \frac{1}{2} \phi} \right) \right\} \text{ or } \\ &= - e \left\{ \frac{2 + e \cos v}{1 - e^2} \cdot \cos v \left(1 + \frac{2 \sin^2 \frac{1}{2} \phi}{\cos \phi} \right) + tg \phi \left(\frac{1}{\cos^2 \phi} + \frac{1}{2 \cos^2 \frac{1}{2} \phi} \right) \right\} \\ &= - e \left\{ \frac{2 + e \cos v}{1 - e^2} \cos v + \frac{tg \phi}{2 \cos^2 \frac{1}{2} \phi} \left(1 + \frac{2 \cos^2 \frac{1}{2} \phi}{\cos^2 \phi} + e \cos v \cdot \frac{2 + e \cos v}{1 - e^2} \right) \right\} \text{ or } \\ 1 - \frac{\cos \phi}{\rho^2} &= - e \left\{ \frac{2 + e \cos v}{1 - e^2} \cos v + tg \frac{1}{2} \phi \left(\frac{2 \cos^2 \frac{1}{2} \phi}{\cos^2 \phi} + \frac{\cos \phi}{\rho^2} \right) \right\}. \end{aligned}$$

Accordingly equations (13) may be written

$$\left. \begin{aligned} du &= \frac{\cos \phi}{\rho^2} (du_0 + (t - t_0)dn) - \left\{ \frac{2 + e \cos v}{1 - e^2} \cos v + tg \frac{1}{2} \phi \left(\frac{2 \cos^2 \frac{1}{2} \phi}{\cos^2 \phi} + \frac{\cos \phi}{\rho^2} \right) \right\} edQ \\ &\quad + \frac{2 + e \cos v}{1 - e^2} \sin v \cdot \omega' de \\ \omega' d\rho &= tg \phi \sin v (du_0 + (t - t_0)dn) - \left\{ 1 + tg \phi tg \frac{1}{2} \phi \right\} \sin v \cdot edQ - \cos v \cdot \omega' de \end{aligned} \right\} \dots\dots\dots (14)$$

Introducing then as elements $\omega' \cdot e \sin Q = q$

$\omega' \cdot e \cos Q = q'$, we obtain

$edQ = \cos Q dq - \sin Q dq'$

$\omega' de = \sin Q dq + \cos Q dq'$

$$\left. \begin{aligned} &\dots\dots\dots \\ &\dots\dots\dots \\ &\dots\dots\dots \end{aligned} \right\} \dots\dots\dots (15)$$

and the equations (14) are changed into

$$\left. \begin{aligned} du &= \frac{\cos \varphi}{\rho^2} (du_0 + (t-t_0) dn) - \left\{ \frac{2+e \cos v}{1-e^2} \cos(Q+v) + \left(\frac{2 \cos^2 \frac{1}{2} \varphi}{\cos^2 \varphi} + \frac{\cos \varphi}{\rho^2} \right) \operatorname{tg} \frac{1}{2} \varphi \cos Q \right\} dq \\ &\quad + \left\{ \frac{2+e \cos v}{1-e^2} \sin(Q+v) + \left(\frac{2 \cos^2 \frac{1}{2} \varphi}{\cos^2 \varphi} + \frac{\cos \varphi}{\rho^2} \right) \operatorname{tg} \frac{1}{2} \varphi \sin Q \right\} dq' \\ \omega' d\rho &= \operatorname{tg} \varphi \sin v (du_0 + (t-t_0) dn) - \left\{ \sin(Q+v) + \sin v \cdot \operatorname{tg} \varphi \operatorname{tg} \frac{1}{2} \varphi \cos Q \right\} dq \\ &\quad - \left\{ \cos(Q+v) - \sin v \cdot \operatorname{tg} \varphi \operatorname{tg} \frac{1}{2} \varphi \sin Q \right\} dq' \end{aligned} \right\} \dots (16)$$

which expressions are still quite rigorous.

If e is of moderate value, so that in the coefficients the terms multiplied by e^2 may be neglected, then these equations become

$$\left. \begin{aligned} du &= (1 + 2e \cos v) (du_0 + (t-t_0) dn) - \left\{ (2 + e \cos v) \cos u + \frac{3q'}{2\omega'} \right\} dq \\ &\quad + \left\{ (2 + e \cos v) \sin u + \frac{3q}{2\omega'} \right\} dq' \\ \omega' d\rho &= e \sin v (du_0 + (t-t_0) dn) - \sin u dq - \cos u dq' \end{aligned} \right\} \dots (17)$$

If e is so small, that also the terms multiplied by e may be disregarded, or if it is $= 0$, then we have

$$\left. \begin{aligned} du &= du_0 + (t-t_0) dn - 2 \cos u dq + 2 \sin u dq' \\ \omega' d\rho &= - \sin u dq - \cos u dq' \end{aligned} \right\} \dots (18)$$

§ 4.

In case the inclination of the satellite's orbit to the equator is moderate, it is preferable to use as element instead of Q itself the combination $N \pm Q$ (analogous to the longitude of the perihelion in planetary or cometary orbits), where the upper or lower sign is to be taken according as J is near 0° or near 180° . Denoting the new elements, which take the place of Q and u_0 , by P and L_0 , so that

$$\left. \begin{aligned} P &= N \pm Q \text{ and} \\ L_0 &= \mu_0 + P, \text{ we have} \\ \mu &= L_0 - P + (t-t_0)n \\ u &= v \pm (P-N) \text{ and the differential-expressions for } du \text{ and } d\rho \text{ become} \\ du &= \frac{\cos \varphi}{\rho^2} (dL_0 + (t-t_0) dn) \mp dN + \left(1 - \frac{\cos \varphi}{\rho^2} \right) dP + \left(\sec^2 \varphi + \frac{1}{\rho} \right) \sin v \cdot \omega' de \\ \omega' d\rho &= \operatorname{tg} \varphi \sin v (dL_0 + (t-t_0) dn) - \operatorname{tg} \varphi \sin v dP - \cos v \cdot \omega' de \end{aligned} \right\} \dots (19)$$

Making, in case e is of moderate amount,

$$\left. \begin{aligned} \omega' e \sin P &= c \\ \omega' e \cos P &= c' \end{aligned} \right\}$$

we obtain, neglecting in the coefficients those terms, which are multiplied by e^2 ,

$$\left. \begin{aligned} du &= (1 + 2e \cos v) (dL_0 + (t-t_0) dn) \mp dN - \left\{ (2 + e \cos v) \cos(P+v) + \frac{3c'}{2\omega'} \right\} dc \\ &\quad + \left\{ (2 + e \cos v) \sin(P+v) + \frac{3c}{2\omega'} \right\} dc' \\ \omega' d\rho &= e \sin v (dL_0 + (t-t_0) dn) - \sin(P+v) dc - \cos(P+v) dc' \end{aligned} \right\} \dots (20)$$

or by neglecting also those terms, which contain the first power of e ,

$$\left. \begin{aligned} du &= dL_0 + (t-t_0) dn \mp dN - 2 \cos(P+v) dc + 2 \sin(P+v) dc' \\ \omega' d\rho &= - \sin(P+v) dc - \cos(P+v) dc' \end{aligned} \right\} \dots (21)$$

§ 5.

If the inclination of the satellite's orbit to the plane of the equator is so inconsiderable, that the smallness of the coefficients would make the determination of dN from equations of condition very uncertain, it becomes necessary either to refer the orbit to another plane, than that of the equator, or to introduce into the equations of condition instead of dJ and dN functions of these quantities. Taking this last course, the most convenient functions appear to be $\operatorname{tg} \frac{1}{2} J \sin N$ and $\operatorname{tg} \frac{1}{2} J \cos N$. Thus, making

$$\begin{aligned} 2\omega' \operatorname{tg} \frac{1}{2} J \sin N &= b \\ 2\omega' \operatorname{tg} \frac{1}{2} J \cos N &= b' \end{aligned}$$

where again $\omega' = \frac{1}{\operatorname{arc} 1}$, we have

$$\left. \begin{aligned} dJ &= \sin N \cos^2 \frac{1}{2} J db + \cos N \cos^2 \frac{1}{2} J db' \\ \sin J dN &= \cos N \cos^2 \frac{1}{2} J db - \sin N \cos^2 \frac{1}{2} J db' \end{aligned} \right\} \quad (22)$$

In these equations it is better, not to regard db and db' , but $\cos^2 \frac{1}{2} J db$ and $\cos^2 \frac{1}{2} J db'$ as the variables, since then

$$\left. \begin{aligned} \frac{\sin^2 \sigma dp}{\cos^2 \frac{1}{2} J db} &= \sinh \sin u \sin(N+H+u) - (\sin D \mp \cosh) \frac{\cos N}{\sin J} \\ \frac{\sin^2 \sigma dp}{\cos^2 \frac{1}{2} J db'} &= \sinh \sin u \cos(N+H+u) + (\sin D \mp \cosh) \frac{\sin N}{\sin J} \\ \frac{\sin \sigma d\sigma}{\cos^2 \frac{1}{2} J db} &= -\cosh \sin u \sin N - \{ \cos D \sin \sigma \sin p \mp \sinh \cos(H+u) \} \frac{\cos N}{\sin J} \\ \frac{\sin \sigma d\sigma}{\cos^2 \frac{1}{2} J db'} &= -\cosh \sin u \cos N + \{ \cos D \sin \sigma \sin p \mp \sinh \cos(H+u) \} \frac{\sin N}{\sin J} \end{aligned} \right\} \dots\dots\dots (23)$$

In these expressions is supposed, that L_0, b, b' and P, e or functions of P, e are considered as elements of the orbit. The sign $\mp$ is to be taken with regard to $P = N \pm Q$.

These formulae might be employed. But I shall dwell upon them the less, as in those cases, in which they are to be made use of, it may not be more inconvenient to change the fundamental-plane, to which the situation of the orbit is referred. The choice of the ecliptic instead of the equator would indeed be generally preferable, if the ephemerides contained geocentric longitudes and latitudes of the planets instead of rightascensions and declinations. The formulae, which have been given before for finding the apparent positions of the satellite from given elements, may be used generally whatever be the fundamental-plane to which the elements refer, but they give of course angles of position in reference to the circle of latitude in the system adopted.

Let u', h', H', K' be quantities analogous to u, h, H, K but referring to the ecliptic and not to the equator, also let L and B denote the apparent longitude and latitude (on the ecliptic) of the planet; then we have

$$\left. \begin{aligned} \sin \frac{1}{2}(90^\circ - B) \sin \frac{1}{2}(90^\circ - L + E) &= \cos \frac{1}{2}(90^\circ + A) \sin \frac{1}{2}(90^\circ - D - \omega) \\ \sin \frac{1}{2}(90^\circ - B) \cos \frac{1}{2}(90^\circ - L + E) &= \sin \frac{1}{2}(90^\circ + A) \sin \frac{1}{2}(90^\circ - D + \omega) \\ \cos \frac{1}{2}(90^\circ - B) \sin \frac{1}{2}(90^\circ - L - E) &= \cos \frac{1}{2}(90^\circ + A) \cos \frac{1}{2}(90^\circ - D - \omega) \\ \cos \frac{1}{2}(90^\circ - B) \cos \frac{1}{2}(90^\circ - L - E) &= \sin \frac{1}{2}(90^\circ + A) \cos \frac{1}{2}(90^\circ - D + \omega) \end{aligned} \right\} \dots\dots\dots (26)$$

I now return again to the equator as the fundamental-plane.

