The Power Series for $\sin x$, $\cos x$

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Abundant illustrations of the application to statistics of the properties of the Normal Curve of Error

$$y = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

may be found in Professor Pearson's *Grammar of Science*, and the series of papers in the *Phil. Trans.* already alluded to. Mr. W. F. Sheppard proves many of the properties by elementary methods in a paper in the *Philosophical Transactions*, vol. 192A, p. 101. C. S. JACKSON.

THE POWER SERIES FOR $\sin x$, $\cos x$.

THE hopeless complexity, from the elementary didactic point of view, of the hitherto accepted accurate trigonometrical proofs of the power series for $\sin x$, $\cos x$ has been so fully insisted upon in these pages (this volume, pp. 85, 86), that a more elementary method seems urgently called for. The series in question follow at once from the two theorems

$$\sin x \text{ lies between } S_n, S_{n+1}, \dots \dots \dots (1)$$

$$\cos x \text{ lies between } C_n, C_{n+1}, \dots \dots \dots (2)$$

where

$$S_n = \frac{x}{1} - \frac{x^3}{3} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1},$$

$$C_n = 1 - \frac{x^2}{2} + \dots + (-1)^n \frac{x^{2n}}{2n}.$$

Now it may be shown by elementary methods that (1) follows from (2) for all real values of x . It may also be similarly shown that if (1) be granted, then $\cos x$ lies between C_{n+1} , C_{n+2} provided $-2\pi \leq x \leq +2\pi$. Since then it is known that (2) is true when $n=0$ for all real values of x , we at once obtain an inductive proof of (1), (2) for $-2\pi \leq x \leq +2\pi$.

To show that (1) follows from (2) let there be m angles $\alpha, \beta, \gamma, \dots$, then

$$\cos \alpha \cos \beta \cos \gamma \dots = \frac{1}{2^m} \sum \cos (\pm \alpha \pm \beta \pm \gamma \pm \dots).$$

Hence, according as $\cos x \geq a_0 + \sum a_r x^{2r}$,
we have $\cos \alpha \cos \beta \cos \gamma \dots \geq a_0 + \sum a_r T_r$,

where $T_r = \frac{1}{2^m} \sum (\pm \alpha \pm \beta \pm \gamma \pm \dots)^{2r}$.

Now taking $\alpha, \beta, \gamma, \dots$ to be $\frac{x}{2}, \frac{x}{2}, \dots, \frac{x}{2m}$, we have

$$T_r = \frac{2x^{2r}}{2^{m(2r+1)}} t_r,$$

where t_r is the sum of the $2r^{\text{th}}$ powers of the odd numbers from 1 to $2^m - 1$, so that

$$t_r = \frac{1}{2r+1} \frac{2^{m(2r+1)}}{2} (1 + \epsilon),$$

where ϵ can be made as small as we please by taking m large enough. Thus we have

$$T_r = \frac{x^{2r}}{2r+1} (1 + \epsilon).$$

Hence, since $\cos \frac{x}{2} \cos \frac{x}{2^2} \dots \cos \frac{x}{2^m} = \frac{\sin x}{2^m \sin \frac{x}{2^m}}$

it follows that $\frac{\sin x}{2^m \sin \frac{x}{2^m}} \geq a_0 + \sum \frac{a_r}{2r+1} (1 + \epsilon) x^{2r}$,

according as $\cos x \geq a_0 + \sum a_r x^{2r}$.

Now taking $a_r = \frac{(-1)^r}{2r}$, $r = 0, 1, 2, \dots, n$

and making m infinite, we see that (1) follows from (2).

To prove the other result which has been stated we require the elementary algebraic inequality that if u lies between U_n , U_{n+1} and v lies between V_n , V_{n+1} , where

$$U_n = u_0 - u_1 + u_2 - \dots + (-1)^n u_n,$$

$$V_n = v_0 - v_1 + v_2 - \dots + (-1)^n v_n,$$

then uv lies between W_n , W_{n+1} , where

$$W_n = w_0 - w_1 + w_2 - \dots + (-1)^n w_n,$$

$$w_r = u_0 v_r + u_1 v_{r-1} + \dots + u_r v_0,$$

provided that, for $r > 1$, u_r , v_r are positive and $v_r > v_{r+1}$.

Using this result, it follows from (1) by taking

$$u_r = v_r = \frac{x^{2r+1}}{2^{2r+1} [2r+1]},$$

that $\sin^2 \frac{x}{2}$ lies between W_n , W_{n+1} , where

$$w_r = \frac{x^{2r+2}}{2^{2r+2}} \sum \frac{1}{[2p+1][2q+1]},$$

Σ denoting summation for $p+q=r$, $p=0, 1, 2, \dots, r$, so that

$$w_r = \frac{x^{2r+2}}{2 [2r+2]}.$$

Hence, using the formula $\cos x = 1 - 2 \sin^2 \frac{x}{2}$ it follows from (1) that $\cos x$ lies between C_{n+1} , C_{n+2} , provided $-2\pi \leq x \leq +2\pi$.

E. J. NANSON.