§ 6.

Hitherto I have assumed, that the observations furnish angles of position and apparent distances of the satellite. I have now to take into consideration those cases, in which the results of observation appear as functions of these quantities. Let the observed quantities be x and y , so that

the formulae remain available for all values of J . To the terms, which have to be substituted in (9) for

$$\sinh \cos(H+u) \sin u dJ + \{ \sin J \cos \sigma \sin u - \sin D \} dN \text{ and } \cosh \sin u dJ - \sin \sigma \sin p \cos D dN$$

must still be added those, which depend upon the introduction of P instead of Q , so that the partial differentials, which relate to the situation of the orbit, become

$$\left. \begin{aligned} \sin \sigma \sin(p - (K' + E)) &= \cos(H' + u') \\ \sin \sigma \cos(p - (K' + E)) &= \sin(H' + u') \cos h' \\ \cos \sigma &= \sin(H' + u') \sin h' \end{aligned} \right\} \quad (24)$$

where E , the angle between the circle of latitude and of declination at the apparent place of the planet, is found by

$$\cot g E = -\cot \omega \cos B \sec L + \sin B \operatorname{tg} L$$

ω being the obliquity of the ecliptic. E is in all occurring cases to be taken between -90° and $+90^\circ$ and it is accordingly negative or positive as the planet is situated in the ascending or in the descending signs of the zodiac.

If L and B are not given, they are found from A, D, ω by the known formulae, which may be written most conveniently perhaps in the form

$$\left. \begin{aligned} \sec D \cos B \cos L &= \cos A \\ \sec D \cos B \sin L &= \sin A \cos \omega + \operatorname{tg} D \sin \omega \\ \sec D \sin B &= -\sin A \sin \omega + \operatorname{tg} D \cos \omega \end{aligned} \right\} \dots\dots\dots (25)$$

and E is obtained with all desirable accuracy from

$$\begin{aligned} \sin E &= -\sin \omega \cos A \sec B \\ &= -\sin \omega \cos L \sec D \end{aligned}$$

Or the *Gaussian* formulae may be used

$$\begin{aligned} x &= \psi s \cdot \sin p \\ y &= \psi s \cdot \cos p \end{aligned}$$

where ψs denotes a function of s , which for the present I leave undefined. In order to calculate the corresponding values of x and y from given elements, it will be found in

most cases advantageous to employ again the angle σ , so that, by making

$$\left. \begin{aligned} \xi &= \sin \sigma \sin p \\ \eta &= \sin \sigma \cos p \\ \zeta &= \cos \sigma \\ x &= \frac{\psi s}{\sin \sigma} \cdot \xi \\ y &= \frac{\psi s}{\sin \sigma} \cdot \eta \end{aligned} \right\} \dots \dots \dots (27)$$

where generally ζ will be contained in ψs . The expressions for ξ , η , ζ can be derived from the equations (3). It is however simpler and more to the purpose, to find them in a more direct manner. Denoting by α and δ the planeto-

centric rightascension and declination of the satellite, we have for the determination of σ and p the equations

$$\left. \begin{aligned} \sin \sigma \sin p &= \sin(\alpha - A) \cos \delta \\ \sin \sigma \cos p &= -\cos(\alpha - A) \cos \delta \sin D + \sin \delta \cos D \\ \cos \sigma &= \cos(\alpha - A) \cos \delta \cos D + \sin \delta \sin D \end{aligned} \right\} (28)$$

But α , δ and u , J , N are connected by the relations

$$\left. \begin{aligned} \cos \delta \sin(\alpha - N) &= \sin u \cos J \\ \cos \delta \cos(\alpha - N) &= \cos u \\ \sin \delta &= \sin u \sin J \end{aligned} \right\} \dots \dots \dots (29)$$

we have therefore

$$\left. \begin{aligned} \cos \delta \sin(\alpha - A) &= \sin u \cos J \cos(A - N) - \cos u \sin(A - N) \\ \cos \delta \cos(\alpha - A) &= \sin u \cos J \sin(A - N) - \cos u \cos(A - N) \end{aligned} \right\}$$

and the equations (28) become accordingly

$$\left. \begin{aligned} \sin \sigma \sin p &= -\sin(A - N) \cos u + \cos(A - N) \cos J \sin u \\ \sin \sigma \cos p &= -\cos(A - N) \sin D \cos u - \{\sin(A - N) \cos J \sin D - \sin J \cos D\} \sin u \\ \cos \sigma &= \cos(A - N) \cos D \cos u + \{\sin(A - N) \cos J \cos D + \sin J \sin D\} \sin u \end{aligned} \right\} \dots \dots \dots (30)$$

It is now only requisite to render these equations convenient for calculation. By introducing for that purpose with *Bessel* the known auxiliary angles f , F , g , G , h , H according to

$$\left. \begin{aligned} \sin f \sin F &= -\sin(A - N) \\ \sin f \cos F &= \cos(A - N) \cos J \\ \cos f &= -\cos(A - N) \sin J \\ \sin g \sin G &= -\cos(A - N) \sin D \\ \sin g \cos G &= -\sin(A - N) \cos J \sin D + \sin J \cos D \\ \cos g &= \sin(A - N) \sin J \sin D + \cos J \cos D \\ \sin h \sin H &= \cos(A - N) \cos D \\ \sin h \cos H &= \sin(A - N) \cos J \cos D + \sin J \sin D \\ \cos h &= -\sin(A - N) \sin J \cos D + \cos J \sin D \end{aligned} \right\} (31)$$

the equations (30) are changed into

$$\left. \begin{aligned} \sin \sigma \sin p &= \sin f \sin(F + u) = \xi \\ \sin \sigma \cos p &= \sin g \sin(G + u) = \eta \\ \cos \sigma &= \sin h \sin(H + u) = \zeta \end{aligned} \right\} \dots \dots \dots (32)$$

$$\left. \begin{aligned} \sin f \sin g \sin(F - G) &= \cos h \\ \sin g \sin h \sin(G - H) &= \cos f \\ \sin h \sin f \sin(H - F) &= \cos g \\ \operatorname{tg} f \operatorname{tg} g \cos(F - G) &= -1 \\ \operatorname{tg} g \operatorname{tg} h \cos(G - H) &= -1 \\ \operatorname{tg} h \operatorname{tg} f \cos(H - F) &= -1 \\ \sin h \sin(H + u) &= + \cos g \sin f \cos(F + u) - \cos f \sin g \cos(G + u) \\ \sin h \cos(H + u) &= - \cos g \sin f \sin(F + u) + \cos f \sin g \sin(G + u) \end{aligned} \right\} \dots \dots \dots (33)$$

where the value of u is arbitrary.

The numerical calculation of the auxiliary angles g , G , h , H is simple enough, especially by using the forms

The geometrical signification of the auxiliary angles is easily explained. Through the centre of the planet let five lines be drawn O , T , X , Y , Z , some of which have already been defined in § 1: O the line to the Northpole of the equator, T the line to the (positive) pole of the satellites orbit (rightascension $= N - 90^\circ$, declination $= 90^\circ - J$), X the line from the observer to the planet produced, Y a line perpendicular upon X in the plane of the declination-circle and taken from X in the direction, towards O , Z a line perpendicular upon the declination-circle and taken from X in the direction of increasing rightascensions — then are h , g , f the angles formed by the lines X , Y , Z with the line T , and H , G , F the angles, which the planes passing through X , Y , Z and T include with the plane, which passes through O and T .

Of the relations which subsist between these auxiliary angles J mention the following, since part of them will be afterwards made use of

$$\begin{array}{l|l}
\text{cosec } D \sin g \sin G = -\cos(A-N) \\
\text{cosec } D \sin g \cos G = -\sin(A-N) \cos J + \cotg D \sin J \\
\text{cosec } D \cos g = \sin(A-N) \sin J + \cotg D \cos J \\
\sec D \sin h \sin H = \cos(A-N) \\
\sec D \sin h \cos H = \sin(A-N) \cos J + \tg D \sin J \\
\sec D \cos h = -\sin(A-N) \sin J + \tg D \cos J
\end{array}
\left. \begin{array}{l}
\sin b \sin B = -\sin(A-N) \sin D \\
\sin b \cos B = \cos D \\
\cos b = -\cos(A-N) \sin D \\
\sin c \sin C = \sin(A-N) \cos D \\
\sin c \cos C = \sin D \\
\cos c = \cos(A-N) \cos D
\end{array} \right\} \dots\dots\dots (34)$$

where it must be observed, that the term $\text{cosec } D \sin g$ gives the sign of $\sin D$. But if it appears preferable, the following substitutions may be used

$$\begin{array}{l}
\text{whence } \sin g \sin G = \cos b \\
\sin g \cos G = \sin b \sin(B+J) \\
\cos g = \sin b \cos(B+J) \\
\sin h \sin H = \cos c \\
\sin h \cos H = \sin c \sin(C+J) \\
\cos h = \sin c \cos(C+J)
\end{array}$$

§ 7.

The differentiation of the equations (32), duly regarding (31), gives

$$\begin{aligned}
d\xi &= \sin f \cos(F+u) du - \cos(A-N) \sin J \sin u dJ \\
&\quad + \{ \cos(A-N) \cos u + \sin(A-N) \cos J \sin u \} dN \\
d\eta &= \sin g \cos(G+u) du + \{ \sin(A-N) \sin J \sin D + \cos J \cos D \} \sin u dJ \\
&\quad - \{ \sin(A-N) \cos u - \cos(A-N) \cos J \sin u \} \sin D dN \\
d\zeta &= \sin h \cos(H+u) du - \{ \sin(A-N) \sin J \cos D - \cos J \sin D \} \sin u dJ \\
&\quad - \{ \sin(A-N) \cos u - \cos(A-N) \cos J \sin u \} \cos D dN
\end{aligned}$$

or

$$\begin{aligned}
d\xi &= \sin f \cos(F+u) du + \cos f \sin u dJ + \{ \zeta \cos D - \eta \sin D \} dN \\
d\eta &= \sin g \cos(G+u) du + \cos g \sin u dJ + \xi \sin D dN \\
d\zeta &= \sin h \cos(H+u) du + \cos h \sin u dJ - \xi \cos D dN
\end{aligned}
\left. \dots\dots\dots \right\} \dots\dots\dots (35)$$

$$\begin{aligned}
\text{Further } \sin \sigma dp &= \cos p d\xi - \sin p d\eta \\
\cos \sigma d\sigma &= \sin p d\xi + \cos p d\eta
\end{aligned}$$

By substituting in these equations for $d\xi$ and $d\eta$ their values from (35) and for $\sin p$ and $\cos p$ their values from (32), we obtain again by a circuitous process the equations (9)

$$\begin{aligned}
\sin^2 \sigma dp &= -\cos h du + \sin h \cos(H+u) \sin u dJ + \{ \sin J \cos \sigma \sin u - \sin D \} dN \\
\sin \sigma d\sigma &= -\sin h \cos(H+u) du - \cos h \sin u dJ + \sin f \sin(F+u) \cos D dN.
\end{aligned}$$

Differentiating the equation for x and y , we have

$$\frac{dx}{x} - \frac{d\xi}{\xi} = \frac{dy}{y} - \frac{d\eta}{\eta} = \frac{\psi's}{\psi s} ds - \cotg \sigma d\sigma$$

if $\psi's$ signifies the differential-coefficient of ψs with respect to s . Substituting for ds its value from

$$\frac{ds}{\sin s} = \frac{\cos(\sigma-s)}{\sin \sigma} d\sigma + \frac{\sin(\sigma-s)}{\sin \sigma} \cdot \frac{dr}{r}$$

we obtain

$$\frac{dx}{x} - \frac{d\xi}{\xi} = \frac{dy}{y} - \frac{d\eta}{\eta} = \frac{\sin s}{\psi s} \cdot \psi's \cdot \frac{\sin(\sigma-s)}{\sin \sigma} \cdot \frac{dr}{r} + \left\{ \frac{\sin s}{\psi s} \cdot \psi's \cos(\sigma-s) - \cos \sigma \right\} \frac{d\sigma}{\sin \sigma} = R \dots (36)$$

The form of the function ψs depends of course upon the nature of the observations and is therefore different under different circumstances. The following are the values of R corresponding with some forms of ψs :

$$\begin{aligned}
 \text{For } \psi s &= tgs & \text{is } R &= \left\{ 1 - tgs \cotg \sigma \right\} \frac{dr}{r} + tgs d\sigma \\
 \psi s &= \sin s & R &= \cos s \cdot \frac{\sin(\sigma-s)}{\sin \sigma} \cdot \frac{dr}{r} + \sin s \cdot \frac{\sin(\sigma-s)}{\sin \sigma} d\sigma \\
 \psi s &= 2tg \frac{1}{2}s & R &= \frac{\sin(\sigma-s)}{\sin \sigma} \cdot \frac{dr}{r} + 2\sin \frac{1}{2}s \cdot \frac{\sin(\sigma-\frac{1}{2}s)}{\sin \sigma} d\sigma \\
 \psi s &= 2\sin \frac{1}{2}s & R &= \cos^2 \frac{1}{2}s \cdot \frac{\sin(\sigma-s)}{\sin \sigma} \cdot \frac{dr}{r} + \sin \frac{1}{2}s \cdot \frac{\sin(\sigma-\frac{1}{2}s) + \sin(\sigma-s) \cos \frac{1}{2}s}{\sin \sigma} d\sigma \dots (37)
 \end{aligned}$$

or regarding only the beginning of the developement

$$R = \left\{ 1 - \frac{r}{\Delta} \cos \sigma \right\} \frac{dr}{r} + \frac{r}{\Delta} \sin \sigma d\sigma$$

I shall pursue only the first of these forms. According to it we have

$$\begin{aligned}
 x'' &= \omega'' tgs \sin p \\
 y'' &= \omega'' tgs \cos p
 \end{aligned}$$

or, since sp and the linear distance Δ , of the satellite from the observer are found by

$$\begin{aligned}
 \Delta \sin s \sin p &= r \xi \\
 \Delta \sin s \cos p &= r \eta \\
 \Delta \cos s &= r \zeta + \Delta,
 \end{aligned}$$

$$\begin{aligned}
 &\text{by making again } \frac{\Delta}{r} + \zeta = \frac{1}{\tau} \left. \begin{aligned} &\text{we have} \\ &x'' = \omega'' \tau \xi \\ &y'' = \omega'' \tau \eta \end{aligned} \right\} \dots (38)
 \end{aligned}$$

$$\begin{aligned}
 \text{therefore } \frac{dx''}{60} &= \tau \left\{ d\xi + \tau \xi \sin \sigma d\sigma + \xi (1 - \tau \zeta) \frac{\omega' dr}{r} \right\} \\
 \frac{dy''}{60} &= \tau \left\{ d\eta + \tau \eta \sin \sigma d\sigma + \eta (1 - \tau \zeta) \frac{\omega' dr}{r} \right\}
 \end{aligned}$$

and consequently

$$\begin{aligned}
 \frac{dx''}{60} &= \tau \xi \left\{ (\cotg(F+u) - \tau \zeta \cotg(H+u)) du + (1 - \tau \zeta) \frac{\omega' dr}{r} \right\} \\
 &\quad + \tau \sin u (\cos f - \tau \cos h \cdot \xi) dJ + \tau (\zeta \cos D - \eta \sin D + \tau \cos D \cdot \xi \xi) dN \\
 \frac{dy''}{60} &= \tau \eta \left\{ (\cotg(G+u) - \tau \zeta \cotg(H+u)) du + (1 - \tau \zeta) \frac{\omega' dr}{r} \right\} \\
 &\quad + \tau \sin u (\cos g - \tau \cos h \cdot \eta) dJ + \tau (\xi \sin D + \tau \cos D \cdot \xi \eta) dN \dots (39)
 \end{aligned}$$

There will be scarcely ever any occasion for the application of these rigorous formulae. For, as the angle s is always small, the last terms of the coefficients in these equations may be safely neglected, since their products by $du dJ$ etc. may be considered as terms of the second order.

The differential-expressions become accordingly, if likewise instead of τ its approximate value $\frac{r}{\Delta}$ is introduced and if also for the sake of convenience the equations are multiplied by $\frac{\Delta_0}{a}$

$$\begin{aligned}
 \frac{\Delta_0}{60a} \cdot dx'' &= \frac{\Delta_0}{\Delta} \rho \xi \left\{ \cotg(F+u) du + \frac{\omega' dr}{r} \right\} + \frac{\Delta_0}{\Delta} \rho \sin u \cos f dJ \\
 &\quad + \frac{\Delta_0}{\Delta} \rho (\zeta \cos D - \eta \sin D) dN \\
 \frac{\Delta_0}{60a} \cdot dy'' &= \frac{\Delta_0}{\Delta} \rho \eta \left\{ \cotg(G+u) du + \frac{\omega' dr}{r} \right\} + \frac{\Delta_0}{\Delta} \rho \sin u \cos g dJ \\
 &\quad + \frac{\Delta_0}{\Delta} \rho \xi \sin D dN \dots (40)
 \end{aligned}$$

The same approximate expressions are found for all the above mentioned forms of ψ . It is then only requisite that in calculating the values of x and y the function ψ should be assumed in strict accordance with the manner, in which the observations have been made. This will usually cause some difficulty, as the nature of differential-observations leaves some room for doubt respecting the true form of ψ .

Indeed the translation of the direct data of observation into their true angular values cannot be made rigorously without minute researches into the peculiarities of the instrument, such as perhaps only the Koenigsberg Heliometer has been made the subject of by the hands of *Bessel*.

Practically however these doubts are of no importance, since the angles, which are measured in these differential-

observations are usually so small, that it is in fact almost a matter of indifference, which of the above mentioned forms of ψ is assumed to be the true one.

§ 8.

I have yet to consider the case, in which differences of rightascension and declination have been observed. It may be looked upon as a special case of the kind, which I have just treated. Let the apparent rightascension and declination of the satellite, as seen from the observers place, be A_i and D_i ; then we have between A_i, D_i, A, D, s, p the relations

$$\left. \begin{aligned} \cos D_i \sin(A_i - A) &= \sin s \sin p \\ \cos D_i \cos(A_i - A) &= -\sin s \cos p \sin D + \cos s \cos D \\ \sin D_i &= \sin s \cos p \cos D + \cos s \sin D \end{aligned} \right\} \text{and also}$$

$$\left. \begin{aligned} \sin s \cos p &= \sin D_i \cos D - \cos D_i \sin D \cos(A_i - A) \\ \cos s &= \sin D_i \sin D + \cos D_i \cos D \cos(A_i - A) \end{aligned} \right\} (41)$$

and from these last equations

$$\left. \begin{aligned} \sin(D_i - D) &= \sin s \cos p - \sin s \sin p \operatorname{tg} \frac{1}{2}(A_i - A) \sin D \\ \cos(D_i - D) &= \cos s + \sin s \sin p \operatorname{tg} \frac{1}{2}(A_i - A) \cos D \end{aligned} \right\}$$

Introducing again ξ, η, ζ , these equations may be written

$$\left. \begin{aligned} \frac{\Delta_i}{r} \cos D_i \sin(A_i - A) &= \xi \\ \frac{\Delta_i}{r} \cos D_i \cos(A_i - A) &= \left(\frac{\Delta}{r} + \zeta\right) \cos D - \eta \sin D \\ \frac{\Delta_i}{r} \sin D_i &= \left(\frac{\Delta}{r} + \zeta\right) \sin D + \eta \cos D \\ \frac{\Delta_i}{r} \sin(D_i - D) &= \eta - \xi \operatorname{tg} \frac{1}{2}(A_i - A) \sin D \\ \frac{\Delta_i}{r} \cos(D_i - D) &= \frac{\Delta}{r} + \zeta + \xi \operatorname{tg} \frac{1}{2}(A_i - A) \cos D \end{aligned} \right\} (42)$$

$$\cos D \operatorname{tg}(A_i - A) = \frac{\sin f \sin(F + u)}{\frac{\Delta}{r} + \sin f' \sin(F' + u)}$$

and the equation of condition

$$\left. \begin{aligned} \frac{\Delta_0}{60a} \cos D \cdot d(A_i - A) &= \frac{\Delta_0}{\Delta} \rho \sin f \sin(F + u) \left\{ \cot g(F + u) du + \frac{\omega' dr}{r} \right\} \\ &+ \frac{\Delta_0}{\Delta} \rho \sin u \cos f dJ + \frac{\Delta_0}{\Delta} \rho \sin f' \sin(F' + u) dN \end{aligned} \right\} \dots\dots\dots (46)$$

§ 9.

In order to calculate the values of ξ, η, ζ conveniently, auxiliary angles f, F, g, G, h, H have been hitherto made use of. For the same purpose we may however take also another course. Denoting again by α and δ the planetocentric rightascension and declination of the satellite, we have according to (28)

$$\left. \begin{aligned} \sin \sigma \sin p &= \sin(\alpha - A) \cos \delta \\ \sin \sigma \cos p &= -\cos(\alpha - A) \cos \delta \sin D + \sin \delta \cos D \\ \cos \sigma &= \cos(\alpha - A) \cos \delta \cos D + \sin \delta \sin D \end{aligned} \right\}$$

The three first equations would suffice for the calculation of the absolute values of A_i, D_i, Δ_i . In order to find also the differences of declination accurately, it is necessary to resort to the two last. We have accordingly

$$\left. \begin{aligned} \cos D \operatorname{tg}(A_i - A) &= \frac{\xi}{\frac{\Delta}{r} + \zeta - \eta \operatorname{tg} D} \\ \operatorname{tg}(D_i - D) &= \frac{\eta - \xi \operatorname{tg} \frac{1}{2}(A_i - A) \sin D}{\frac{\Delta}{r} + \zeta + \xi \operatorname{tg} \frac{1}{2}(A_i - A) \cos D} \end{aligned} \right\} (43)$$

or developing and disregarding terms of higher orders

$$\left. \begin{aligned} \cos D(A_i - A) &= \frac{r}{\Delta} \omega'' \xi \left\{ 1 - \frac{r}{\Delta} (\zeta - \eta \operatorname{tg} D) \right\} \\ D_i - D &= \frac{r}{\Delta} \omega'' \eta \left\{ 1 - \frac{r}{\Delta} \zeta + \frac{\xi \xi}{2\eta} \operatorname{tg} D \right\} \end{aligned} \right\} \dots\dots (44)$$

The equations of condition are again given with sufficient accuracy by (40), if $\cos D \cdot d(A_i - A)$ and $d(D_i - D)$ are therein substituted for dx'' and dy'' .

In case differences of rightascension only have been observed, the calculation of η and ζ and consequently of g, G, h, H is rendered unnecessary by introducing two other auxiliary angles f', F' which are more simple to calculate. Putting

$$\left. \begin{aligned} \sin f' \sin F' &= \cos(A - N) \\ \sin f' \cos F' &= \sin(A - N) \cos J \\ \cos f' &= -\sin(A - N) \sin J \end{aligned} \right\} \dots\dots\dots (45)$$

we obtain $\zeta \cos D - \eta \sin D = \sin f' \sin(F' + u)$, therefore

$$\left. \begin{aligned} \cos D \operatorname{tg}(A_i - A) &= \frac{\sin f \sin(F + u)}{\frac{\Delta}{r} + \sin f' \sin(F' + u)} \\ \frac{\Delta_0}{60a} \cos D \cdot d(A_i - A) &= \frac{\Delta_0}{\Delta} \rho \sin f \sin(F + u) \left\{ \cot g(F + u) du + \frac{\omega' dr}{r} \right\} \\ &+ \frac{\Delta_0}{\Delta} \rho \sin u \cos f dJ + \frac{\Delta_0}{\Delta} \rho \sin f' \sin(F' + u) dN \end{aligned} \right\} \dots\dots\dots (46)$$

wherein α and δ are found by means of the formulae (29)

$$\left. \begin{aligned} \cos \delta \sin(\alpha - N) &= \sin u \cos J \\ \cos \delta \cos(\alpha - N) &= \cos u \\ \sin \delta &= \sin u \sin J \end{aligned} \right\}$$

By introducing therefore two auxiliary angles w, W , such that

$$\left. \begin{aligned} \sin w \sin W &= \sin \delta \\ \sin w \cos W &= \cos \delta \cos(\alpha - A) \\ \cos w &= \cos \delta \sin(\alpha - A) \end{aligned} \right\} \dots\dots\dots (47)$$

we obtain ξ, η, ζ from

$$\left. \begin{aligned} \xi &= \sin \sigma \sin p = \cos w \\ \eta &= \sin \sigma \cos p = \sin w \sin(W-D) \\ \zeta &= \cos \sigma = \sin w \cos(W-D) \end{aligned} \right\} \dots (48)$$

If it is only desired to compare observations with given elements, this manner of calculation is very convenient. If it is however requisite, to find also the data for the equations of condition, this course becomes not concise enough, and the other method is preferable.

I may yet mention, as to the geometrical meaning of w and W in (47), that w is the angle between the lines drawn from the planet to the satellite and to the pole of the declination-circle, which passes through the apparent place of the planet, and W the angle included between the plane of these two lines and the plane of the equator.

§ 10.

By means of the formulae above developed the apparent positions of the satellite in reference to the planet may be calculated from given elements and given places of the planet and likewise the equations of condition may be formed, which exist between variations of the so found positions and corresponding variations of the elements upon which they are based. By comparing a series of observed positions with their calculated values and by solving the system of resulting equations of condition properly, the corrections may be determined which the elements require in order to satisfy the observations. But since in this an approximate knowledge of the elements is assumed it is now necessary to point out, in what manner such a knowledge may be gained most easily.

In order to determine the seven elements of a satellite's elliptic orbit, seven data will, theoretically, in general be sufficient. On account however of the proportionally very large uncertainty of the observations an attempt to base an orbit upon such a limited number of positions would only be a waste of labour. If a proper determination of the orbit is really intended, it will be first of all permitted to assume one element, the time of revolution, to be nearly known. At least that value for it, which may be derived from a calculation of all the elements based upon not more than seven data, will probably never come so near the true one than that found without much calculation by a simple comparison of the times, at which the satellite after one or several revolutions regains a similar position with respect to the planet, some proper allowance, if necessary, being made for the motion of the planet in the mean time. In consequence of this assumption, that the time of revolution and accordingly the mean daily motion is known, the number of the unknown elements is reduced to six, which in general

may be determined from three angles of position and three distances, or from four angles and two distances. Disregarding for a moment the apparent motion of the planet in the intervals of the times of observation and referring the situation of the satellite's orbit to the plane, the pole of which is the apparent place of the planet, the observed angles of position are evidently to be considered longitudes on this plane and the distances simple functions of the latitudes ($90^\circ - \sigma$) and radii vectores. By duly regarding the planets apparent motion the formulae of course lose their suppleness or, indeed, it becomes necessary to change entirely the manner of treating the problem. Still the application of the principles of the *Theoria motus* for the derivation of practical prescriptions of calculation will cause no great difficulty, at the circumstance, that the differences of the mean anomalies are considered to be known, gives rise to the chief modification, which is indeed an advantageous one. Any interest however to be derived from the development of the necessary formulae is merely theoretical and so far doubtful, as the principles upon which they are founded contain nothing essentially new. For, on account of the small eccentricities of the satellitary orbits the influence of unavoidable errors of observation is likely to disfigure the elliptic elements thus obtained to such a degree, that they scarcely ever will prove a tolerable approximation to the true values. A circular orbit will probably always be found valuable as a first approximation and the annexed comparison of it with a series of observations will in most cases, even without formation of equations of condition, by simple exhibition of the remaining errors lead to a more accurate hypothesis respecting the amount of eccentricity and the situation of the line of apses, than it would be possible to procure by a direct calculation from six data. Not even in the case of Hyperion, the orbit of which has perhaps the largest eccentricity of all satellitary orbits, would it be advisable to take this course, since for the first approximation it may be fairly assumed, that the satellite moves in the plane of the ring of Saturn.

In consequence of these considerations I shall forbear to give here the formulae for the calculation of an ellipse and pursue only the circular hypothesis. There are then only four unknown elements to be found and two complete observations suffice for this purpose.

§ 11.

For the time t let there be given the quantities p, s, A, D, Δ , as before defined, and for the time t' corresponding ones p', s', A', D', Δ' . The mean daily motion of the satellite in its orbit I denote again by n .

Then first of all let angles ε , ζ , ζ' be calculated by the equations

$$\left. \begin{aligned} \sin \lambda \sin(\mu + \tfrac{1}{2}(p+p')) &= \sin \tfrac{1}{2}(A-A') \cos \tfrac{1}{2}(D+D') \\ \sin \lambda \cos(\mu + \tfrac{1}{2}(p+p')) &= \cos \tfrac{1}{2}(A-A') \sin \tfrac{1}{2}(D+D') \\ \cos \lambda \sin(\nu + \tfrac{1}{2}(p-p')) &= \sin \tfrac{1}{2}(A-A') \sin \tfrac{1}{2}(D+D') \\ \cos \lambda \cos(\nu + \tfrac{1}{2}(p-p')) &= \cos \tfrac{1}{2}(A-A') \cos \tfrac{1}{2}(D+D') \\ \sin \tfrac{1}{2}\varepsilon \sin \tfrac{1}{2}(\zeta + \zeta') &= \sin \lambda \sin \mu \\ \sin \tfrac{1}{2}\varepsilon \cos \tfrac{1}{2}(\zeta + \zeta') &= \cos \lambda \sin \nu \\ \cos \tfrac{1}{2}\varepsilon \sin \tfrac{1}{2}(\zeta - \zeta') &= \sin \lambda \cos \mu \\ \cos \tfrac{1}{2}\varepsilon \cos \tfrac{1}{2}(\zeta - \zeta') &= \cos \lambda \cos \nu \end{aligned} \right\} (49)$$

ζ and ζ' are the arcs contained between the apparent places of the planet and the point of intersection of the two great circles, which pass through the apparent places of the planet and satellite in both observations, and ε is the angle between the two arcs.

After ε , ζ , ζ' have been calculated, the radius a of the circular orbit is to be found by trials, so that, if with an hypothetical value of a two angles σ and σ' are determined according to

$$\left. \begin{aligned} \sin(\sigma - s) &= \frac{\Delta \sin s}{a} \\ \sin(\sigma' - s') &= \frac{\Delta' \sin s'}{a} \end{aligned} \right\} (50)$$

there may be satisfied at the same time the equation*)

$$\cos((t'-t)n) = \cos(\zeta - \sigma) \cos(\zeta' - \sigma') + \sin(\zeta - \sigma) \sin(\zeta' - \sigma') \cos \varepsilon$$

The angles σ and σ' lie between 0° and 180° . Since they are determined by their sines, it remains undecided in what quadrant they are to be taken. But this ambiguity

*) This form of the equation is well adopted for the present purpose, since the requisite observations may be so chosen as to give for the angle $(t' - t)n$ a value of about a quadrant. In calculating a circular planetary orbit, where this angle is of small amount, the form of the corresponding equation must be changed. It may then be written as a double one

$$\left. \begin{aligned} \sin\left(\frac{k(t'-t)}{2a^2}\right) \sin z &= \sin \tfrac{1}{2}\varepsilon \sin \tfrac{1}{2}(\zeta + \zeta') \\ \sin\left(\frac{k(t'-t)}{2a^2}\right) \cos z &= \cos \tfrac{1}{2}\varepsilon \sin \tfrac{1}{2}(\zeta - \zeta') \end{aligned} \right\}$$

in which the angle z itself is not wanted, or, if it is thought preferable, in the form

$$\sin^2\left(\frac{k(t'-t)}{2a^2}\right) = \sin^2 \tfrac{1}{2}\varepsilon \sin^2 \tfrac{1}{2}(\zeta + \zeta') + \cos^2 \tfrac{1}{2}\varepsilon \sin^2 \tfrac{1}{2}(\zeta - \zeta')$$

where ε is again (Theor. mot., art. 137) the angle at the point of intersection of the two great circles, which pass through the geocentric places of the sun and of the planet in both observations, ζ and ζ' the angular distances of this point of intersection from the heliocentric places of the planet, k the Gaussian constant and a the radius of the orbit.

arises out of the very nature of the subject, since two or even many observations, during which the apparent place of the planet has not considerably changed, leave it entirely undecided, whether the motion of the satellite takes place in a direct or retrograde direction. As long as there is nothing more known, both hypotheses must of course be pursued. In order to proceed accordingly in choosing the quadrant of σ correctly, the safest guide may be the simple consideration, that the transit of the satellite from the side of the apparent upper conjunction to the side of the lower conjunction or the transition of σ from the first into the second quadrant takes place in general for J between 0° and 90° on the eastern side and for J between 90° and 180° on the western side of the declination-circle. The exceptions from this rule, occurring if $\sin(A-N) \tan D > \cot J$, which for instance may take place in the case of the satellites of Uranus, will make no difficulty.

After having found a , the planetocentric rightascension and declinations α , δ , α' , δ' are determined by the formulae

$$\left. \begin{aligned} \operatorname{cosec} \sigma \cos \delta \sin(\alpha - A) &= \sin p \\ \operatorname{cosec} \sigma \cos \delta \cos(\alpha - A) &= \cot g \sigma \cos D - \cos p \sin D \\ \operatorname{cosec} \sigma \sin \delta &= \cot g \sigma \sin D + \cos p \cos D \\ \operatorname{cosec} \sigma' \cos \delta' \sin(\alpha' - A') &= \sin p' \text{ etc.} \end{aligned} \right\} (51)$$

instead of which may be used, if it is thought preferable,

$$\left. \begin{aligned} \sin w \sin(W - D) &= \sin \sigma \cos p \\ \sin w \cos(W - D) &= \cos \sigma \\ \cos w &= \sin \sigma \sin p \\ \cos \delta \sin(\alpha - A) &= \cos w \\ \cos \delta \cos(\alpha - A) &= \sin w \cos W \\ \sin \delta &= \sin w \sin W \\ \sin w' \sin(W - D) &= \sin \sigma' \cos p' \text{ etc.} \end{aligned} \right\} \dots\dots (52)$$

Then J and N result from the equations

$$\left. \begin{aligned} \tan J \sin(\alpha - N) &= \tan \delta \\ \tan J \cos(\alpha - N) &= \frac{\tan \delta' - \tan \delta \cos(\alpha' - \alpha)}{\sin(\alpha' - \alpha)} \end{aligned} \right\} \dots\dots (53)$$

and u and u' from

$$\left. \begin{aligned} \tan u &= \tan(\alpha - N) \sec J \\ \tan u' &= \tan(\alpha' - N) \sec J \end{aligned} \right\} \dots\dots\dots (54)$$

whereby a check is gained, since the difference $u' - u$ must give again the original value $(t' - t)n$.

§ 12.

In case the apparent motion of the planet in the interval of the times is very small, if for instance the planet is near its stationary point, this motion may be neglected entirely, so that A' , D' become identical with A , D . The formulae are accordingly simplified.

Again let a be found by trials, so that the equations

$$\left. \begin{aligned} \sin(\sigma - s) &= \frac{\Delta \sin s}{a} \\ \sin(\sigma' - s') &= \frac{\Delta' \sin s'}{a} \\ \cos((t' - t)n) &= \cos \sigma \cos \sigma' + \sin \sigma \sin \sigma' \cos(p - p') \end{aligned} \right\} \dots (55)$$

are satisfied. Then the formulae (51) etc. may again be employed. But it appears preferable to find J , N , u , u' in one of the two following ways. First, putting $\frac{\sin((t' - t)n)}{\sin \sigma \sin \sigma'} = \mu$, let m and M be determined by the equations

$$\left. \begin{aligned} \mu \sin m \sin M &= \cotg \sigma \sin p' - \cotg \sigma' \sin p \\ \mu \sin m \cos M &= \sin(p - p') \\ \mu \cos m &= \cotg \sigma \cos p' - \cotg \sigma' \cos p \end{aligned} \right\} \dots (56)$$

then are J and N deduced from

$$\left. \begin{aligned} \sin J \sin(N - A) &= \sin m \cos(M + D) \\ \sin J \cos(N - A) &= \cos m \\ \cos J &= \sin m \sin(M + D) \end{aligned} \right\} \dots (57)$$

and u , u' by means of the formulae

$$\left. \begin{aligned} \cotg(\alpha - A) &= \frac{\cotg \sigma \cos D - \cotg p \sin D}{\sin p} \\ \tg u &= \tg(\alpha - N) \sec J \\ \cotg(\alpha' - A) &= \text{etc.} \end{aligned} \right\} \dots (58)$$

where the quadrant for $\alpha - A$ must be taken so that $\sin(\alpha - A)$ and $\sin p$ may have the same sign.

Or, secondly, putting again $\frac{\sin((t' - t)n)}{\sin \sigma \sin \sigma'} = \mu$, let h and K be calculated by the equations

$$\left. \begin{aligned} \mu \sin h \sin K &= \cotg \sigma \cos p' - \cotg \sigma' \cos p \\ \mu \sin h \cos K &= -\cotg \sigma \sin p' + \cotg \sigma' \sin p \\ \mu \cos h &= \sin(p - p') \end{aligned} \right\} \dots (59)$$

then are J and N and also H found from

$$\left. \begin{aligned} \sin J \sin(N - A) &= \cos K \sin h \sin D + \cos h \cos D \\ \sin J \cos(N - A) &= \sin K \sin h \\ \sec D \sin J \sin H &= \sin K \\ \sec D \sin J \cos H &= \cos K \cos h + \tg D \sin h \\ \sec D \cos J &= -\cos K \sin h + \tg D \cos h \end{aligned} \right\} (60)$$

or from

$$\left. \begin{aligned} \sin \frac{1}{2} J \sin \frac{1}{2} (H + O) &= \sin \frac{1}{2} K \sin \frac{1}{2} (90^\circ - D - h) \\ \sin \frac{1}{2} J \cos \frac{1}{2} (H + O) &= \cos \frac{1}{2} K \sin \frac{1}{2} (90^\circ - D + h) \\ \cos \frac{1}{2} J \sin \frac{1}{2} (H - O) &= \sin \frac{1}{2} K \cos \frac{1}{2} (90^\circ - D - h) \\ \cos \frac{1}{2} J \cos \frac{1}{2} (H - O) &= \cos \frac{1}{2} K \cos \frac{1}{2} (90^\circ - D + h) \\ N &= A + O + 90^\circ \end{aligned} \right\} \dots (61)$$

and u , u' from

$$\left. \begin{aligned} \tg(H + u) &= \cotg(p - K) \sec h \\ \tg(H + u') &= \cotg(p' - K) \sec h \end{aligned} \right\} \dots (62)$$

where the quadrant of $H + u$ and $H + u'$ is to be taken so that $\cos(H + u)$ and $\sin(p - K)$ etc. may have the same sign. The difference $u' - u$ is again serviceable as a check, since it must come out $= (t' - t)n$.

§ 13.

The trustworthiness of the elements deduced in this manner depends entirely upon the accuracy of the observations, upon which they are based, since the errors of these enter completely into the results. In order to become more independent of fortuitous inaccuracies of single observations it may appear sometimes advantageous, in case a sufficient number of observations is extant, to found the first draft of the orbit upon the totality of measures by constructing graphically the ellipse, which corresponds to the apparent motion of the satellite round the planet. This cannot of course be done except in case the apparent motion of the planet in the interlapse of time has not been considerable. The next assumption, that the centre of the apparent ellipse coincides with the centre of the planet, is fully justified, since the neglect depending upon it, namely, putting $\sin s = \frac{r}{\Delta} \sin \sigma$ instead of $= \frac{r}{\Delta} \sin(\sigma - s)$, is very insignificant, as I have already shown.

The prescriptions for the calculation of J and N from the data given by graphical construction may be brought into a very simple form. Let p_0 signify the angle of position of the major axis of the apparent ellipse, s_0 its semi-major axis, s its semi-minor axis. Then, by making

$$\left. \begin{aligned} \sin \sigma &= \frac{\sin s}{\sin s_0} \quad \text{or} \quad = \frac{s}{s_0} \\ \cos c &= \cos \sigma \cos p_0 \\ \sin c \sin C &= \cos \sigma \sin p_0 \\ \sin c \cos C &= \mp \sin \sigma \end{aligned} \right\} \dots (63)$$

where the upper or lower sign must be taken according as the apparent motion of the satellite round the planet takes place in the direction of increasing or decreasing angles of position, we obtain J and N by means of the equations

$$\left. \begin{aligned} \sin J \sin(N - A) &= \sin c \cos(C + D) \\ \sin J \cos(N - A) &= \cos c \\ \cos J &= \sin c \sin(C + D) \end{aligned} \right\} \dots (64)$$

if A and D denote again the apparent rightascension and declination of the planet.

Since p_0 signifies the angle of position of the major axis of the apparent ellipse, its value is ambiguous by 180° . By substituting both assumptions for it in the formulae, double value are found for J and N , of which one belongs to direct, the other to retrograde motion. Which of the two is the correct one cannot be determined without further observations. The simple consideration, whether the apparent ellipse opens or closes in the course of time, decides on the true direction of the motion. Assuming $\cos \sigma$ always positive, the angle p_0 is to be taken so that it belongs to that point of the orbit, at which the satellite passes from

the side of the apparent upper conjunction to the side of the lower conjunction. In accordance with the explanation already given let it be observed, that this transit takes place in most cases for $J < 90^\circ$ to the East and for $J > 90^\circ$ to the West of the circle of declination.

After J and N have been determined, let again h, H, K be calculated by means of the equations (1) or (2). Then, if p is the angle of position of the satellite for the time t of any of the given observations, the value of u , belonging to the time t_0 , is found by means of

$$\left. \begin{aligned} tg U &= cotg(p-K) sec h \\ u_0 &= U-H-(t-t_0)n \end{aligned} \right\} \dots\dots\dots (65)$$

where the quadrant of U must be chosen so that $cos U$ and $sin(p-K)$ may have the same sign.

A special case still deserves to be mentioned, namely, if the apparent ellipse becomes a straight line. The determination of the situation of the orbits, which the satellites of Uranus describe, as made by Sir *William Herschel*, rests upon this supposition. The formulae become

$$\left. \begin{aligned} tg J \sin(N-A) &= -tg D \\ tg J \cos(N-A) &= cotg p_0 sec D \end{aligned} \right\} \dots\dots\dots (66)$$

where the sign of $tg J$ is ambiguous.

§ 14.

Referring now the situation of the satellite's orbit to the plane of the planet's orbit, let γ denote its inclination, Θ the longitude of its ascending node, as reckoned from the ascending node of the orbit of the planet on the plane of the (terrestrial) equator, I' the angle between the line of apsides and the line of nodes. Also, let J_0 and N_0 signify the inclination and the ascending node of the planetary orbit upon the equator. Then are γ, Θ, I' found from J, N, Q, J_0, N_0 by means of the equations

$$\left. \begin{aligned} \sin \gamma \sin \Theta &= \sin(N-N_0) \sin J \\ \sin \gamma \cos \Theta &= \cos(N-N_0) \sin J \cos J_0 - \cos J \sin J_0 \\ \cos \gamma &= \cos(N-N_0) \sin J \sin J_0 + \cos J \cos J_0 \\ \sin \gamma \sin(Q-I') &= \sin(N-N_0) \sin J_0 \\ \sin \gamma \cos(Q-I') &= -\cos(N-N_0) \sin J_0 \cos J + \cos J_0 \sin J \end{aligned} \right\} (67)$$

or by the formulae

$$\left. \begin{aligned} \sin \frac{1}{2} \gamma \sin \frac{1}{2}(\Theta+Q-I') &= \sin \frac{1}{2}(N-N_0) \sin \frac{1}{2}(J+J_0) \\ \sin \frac{1}{2} \gamma \cos \frac{1}{2}(\Theta+Q-I') &= \cos \frac{1}{2}(N-N_0) \sin \frac{1}{2}(J-J_0) \\ \cos \frac{1}{2} \gamma \sin \frac{1}{2}(\Theta-Q+I') &= \sin \frac{1}{2}(N-N_0) \cos \frac{1}{2}(J+J_0) \\ \cos \frac{1}{2} \gamma \cos \frac{1}{2}(\Theta-Q+I') &= \cos \frac{1}{2}(N-N_0) \cos \frac{1}{2}(J-J_0) \end{aligned} \right\} (68)$$

The differential-expressions between $d\gamma, d\Theta, dI'$ and dJ, dN, dQ are these

$$\left. \begin{aligned} d\gamma &= \cos(Q-I') dJ + \sin(Q-I') \sin J dN \\ \sin \gamma d\Theta &= -\sin(Q-I') dJ + \cos(Q-I') \sin J dN \\ \sin \gamma dI' &= \sin(Q-I') \cos \gamma dJ - \cos I' \sin J_0 dN + \sin \gamma dQ \end{aligned} \right\} (69)$$

§ 15.

Hitherto I have tacitly supposed, that the motion of the satellite takes place in a fixed and purely elliptical orbit. It remains now to investigate the changes which the elements undergo, so that they may be properly transferred to other epochs.

But before doing this I must call to mind the relations existing between the motions of the satellite and the forces, by which they are produced.

Let $\odot$ denote the proportion of the mass of the Sun to the sum of the masses of the planet and satellite;

η the ellipticity of the planet, here assumed to be an ellipsoid of rotation;

2ζ the proportion of the centrifugal force to the force of gravity at the equator of the planet;

α the proportion of the major axis of the satellite's orbit to the equatorial-diameter of the planet;

ε the latitude of the satellite above the plane of the planet's equator —

then are

$$\left\{ \frac{1}{\odot a^2 \rho^2} \left\{ 1 - \frac{\eta - \zeta}{\alpha^2} \cdot \frac{1 - 3 \sin^2 \varepsilon}{\rho^2} \right\} + \text{perturbing forces of the Sun and other bodies} \right\}$$

the forces, which produce the motion of the satellite round the centre of the planet. The purely elliptical part of satellite's motion in an orbit, the semi-major axis of which is $= a$, is the effect of the force

$$\frac{1}{\odot a^2 \rho^2} \left\{ 1 - \frac{\eta - \zeta}{\alpha^2} \right\}$$

The variable remainder of the acting forces

$$\left\{ -\frac{3(\eta - \zeta)}{\odot a^2 x^2} \cdot \frac{\sin^2 \varepsilon}{\rho^4} + \text{perturbing forces of the Sun etc.} \right\}$$

gives rise to the perturbations of the purely elliptical motion. Denoting, as before, by n the mean sidereal motion of the satellite in its orbit and by k the *Gaussian* constant, we have between $\odot$ and a the relation

$$\odot = \frac{k^2}{a^3 n^2} \left\{ 1 + \frac{\eta - \zeta}{\alpha^2} \right\}$$

where n and k are of course to be expressed by the same unit. If $\odot_0$ signifies the mass of the Sun in proportion to the sum of the masses of all the bodies belonging to the planet, then is

$$\odot_0 = \frac{k^2}{a^3 n^2} \left\{ 1 - \frac{\eta - \zeta}{\alpha^2} \right\} \cdot \frac{1+m}{1+\Sigma m}$$

denoting by m the mass of the satellite and by Σm the sum of the masses of all the secondary bodies belonging to the planet's system, both expressed in units of the mass of the planet.

The differentiation of this equation gives, if only $\odot_0, a, n$ are considered variables,

$$\frac{da}{a} + \frac{d\odot_0}{3\odot_0} + \frac{2}{3} \cdot \frac{dn}{n} = 0 \dots\dots\dots(70)$$

Accordingly $d\odot_0$ may be introduced instead of da in the equations of condition (11), (39), (40), which contain $\frac{dr}{r} = \frac{da}{a} + \frac{d\rho}{\rho}$. Since this will be necessary sometimes I shall adopt it in general. The necessity for it occurs in those cases, in which it appears desirable to deduce from a system of observations of several satellites the most probable values of the mass of the planet and of the elements of their orbits. Mr. *Lassell's* Valetta-observations of Oberon and Titania, for example, are made under very similar circumstances. In order to deduce the most probable values for the corrections of their assumed elements, it will (or might) be requisite to combine all the resulting equations of condition into one system of thirteen or, in case the mean motions are supposed to be known, of eleven final equations, which system has the peculiarity that only one of these equations contains all the unknown quantities. Of course, before doing this, it is necessary to combine the equations of condition for each satellite separately, so that we obtain two systems of seven (or six) final equations, both of which have one unknown quantity. The solution of these separate systems decides then, whether it is safely allowed to combine both systems into one and what weight is properly to be given to each.

§ 16.

I have now to take into consideration the effect of the forces, which disturb the purely elliptical motion of the satellite. But since the masses of the disturbing satellites and also the peculiarities of the outer planets are unknown and since therefore the forces, which depend thereon, cannot be determined a priori, I shall disregard them at present altogether and I shall accordingly pursue only that part of

The perturbing force is

$$\Omega = \frac{m}{1+m} \left\{ \{(x,-x)^2 + (y,-y)^2 + (z,-z)^2\}^{-\frac{1}{2}} - \frac{x,x + y,y + z,z}{(x,-x)^2 + (y,-y)^2 + (z,-z)^2} \right\} + \frac{m_n}{1+m} \left\{ \dots \right.$$

where m, x, y, z signify the mass and the rectangular coordinates of the disturbed body

m, x, y, z , those of the disturbing body,

the unit of the masses being the mass of the primary body and the origin of the coordinates being in coincidence with the centre of it. Since the data are wanted, from which for the satellites of the outer planets the situation of the planes may be determined, to which in the theory of satellites the term of fundamental-planes properly belongs, I shall choose as plane of the x, y the plane of the planetary orbit and in consequence I shall make use again of the former no-

the disturbing forces, which depends upon the attraction of the Sun. The development of the formulae for this purpose causes no great difficulty. There is however no practical interest in taking more of it into account than that part, which is independent of the place of the satellite in its orbit. For even in the case of *Japetus*, upon which the influence of the disturbing force of the sun is the largest, the perturbations, which depend on the place of the satellite, do not affect the geocentric place of it by more than about $\frac{1}{20}''$, so that it would be a waste of labour to calculate them. I shall accordingly restrict the present development of the perturbations to those, which do not contain the longitude of the satellite in its orbit.

Again, let a denote the semi major axis of the orbit, n the mean sidereal motion, e the eccentricity, μ_0 the mean anomaly of the satellite for the epoch t_0 ; ω the angle between the line of apses and the line of nodes, i the inclination of the orbit, θ the longitude of the ascending node, these three last elements being referred to an arbitrary fundamental-plane; then, if, as usual, the perturbing force is represented by Ω , the well known formulae of *Lagrange* may be written

$$\left. \begin{aligned} \frac{da}{dt} &= 2a^2n \left(\frac{d\Omega}{d\mu_0} \right) \\ \frac{d\mu_0}{dt} &= -2a^2n \left(\frac{d\Omega}{da} \right) - an \cdot \frac{1-e^2}{e} \left(\frac{d\Omega}{de} \right) \\ \frac{de}{dt} &= an \cdot \frac{1-e^2}{e} \left(\frac{d\Omega}{d\mu_0} \right) - an \cdot \frac{\sqrt{1-e^2}}{e} \left(\frac{d\Omega}{d\omega} \right) \\ \frac{d\omega}{dt} &= an \cdot \frac{\sqrt{1-e^2}}{e} \left(\frac{d\Omega}{de} \right) - an \cdot \frac{\cotgi}{\sqrt{1-e^2}} \left(\frac{d\Omega}{di} \right) \\ \frac{di}{dt} &= an \cdot \frac{\cotgi}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\omega} \right) - an \cdot \frac{\operatorname{cosec} i}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\theta} \right) \\ \frac{d\theta}{dt} &= an \cdot \frac{\operatorname{cosec} i}{\sqrt{1-e^2}} \left(\frac{d\Omega}{di} \right) \end{aligned} \right\} \quad (71)$$

tations (§ 14) T, γ, Θ instead of ω, i, θ . Putting the axis of X into the line of nodes etc. and denoting the heliocentric longitude of the planet in its orbit, reckoned from the ascending node of it in the equator by L and the radius vector of the planet by R , we have, if v and r signify again the true anomaly and the radius vector of the satellite,

$$\left. \begin{aligned} x &= r \cos(T+v) \\ y &= r \sin(T+v) \cos \gamma \\ z &= r \sin(T+v) \sin \gamma \\ x' &= -R \cos(L-\Theta) \\ y' &= -R \sin(L-\Theta) \\ z' &= 0 \end{aligned} \right\} \dots\dots\dots(72)$$

and Ω becomes accordingly, $\sin \frac{m}{1+m} = \odot$,

$$\Omega = \odot \left\{ \left\{ R^2 + r^2 + 2Rr(\cos(\Gamma+v)\cos(L-\Theta) + \sin(\Gamma+v)\sin(L-\Theta)\cos\gamma) \right\}^{-\frac{1}{2}} + \frac{r}{R^2}(\cos(\Gamma+v)\cos(L-\Theta) + \sin(\Gamma+v)\sin(L-\Theta)\cos\gamma) \right\} \dots\dots\dots (73)$$

or, putting $\cos(\Gamma+v)\cos(L-\Theta) + \sin(L+v)\sin(L-\Theta)\cos\gamma = V$,

$$\Omega = \frac{\odot}{R} \left\{ \left(1 + \frac{2r}{R}V + \frac{r^2}{R^2} \right)^{-\frac{1}{2}} + \frac{r}{R}V \right\}$$

and developing according the powers of $\frac{r}{R}$,

$$\Omega = \frac{\odot}{R} \left\{ 1 + \frac{1}{2} \left(\frac{r}{R} \right)^2 (2V^2 - 1) + \frac{3}{2} \left(\frac{r}{R} \right)^3 V(1 - \frac{5}{3}V^2) + \dots \right\}$$

By making then

$$\begin{aligned} \cos B \sin U &= \sin(L-\Theta) \cos \gamma \\ \cos B \cos U &= \cos(L-\Theta) \\ \sin B &= \sin(L-\Theta) \sin \gamma \end{aligned}$$

we obtain

$$V = \cos B \cos(\Gamma - U + v)$$

and therefore

$$\Omega = \frac{\odot}{R} \left\{ 1 + \frac{1}{2} \left(\frac{r}{R} \right)^2 \{ 3 \cos^2 B \cos^2(\Gamma - U + v) - 1 \} + \dots \right\}$$

or, since $r = a\rho$, putting

$$\begin{aligned} W &= \rho^2 \left\{ \frac{1}{3} + \cos^2 B \cos 2(\Gamma - U + v) - \sin^2 B \right\}, \\ \Omega &= \frac{\odot}{R} \left\{ 1 + \frac{3}{4} \left(\frac{a}{R} \right)^2 W \right\} \end{aligned} \quad (75)$$

Hence follow the differentials

$$\left. \begin{aligned} \left(\frac{d\Omega}{da} \right) &= \frac{3}{4} \cdot \frac{\odot a^2}{R^3} \cdot \frac{2W}{a} \\ \left(\frac{d\Omega}{d\mu_0} \right) &= \frac{3}{4} \cdot \frac{\odot a^2}{R^3} \left(\frac{dW}{d\mu_0} \right) \\ \left(\frac{d\Omega}{de} \right) &= \frac{3}{4} \cdot \frac{\odot a^2}{R^3} \left(\frac{dW}{de} \right) \\ \left(\frac{d\Omega}{d\Gamma} \right) &= \text{etc.} \end{aligned} \right\} \dots\dots\dots (76)$$

Introducing again the eccentric anomaly ε , we have

$$\begin{aligned} \rho &= 1 - e \cos \varepsilon \\ \rho^2 \sin 2v &= 2 \sqrt{1-e^2} \sin \varepsilon (\cos \varepsilon - e) \\ \rho^2 \cos 2v &= (\cos \varepsilon - e)^2 - (1-e^2) \sin^2 \varepsilon, \end{aligned}$$

the expression for W may therefore be written

$$W = \left(\frac{1}{3} - \sin^2 B \right) \left\{ 1 + \frac{1}{2} e^2 - 2e \cos \varepsilon + \frac{1}{2} e^2 \cos 2\varepsilon \right\} + \cos^2 B \cos 2(\Gamma - U) \left\{ \frac{3}{2} e^2 - 2e \cos \varepsilon + (1 - \frac{1}{2} e^2) \cos 2\varepsilon \right\} + \cos^2 B \sin 2(\Gamma - U) \cdot 2 \sqrt{1-e^2} \sin \varepsilon (e - \cos \varepsilon) \quad (77)$$

As I have mentioned before, I shall develop only that part of the solar perturbations, which is independent of the place of the satellite in its orbit. I shall retain in W accordingly only those terms, which give this part of the perturbations. Now, the non-periodical part of any function Φ of ρ , v , ε , μ in an ellipse is

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi d\mu = \frac{1}{2\pi} \int_0^{2\pi} \Phi \rho d\varepsilon \text{ or } = \frac{1}{2\pi} \int_0^{2\pi} \Phi (1 - e \cos \varepsilon) d\varepsilon$$

Designating therefore by W_0 that part of W , which gives the non-periodical terms of the perturbations, we have $W_0 = (1 + \frac{3}{2} e^2) (\frac{1}{3} - \sin^2 B) + \frac{5}{2} e^2 \cos^2 B \cos 2(\Gamma - U)$ (78) and it only remains now to differentiate this expression.

W_0 appears here as a function of e , Γ , U and B . We have

$$\left. \begin{aligned} \left(\frac{dW_0}{de} \right) &= e \{ 1 - 3 \sin^2 B + 5 \cos^2 B \cos 2(\Gamma - U) \} \\ \left(\frac{dW_0}{d\Gamma} \right) &= - \left(\frac{dW_0}{dU} \right) = - 5 e^2 \cos^2 B \sin 2(\Gamma - U) \\ \left(\frac{dW_0}{dB} \right) &= - \sin 2B \left\{ 1 + \frac{3}{2} e^2 + \frac{5}{2} e^2 \cos 2(\Gamma - U) \right\} \end{aligned} \right\} \quad (79)$$

U and B are functions of γ , Θ and L . By differentiation of the equations (74) we obtain, L being constant,

$$\left. \begin{aligned} \left(\frac{dU}{d\gamma} \right) &= - \cos U \operatorname{tg} B \\ \left(\frac{dU}{d\Theta} \right) &= - \cos \gamma \sec^2 B \\ \left(\frac{dB}{d\gamma} \right) &= \sin U \\ \left(\frac{dB}{d\Theta} \right) &= - \cos U \sin \gamma \end{aligned} \right\} \dots\dots\dots (80)$$

and consequently by the usual algorithm of partial differentials

$$\left. \begin{aligned} \left(\frac{dW_0}{d\gamma}\right) &= -\sin 2B \left\{ \left(1 + \frac{3}{2}e^2\right) \sin U + \frac{5}{2}e^2 \sin(2\Gamma-U) \right\} \\ \left(\frac{dW_0}{d\Theta}\right) &= +\sin 2B \left\{ \left(1 + \frac{3}{2}e^2\right) \cos U + \frac{5}{2}e^2 \cos(2\Gamma-U) \right\} \sin \gamma \\ &\quad - 5e^2 \cos^2 B \sin 2(\Gamma-U) \cos \gamma \end{aligned} \right\} \dots\dots\dots (81)$$

The differential-expressions of the perturbing function become accordingly

$$\left. \begin{aligned} \left(\frac{d\Omega}{da}\right) &= \frac{\odot a}{2R^3} \left\{ \left(1 + \frac{3}{2}e^2\right) (1 - 3\sin^2 B) + \frac{1}{2}e^2 \cos^2 B \cos 2(\Gamma-U) \right\} \\ \left(\frac{d\Omega}{d\mu_0}\right) &= 0 \\ \left(\frac{d\Omega}{de}\right) &= + \frac{3}{4} \frac{\odot a^2}{R^3} e \left\{ 1 - 3\sin^2 B + 5\cos^2 B \cos 2(\Gamma-U) \right\} \\ \left(\frac{d\Omega}{d\Gamma}\right) &= - \frac{3}{4} \frac{\odot a^2}{R^3} 5e^2 \cos^2 B \sin 2(\Gamma-U) \\ \left(\frac{d\Omega}{d\gamma}\right) &= - \frac{3}{4} \frac{\odot a^2}{R^3} \sin 2B \left\{ \left(1 + \frac{3}{2}e^2\right) \sin U + \frac{5}{2}e^2 \sin(2\Gamma-U) \right\} \\ \left(\frac{d\Omega}{d\Theta}\right) &= + \frac{3}{4} \frac{\odot a^2}{R^3} \left\{ \sin 2B \left\{ \left(1 + \frac{3}{2}e^2\right) \cos U + \frac{5}{2}e^2 \cos(2\Gamma-U) \right\} \sin \gamma \right. \\ &\quad \left. - 5e^2 \cos^2 B \sin 2(\Gamma-U) \cos \gamma \right\} \end{aligned} \right\} \dots\dots\dots (82)$$

by substituting which in the equations (71) we obtain the variations of the elements

$$\left. \begin{aligned} \frac{da}{dt} &= 0 \\ \frac{d\mu_0}{dt} &= - \frac{\odot a^3 n}{R^3} \left\{ 1 - 3\sin^2 B + \frac{3}{4}(1 + e^2) \left\{ 1 - 3\sin^2 B + 5\cos^2 B \cos 2(\Gamma-U) \right\} \right\} \\ \frac{de}{dt} &= + \frac{3}{4} \frac{\odot a^3 n}{R^3} 5e \sqrt{1-e^2} \cos^2 B \sin 2(\Gamma-U) \\ \frac{d\Gamma}{dt} &= + \frac{3}{4} \frac{\odot a^3 n}{R^3} \sqrt{1-e^2} \left\{ 1 - 3\sin^2 B + 5\cos^2 B \cos 2(\Gamma-U) \right. \\ &\quad \left. + \frac{\sin 2B}{1-e^2} \left\{ \left(1 + \frac{3}{2}e^2\right) \sin U + \frac{5}{2}e^2 \sin(2\Gamma-U) \right\} \cot \gamma \right\} \\ \frac{d\gamma}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{R^3} \frac{\sin 2B}{\sqrt{1-e^2}} \left\{ \left(1 + \frac{3}{2}e^2\right) \cos U + \frac{5}{2}e^2 \cos(2\Gamma-U) \right\} \\ \frac{d\Theta}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{R^3} \frac{\sin 2B}{\sqrt{1-e^2}} \left\{ \left(1 + \frac{3}{2}e^2\right) \sin U + \frac{5}{2}e^2 \sin(2\Gamma-U) \right\} \operatorname{cosec} \gamma \end{aligned} \right\} \dots\dots\dots (83)$$

The equation for $\frac{d\Gamma}{dt}$ may also be written

$$\frac{d\Gamma}{dt} = + \frac{3}{4} \frac{\odot a^3 n}{R^3} \sqrt{1-e^2} \left\{ 1 - 3\sin^2 B + 5\cos^2 B \cos 2(\Gamma-U) \right\} - \cos \gamma \cdot \frac{d\Theta}{dt}$$

Introducing instead of Γ and μ_0 as elements

$$\Pi = \Theta \pm \Gamma \text{ and}$$

$$A_0 = \mu_0 + \Pi,$$

the corresponding differential-expressions become

$$\left. \begin{aligned} \frac{d\Pi}{dt} &= \pm \frac{3}{4} \frac{\odot a^3 n}{R^3} \sqrt{1-e^2} \left\{ 1 - 3\sin^2 B + 5\cos^2 B \cos 2(\Gamma-U) \right\} + (1 \mp \cos \gamma) \frac{d\Theta}{dt} \\ \frac{dA_0}{dt} &= - \frac{\odot a^3 n}{R^3} \left\{ 1 - 3\sin^2 B + \frac{3}{4}(1 + e^2 \mp \sqrt{1-e^2}) \left\{ 1 - 3\sin^2 B + 5\cos^2 B \cos 2(\Gamma-U) \right\} \right\} \\ &\quad + (1 \mp \cos \gamma) \frac{d\Theta}{dt} \end{aligned} \right\} \dots\dots\dots (84)$$

§ 17.

I have written these equations in the form which seems best adopted for their integration by quadratures; since I shall make use of this method for the modern observations, for which the requisite calculation is sufficiently easy and convenient, so that it is unnecessary to make any sacrifice in point of accuracy. For the old observations it is however preferable to integrate approximately in the usual manner. For this purpose I have now to write the differential-expressions in another form and to separate the secular and periodical terms. Clearing the equations (83) and (84) again of the auxiliary angles U, B , we obtain

$$\begin{aligned}
 \frac{dA_0}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{R^3} \left\{ \left(\frac{1}{3} + e^2 \mp \sqrt{1-e^2} \right) \left\{ 1 - \frac{3}{2} \sin^2 \gamma + \frac{3}{2} \sin^2 \gamma \cos 2(L-\Theta) \right\} \right. \\
 &\quad \left. + \frac{5}{2} (1 + e^2 \mp \sqrt{1-e^2}) \left\{ \sin^2 \gamma \cos 2\Gamma + 2 \cos \gamma \sin 2\Gamma \sin 2(L-\Theta) \right. \right. \\
 &\quad \left. \left. + (1 + \cos^2 \gamma) \cos 2\Gamma \cos 2(L-\Theta) \right\} \right\} \\
 &\quad + (1 \mp \cos \gamma) \frac{d\Theta}{dt} \\
 \frac{de}{dt} &= + \frac{3}{4} \frac{\odot a^3 n}{R^3} \frac{5}{2} e \sqrt{1-e^2} \left\{ \sin^2 \gamma \sin 2\Gamma + (1 + \cos^2 \gamma) \sin 2\Gamma \cos 2(L-\Theta) \right. \\
 &\quad \left. - 2 \cos \gamma \cos 2\Gamma \sin 2(L-\Theta) \right\} \\
 \frac{d\Pi}{dt} &= \pm \frac{3}{4} \frac{\odot a^3 n}{R^3} \sqrt{1-e^2} \left\{ 1 - \frac{3}{2} \sin^2 \gamma + \frac{3}{2} \sin^2 \gamma \cos 2(L-\Theta) + \frac{5}{2} \sin^2 \gamma \cos 2\Gamma \right. \\
 &\quad \left. + 5 \cos \gamma \sin 2\Gamma \sin 2(L-\Theta) + \frac{5}{2} (1 + \cos^2 \gamma) \cos 2\Gamma \cos 2(L-\Theta) \right\} \\
 &\quad + (1 \mp \cos \gamma) \frac{d\Theta}{dt} \\
 \frac{d\gamma}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{R^3} \frac{\sin \gamma}{\sqrt{1-e^2}} \left\{ \frac{5}{2} e^2 \cos \gamma \sin 2\Gamma (1 - \cos 2(L-\Theta)) \right. \\
 &\quad \left. + (1 + \frac{3}{2} e^2 + \frac{5}{2} e^2 \cos 2\Gamma) \sin 2(L-\Theta) \right\} \\
 \frac{d\Theta}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{R^3} \frac{1}{\sqrt{1-e^2}} \left\{ (1 + \frac{3}{2} e^2 - \frac{5}{2} e^2 \cos 2\Gamma) \cos \gamma (1 - \cos 2(L-\Theta)) \right. \\
 &\quad \left. + \frac{5}{2} e^2 \sin 2\Gamma \sin 2(L-\Theta) \right\}
 \end{aligned} \tag{85}$$

In order to separate in these equations first the secular terms, it is again to be born in mind, that the non-periodical part of any function Φ of ρ, v, μ in an ellipse is

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi d\mu = \frac{1}{2\pi \sqrt{1-e^2}} \int_0^{2\pi} \Phi \rho^2 dv.$$

The ellipse here under consideration is the planetary

orbit. Designating by a^0 its semi-major axis, by e_0 its eccentricity and by v_0 the true anomaly of the planet, we obtain the non-periodical part of $\Phi(v, \rho) = \frac{1}{R^3}$, since $R = \frac{a_0(1-e_0^2)}{1+e_0 \cos v_0}$,

$$= \frac{1}{a_0^3(1-e_0^2)^{\frac{3}{2}}}$$

and consequently the secular variations of the elements of the satellite's orbit become

$$\begin{aligned}
 \frac{dA_0}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{a_0^3(1-e_0^2)^{\frac{3}{2}}} \left\{ \left(\frac{1}{3} + e^2 \mp \sqrt{1-e^2} \right) (1 - \frac{3}{2} \sin^2 \gamma) + \frac{5}{2} (1 + e^2 \mp \sqrt{1-e^2}) \sin^2 \gamma \cos 2\Gamma \right\} \\
 &\quad + (1 \mp \cos \gamma) \frac{d\Theta}{dt} \\
 \frac{de}{dt} &= + \frac{3}{4} \frac{\odot a^3 n}{a_0^3(1-e_0^2)^{\frac{3}{2}}} \frac{5}{2} e \sqrt{1-e^2} \sin^2 \gamma \sin 2\Gamma \\
 \frac{d\Pi}{dt} &= \pm \frac{3}{4} \frac{\odot a^3 n}{a_0^3(1-e_0^2)^{\frac{3}{2}}} \sqrt{1-e^2} \left\{ 1 - \frac{3}{2} \sin^2 \gamma + \frac{5}{2} \sin^2 \gamma \cos 2\Gamma \right\} + (1 \mp \cos \gamma) \frac{d\Theta}{dt} \\
 \frac{d\gamma}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{a_0^3(1-e_0^2)^{\frac{3}{2}}} \frac{\sin \gamma}{\sqrt{1-e^2}} \frac{5}{2} e^2 \cos \gamma \sin 2\Gamma \\
 \frac{d\Theta}{dt} &= - \frac{3}{4} \frac{\odot a^3 n}{a_0^3(1-e_0^2)^{\frac{3}{2}}} \frac{\cos \gamma}{\sqrt{1-e^2}} \left\{ 1 + \frac{3}{2} e^2 - \frac{5}{2} e^2 \cos 2\Gamma \right\}
 \end{aligned} \tag{86}$$

The inequalities of these secular variations, depending upon the rapidity of the variation of the angle 2Γ , will in most cases be scarcely appreciable except for $d\Pi$ and perhaps de , since for the other elements they are multiplied by e^2 . The integration is then effected by the aid of the well known substitution.

Of the periodical terms in the integrals of (85), which depend on the place of the planet in its orbit, those, which are multiplied by the eccentricity of the planetary orbit, become quite insensible and may be neglected. The others are to be taken into account in the usual manner. But this remains a matter for consideration in the special calculations.

So much respecting the mere solar perturbations. I have purposely abstained from mixing them with those, which depend upon the spheroidity of the planet and the attraction of other bodies, but which for want of the requisite data cannot be determined a priori. The chief effect of the planet's spheroidity and, in the case of Saturn, of the attraction of the rings will be a progressive motion of the line of apses and also a motion of the line of nodes, the amount of which is however left to be determined empirically from

$$\left. \begin{aligned} \sin J \sin(N - N_0) &= \sin(\Theta + \delta\Theta) \sin\gamma \\ \sin J \cos(N - N_0) &= \cos(\Theta + \delta\Theta) \sin\gamma \cos J_0 + \cos\gamma \sin J_0 \\ \cos J &= -\cos(\Theta + \delta\Theta) \sin\gamma \sin J_0 + \cos\gamma \cos J_0 \\ \sin J \sin(Q - I) &= \sin(\Theta + \delta\Theta) \sin J_0 \\ \sin J \cos(Q - I) &= \cos(\Theta + \delta\Theta) \sin J_0 \cos\gamma + \cos J_0 \sin\gamma \end{aligned} \right\} \dots\dots\dots (87)$$

or, if it is thought preferable, by the formulae

$$\left. \begin{aligned} \sin \frac{1}{2} J \sin \frac{1}{2} (N - N_0 - Q + I) &= \sin \frac{1}{2} (\Theta + \delta\Theta) \sin \frac{1}{2} (\gamma - J_0) \\ \sin \frac{1}{2} J \cos \frac{1}{2} (N - N_0 - Q + I) &= \cos \frac{1}{2} (\Theta + \delta\Theta) \sin \frac{1}{2} (\gamma + J_0) \\ \cos \frac{1}{2} J \sin \frac{1}{2} (N - N_0 + Q - I) &= \sin \frac{1}{2} (\Theta + \delta\Theta) \cos \frac{1}{2} (\gamma - J_0) \\ \cos \frac{1}{2} J \cos \frac{1}{2} (N - N_0 + Q - I) &= \cos \frac{1}{2} (\Theta + \delta\Theta) \cos \frac{1}{2} (\gamma + J_0) \end{aligned} \right\} \dots\dots\dots (88)$$

In concluding this paper I should perhaps collect the formulae of calculation in the form, in which I actually make use of them in the different cases of my following researches. But I defer doing so till I shall have the honour to submit to the notice of astronomers the results of my numerical calculations.

Durham Observatory, in the spring 1856.

A. Marth.

Schreiben des Herrn Dr. Förster an den Herausgeber.

In Folgendem erlaube ich mir, Ihnen die Resultate einiger Planeten-Rechnungen zu übersenden. Sie betreffen Thalia und Atalante.

Von Thalia sind bis jetzt 3 Erscheinungen beobachtet, deren letzte jedoch so ungünstig war ($\delta = -30''$), dass nur hier in Berlin zwei Beobachtungen gelungen sind, die folgendermassen unter einander übereinstimmen:

R-B		
1855 Juli 19	-54''1	-20''1
23	-56,1	-14,9

Da aber bei einer so geringen Zahl von Beobachtungen selbst eine noch bessere Uebereinstimmung illusorisch sein kann, schien es mir nicht rätlich, eine definitive Verbesserung der Elemente vorzunehmen. Ich suchte vielmehr unter

observation. Constant terms from this source enter consequently into the equations for $\frac{dI}{dt}$ or $\frac{dII}{dt}$ and $\frac{d\Theta}{dt}$ in (83), (84) and (86) and their integrals must be taken accordingly.

§ 18.

After A_0 , e , II , γ , Θ , $A_0 - II = \mu_0$, $II \mp \Theta = I'$ corresponding to the time t having been found, it is now only left to determine from them again the elements of the satellite's orbit in reference to the plane of the equator.

Let J_0 and N_0 denote the inclination and the longitude of the ascending node of the planetary orbit on the equator for the time t and $\delta\Theta$ the angular distance of the equator for the time t from the equator for the time t_0 , measured along the planetary orbit. Then are J , N and Q obtained by means of the equations

gehöriger Berücksichtigung der Gewichte durch Variation der Distanzen 5 Normal-Oertern aus den 3 Erscheinungen mich möglichst zu nähern. Die 5 Normal-Oerter sind:

		α	δ
I.	1852 Decbr. 20,0	47° 18' 47''7	+17° 6' 46''4
II.	1853 Jan. 14,0	46 46 6,0	+19 10 5,1
III.	1853 April 10,5	78 18 31,7	+28 21 53,2
IV.	1854 Mai 12,0	230 11 48,3	-13 32 27,0
V.	1855 Juli 21,5	312 26 45,7	-30 33 29,6

sämmtlich auf das mittlere Aeq. 1853,0 bezogen und für Störungen corrigirt.

Ich wählte die Oerter I und IV als die sichersten zur genauen Darstellung und fand durch möglichst genauen Anschluss an die übrigen folgendes Elementen-System: