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LXIII. *On the Emission of Sound by a Source on the Axis of a Cylindrical Tube.* By MARY TAYLOR, M.Sc. (Manchr.)*.

THE following problem was suggested by Lord Rayleigh in his 'Theory of Sound,' Art. 301. It is here worked out; and attention is drawn to a point of some interest. It appears that the emission of energy by the source is a *discontinuous* function of the frequency. If the frequency be supposed to be gradually increased, the emission tends to become infinite as the frequency approaches any one of a series of critical values, but becomes finite again *immediately* on passing these. This is on the hypothesis of no dissipative forces. An attempt is also made to show the manner in which such forces will affect the above results.

First consider the case of an incompressible fluid. The motion is evidently symmetrical about the axis of the cylinder. Hence, using cylindrical coordinates r, θ, z , the origin being at the position of the source, and the axis of z along the axis of the cylinder, the equation of motion, which is satisfied at all points except the origin, is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad . \quad . \quad . \quad (1)$$

ϕ being the velocity potential.

Considering only that part of the cylinder which lies on the positive side of the plane $z=0$, assume that ϕ varies as e^{-kz} , the minus sign being taken to insure finiteness at infinity. The equation (1) then becomes

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} + k^2 \phi = 0. \quad . \quad . \quad . \quad (2)$$

The most general solution of (2) which is finite for $r=0$, is $\phi = A J_0(kr) \cdot e^{-kz}$, in the notation of Bessel's functions. The condition of zero normal velocity over the surface of the cylinder gives

$$J_0'(ka) = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where a is the radius of the cylinder. Thus the complete solution of (1) applicable to the present case is

$$\phi = Az + \Sigma A_s J_0(k_s r) e^{-k_s z}, \quad . \quad . \quad . \quad (4)$$

where the summation extends over all values of k_s , other than zero, which satisfy (3).

* Communicated by Prof. H. Lamb, F.R.S.

The constants A and A_s are to be determined from consideration of the flux across the section $z=0$. Over this section $\partial\phi/\partial z$ will clearly vanish except for $r=0$, but the total flux across it must be $\frac{1}{2}$, the strength of the source being taken to be unity. Hence we must have

$$\int_0^a \left(-\frac{\partial\phi}{\partial z} \right)_{z=0} 2\pi r dr = \frac{1}{2}. \quad . \quad . \quad . \quad (5)$$

Now, from (4), by the conjugate property of Bessel's functions,

$$\begin{aligned} \int_0^a \left(-\frac{\partial\phi}{\partial z} \right)_{z=0} J_0(k_s r) 2\pi r dr &= k_s A_s \int_0^a \{J_0(k_s r)\}^2 2\pi r dr \\ &= \pi a^2 k_s A_s \{J_0(k_s a)\}^2, \quad . \quad (6) \end{aligned}$$

in virtue of the equation (3)*.

But

$$\int_0^a \left(-\frac{\partial\phi}{\partial z} \right)_{z=0} J_0(k_s r) 2\pi r dr = \int_0^a \left(-\frac{\partial\phi}{\partial z} \right)_{z=0} 2\pi r dr = \frac{1}{2},$$

since $(-\partial\phi/\partial z)_{z=0}$ vanishes except for infinitesimal values of r . Hence

$$A_s = \frac{1}{2\pi a^2 k_s \{J_0(k_s a)\}^2}, \quad . \quad . \quad . \quad (7)$$

and similarly

$$A = -\frac{1}{2\pi a^2}. \quad . \quad . \quad . \quad (8)$$

Hence

$$\phi = \frac{1}{2\pi a^2} \left[-z + \sum \frac{J_0(k_s r) e^{-k_s z}}{k_s \{J_0(k_s a)\}^2} \right]. \quad . \quad . \quad . \quad (9)$$

For large values of z the first term in this solution is preponderant, and $-\partial\phi/\partial z$ assumes approximately the constant value $1/2\pi a^2$.

Next consider the case when the fluid is gaseous. The equation of motion now is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi, \quad . \quad . \quad . \quad (10)$$

where c is the velocity of sound in the fluid. If ϕ varies as

* Rayleigh, 'Theory of Sound,' § 203.

e^{ikct} , this becomes

$$\nabla^2\phi + k^2\phi = 0; \quad . \quad . \quad . \quad . \quad . \quad (11)$$

or, with the same coordinates as before,

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + k^2\phi = 0. \quad . \quad . \quad . \quad (12)$$

Considering only that part of the cylinder which lies on the positive side of the plane $z=0$, we assume that ϕ varies as $e^{-m_s z}$. The equation (12) then becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + k_s^2\phi = 0, \quad . \quad . \quad . \quad . \quad (13)$$

where $k_s^2 = k^2 + m_s^2. \quad . \quad . \quad . \quad . \quad . \quad (14)$

The most general solution of (14) satisfying the conditions of the problem is found, in the same manner as before, to be

$$\phi = \left[\frac{1}{2\pi i k a^2} e^{-ikz} + \sum \frac{J_0(k_s r) e^{-m_s z}}{2\pi a^2 m_s \{J_0(k_s a)\}^2} \right] e^{ikct}, \quad . \quad (15)$$

where the summation extends over all values of k_s which satisfy the equation

$$J_0'(k_s a) = 0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Two cases arise in this problem according as the values of m_s are or are not all real. First suppose k to be less than the least value of k_s (other than zero) which satisfies (16); that is to say, we assume the period of the disturbing force to be greater than that of the gravest mode of free radial vibrations. The values of m_s are then all real, and the solution, written in real form, is

$$\phi = \frac{1}{2\pi a^2} \left[\frac{\sin k(ct-z)}{k} + \sum \frac{J_0(k_s r) e^{-m_s z} \cos kct}{m_s \{J_0(k_s a)\}^2} \right]. \quad . \quad (17)$$

For large values of z the disturbance reduces practically to the system of plane waves,

$$\phi = \frac{1}{2\pi k a^2} \sin k(ct-z). \quad . \quad . \quad . \quad . \quad (18)$$

The energy transmitted per unit time across unit area of the wave front is then $\rho c / 8\pi^2 a^4$, and the rate of emission of energy by the source is therefore $\frac{\rho c}{4\pi a^2}$. The rate at which energy

is emitted by a unit source in free space is $k^2 c \rho / 8\pi$; so that the presence of the cylindrical wall alters the total emission of energy in the ratio $2/k^2 a^2$.

Next suppose that the period of the forced vibrations is less than the p th free period but greater than the $(p+1)$ th, so that the first p values of m_s^2 are negative, say $m_s^2 = -\mu_s^2$. The solution (15) may then be written

$$\phi = \left[\frac{1}{2i\pi k a^2} e^{-ikz} + \sum_1^p \frac{J_0(k_s r) e^{-i\mu_s z}}{2\pi a^2 i \mu_s \{J_0(k_s a)\}^2} + \sum_{p+1}^{\infty} \frac{J_0(k_s r) e^{-m_s^2 z}}{2\pi a^2 m_s \{J_0(k_s a)\}^2} \right] e^{ikct}, \quad (19)$$

or, in real form,

$$\phi = \frac{\sin k(ct-z)}{2\pi k a^2} + \sum_1^p \frac{J_0(k_s r) \sin(kct - \mu_s z)}{2\pi a^2 \mu_s \{J_0(k_s a)\}^2} + \sum_{p+1}^{\infty} \frac{J_0(k_s r) e^{-m_s^2 z} \cos kct}{2\pi a^2 m_s \{J_0(k_s a)\}^2}; \quad (20)$$

so that the disturbance no longer reduces to a plane wave for large values of z .

Now suppose that the forced period is only very slightly less than the p th free period, so that μ_p is very small. The term into which this quantity enters will then be very great, and will tend to infinity as μ_p tends to zero. The velocity potential will of course also tend to infinity as the forced period approaches coincidence with a free period from a slightly greater value. There is, however, a remarkable difference between the emission of energy in the two cases.

The energy given out by the source is obtained by integrating $-p \partial \phi / \partial z$ over the section of the cylinder, where p denotes the pressure of the fluid. Remembering that $p = p_0 + \rho \partial \phi / \partial t$, where p_0 denotes the undisturbed pressure and ρ the density, the mean value of the energy emitted per unit time is readily found from equation (20) to be

$$\frac{\rho c}{8\pi a^2} + \sum_1^p \frac{\rho k c}{8\pi a^2 \mu_s \{J_0(k_s a)\}^2}.$$

Thus, if the forced period is only slightly less than the p th free period, so that μ_p is very small, the expression for the energy emitted becomes very great and tends to infinity as μ_p tends to zero. If, however, the forced period is slightly

greater than the p th free period, the energy expression is

$$\frac{\rho c}{8\pi a^2} + \sum_1^{p-1} \frac{\rho k c}{8\pi a^2 \mu_s \{J_0(k_s a)\}^2},$$

and is finite. Thus the emission of energy is discontinuous for values of the forced period coinciding with a free period; viz., it tends to a finite or infinite limit according as the critical period is approached from a higher or a lower value.

If we had started on the supposition that the period of the source is coincident with one of the free periods, say $k=k_p$, we should have found

$$\begin{aligned} \phi = & \frac{1}{2\pi k a^2} \sin k(ct-z) + \left\{ A - \frac{z}{2\pi a^2 \{J_0(ka)\}^2} \right\} J_0(kr) \cos kct \\ & + \sum_1^{p-1} \frac{J_0(k_s r) \sin(kct - \mu_s z)}{2\pi a^2 \mu_s \{J_0(k_s a)\}^2} + \sum_{p+1}^{\infty} \frac{J_0(k_s r) e^{-m_s z} \cos kct}{2\pi a^2 m_s \{J_0(k_s a)\}^2}, \quad (21) \end{aligned}$$

where A is arbitrary. The rate of emission of energy is therefore

$$\frac{\rho c}{8\pi a^2} + \sum_1^{p-1} \frac{\rho k_p c}{8\pi a^2 \mu_s \{J_0(k_s a)\}^2},$$

which is the same as the limit to which the emission tends as the forced period approaches coincidence with the p th free period from a slightly greater value.

The reason why the emission of energy is very great when the forced period is slightly less than the free period, but not when it is slightly greater, is found in the difference in the relationship between the phases of the pressure and velocity in the two cases. When the forced period is slightly the smaller, the very large term in the expression for the pressure is in the same phase with the corresponding term in the velocity, and is therefore in a condition favourable for doing work. But as k increases through a root of $J_0'(k_s a) = 0$, the important term in the expression for $\partial\phi/\partial t$ undergoes a change of phase of a quarter period, and thus differs in phase by this amount from the corresponding velocity. Thus it does no work on the whole.

It is obvious that in any actual case the discontinuities noticed above cannot occur; and the problem therefore needs some modification in the case of approximate isochronism. This is effected by the introduction of dissipative forces of the nature of friction, viscosity, or heat conduction, which have hitherto been neglected. They are in fact, as a rule, unimportant except in the critical cases. The simplest hypothesis

which we can make in the present problem is that the motion of the particles of the fluid is resisted by a frictional force proportional to the absolute velocity. This is of course not the true law of friction, but it will serve to indicate the kind of modification which dissipative forces will introduce.

If u, v, w are the component velocities at a point in the fluid, parallel to coordinate axes of x, y, z , the equations of motion become

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - \mu' u, \text{ \&c., \&c., } \dots \quad (22)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{\partial s}{\partial t} = 0, \dots \quad (23)$$

where s denotes the "condensation." Again, if the loss of heat by radiation and conduction be neglected, we have

$$p = p_0 + c^2 \rho s. \dots \quad (24)$$

Now suppose that u, v, w, s , and p all vary as e^{ikct} . We have, then,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + ikcs = 0, \dots \quad (25)$$

while the equations (22) become

$$(\rho ikc + \mu')u = -\frac{\partial p}{\partial x}, \text{ \&c., \&c., } \dots \quad (26)$$

or, if $\mu' = \rho\mu$,

$$\rho(ikc + \mu)u = -\frac{\partial p}{\partial x}, \text{ \&c., \&c., } \dots \quad (27)$$

Differentiating the equations (27) with respect to x, y, z in order, and adding, we get

$$\rho(ikc + \mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\nabla^2 p = -c^2 \rho \nabla^2 s, \quad (28)$$

which, combined with (25), gives

$$\rho ikcs(ikc + \mu) = c^2 \rho \nabla^2 s,$$

or

$$(\nabla^2 + h^2)s = 0,$$

provided

$$h^2 = k^2 - \frac{ik\mu}{c}. \dots \quad (29)$$

The equations (27) will be satisfied by

$$u = -\frac{\partial \phi}{\partial r}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}, \quad \dots \quad (30)$$

provided

$$\phi = \frac{c^2 s}{ikc + \mu} = -\frac{ikcs}{h^2} \dots \dots \dots (31)$$

Thus ϕ must satisfy the equation $(\nabla^2 + h^2)\phi = 0$, or, with the same coordinates as before,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + h^2 \phi = 0. \quad \dots \quad (32)$$

Assuming ϕ to vary as $e^{-m_s z}$, the typical solution of (32) is

$$\phi = A J_0(k_s r) e^{-m_s z} e^{ikct}, \quad \dots \quad (33)$$

where

$$k_s^2 = m_s^2 + h^2. \quad \dots \quad (34)$$

The condition of zero normal velocity over the surface of the cylinder gives as before

$$J_0'(k_s a) = 0. \quad \dots \quad (35)$$

Thus the solution of (32) which is appropriate to a positive progressive wave, and gives the correct value for the flux across the section $z=0$, is

$$\phi = \left[\frac{e^{-ihz}}{2i\pi ha^2} + \sum \frac{e^{-m_s z} J_0(k_s r)}{2\pi a^2 m_s \{J_0'(k_s a)\}^2} \right] e^{ikct}. \quad \dots \quad (36)$$

$$\text{Now write} \quad m_s = \alpha_s + i\beta_s, \quad ih = \alpha + i\beta, \quad \dots \quad (37)$$

whence

$$\alpha_s^2 - \beta_s^2 = k_s^2 - k^2, \quad 2\alpha_s \beta_s = k\mu/c, \quad \alpha^2 - \beta^2 = -k^2, \quad 2\alpha\beta = k\mu/c, \quad \dots \quad (38)$$

Substituting from (37) in (36) we get for the value of ϕ in real terms

$$\begin{aligned} \phi = & \frac{e^{-\alpha z} \{ \alpha \cos(kct - \beta z) + \beta \sin(kct - \beta z) \}}{2\pi a^2 (\alpha^2 + \beta^2)} \\ & + \sum \frac{e^{-\alpha_s z} \{ \alpha_s \cos(kct - \beta_s z) + \beta_s \sin(kct - \beta_s z) \} J_0(k_s r)}{2\pi a^2 (\alpha_s^2 + \beta_s^2) \{J_0'(k_s a)\}^2}. \end{aligned} \quad (39)$$

The values of α_s and α found from (38) must of course be taken with the positive sign, to insure finiteness at infinity. It is easily seen that in the case of no friction when $\mu=0$, the above result reduces to that previously obtained.

Now suppose that the period of the forced vibration coincides with one of the periods of free radial vibration in the case of no friction, say $k=k_p$. Then $\alpha_p=\beta_p=\sqrt{(k\mu/2c)}$, and the corresponding term in the velocity potential, expressed as a real quantity, is

$$\frac{e^{-\sqrt{(k\mu/2c)}z} J_0(kr) \{ \cos(kct - \sqrt{(k\mu/2c)}z) + \sin(kct - \sqrt{(k\mu/2c)}z) \}}{4\pi a^2 \sqrt{(k\mu/2c)} \{ J_0(k_s a) \}^2} \quad \dots \quad (40)$$

This is finite, even in the case of isochronism, though it tends to infinity as μ tends to zero.

To find the energy emitted by the source we must take the integral of the expression $-p\partial\phi/\partial z$ over the cross-section $z=0$. We find, from (36)

$$-\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = \frac{1}{2\pi a^2} \cos kct + \Sigma \frac{J_0(k_s r) \cos kct}{2\pi a^2 \{ J_0(k_s a) \}^2} \quad \dots \quad (41)$$

Also $p = p_0 + c^2 \rho s = p_0 + \rho(ikc + \mu)\phi$.

Hence $(p)_{z=0} = p_0 + \rho\mu \left[\frac{\alpha \cos kct + \beta \sin kct}{2\pi a^2 (\alpha^2 + \beta^2)} + \Sigma \frac{J_0(k_s r) (\alpha_s \cos kct + \beta_s \sin kct)}{2\pi a^2 (\alpha_s^2 + \beta_s^2) \{ J_0(k_s a) \}^2} \right]$
 $+ \rho kc \left[\frac{\beta \cos kct - \alpha \sin kct}{2\pi a^2 (\alpha^2 + \beta^2)} + \Sigma \frac{J_0(k_s r) (\beta_s \cos kct - \alpha_s \sin kct)}{2\pi a^2 (\alpha_s^2 + \beta_s^2) \{ J_0(k_s a) \}^2} \right].$
 $\dots \dots \dots (42)$

Hence the mean value of the energy emitted per unit time is

$$\frac{\rho\mu\alpha}{8\pi a^2 (\alpha^2 + \beta^2)} + \frac{\rho kc\beta}{8\pi a^2 (\alpha^2 + \beta^2)} + \Sigma \frac{\rho(\mu\alpha_s + kc\beta_s)}{8\pi a^2 (\alpha_s^2 + \beta_s^2) \{ J_0(k_s a) \}^2} \quad \dots \quad (43)$$

If the forced period differs only slightly from the p th free period, so that $k_p^2 - k^2$ is small, we can write

$$\alpha_p^2 - \beta_p^2 = k_p^2 - k^2 = \pm \epsilon p^2, \quad \dots \quad (44)$$

the upper or lower sign being taken according as the forced or free period is the greater. Also

$$2\alpha_p\beta_p = k\mu/c.$$

Hence

$$\begin{aligned}\alpha_p^2 + \beta_p^2 &= \left\{ \frac{k^2 \mu^2}{c^2} + \epsilon_p^4 \right\}^{\frac{1}{2}} \\ \alpha_p &= \frac{1}{\sqrt{2}} \left\{ \left(\frac{k^2 \mu^2}{c^2} + \epsilon_p^4 \right)^{\frac{1}{2}} \pm \epsilon_p^2 \right\}^{\frac{1}{2}}, \\ \beta_p &= \frac{1}{\sqrt{2}} \left\{ \left(\frac{k^2 \mu^2}{c^2} + \epsilon_p^4 \right)^{\frac{1}{2}} \mp \epsilon_p^2 \right\}^{\frac{1}{2}}, \quad \dots \quad (45)\end{aligned}$$

The part of the emission due to the p th term of the sum in (43) is

$$\frac{\rho}{\sqrt{2}} \cdot \frac{1}{8\pi a^2 \{J_0(k_p a)\}^2} \cdot \frac{\mu \left\{ \left(\frac{k^2 \mu^2}{c^2} + \epsilon_p^4 \right)^{\frac{1}{2}} \pm \epsilon_p^2 \right\}^{\frac{1}{2}} + kc \left\{ \left(\frac{k^2 \mu^2}{c^2} + \epsilon_p^4 \right)^{\frac{1}{2}} \mp \epsilon_p^2 \right\}^{\frac{1}{2}}}{\left\{ \frac{k^2 \mu^2}{c^2} + \epsilon_p^4 \right\}^{\frac{1}{2}}},$$

or, writing $\mu/kc = \nu$, $\epsilon_p/k = x$,

$$\frac{\rho}{\sqrt{2}} \cdot \frac{c}{8\pi a^2 \{J_0(k_p a)\}^2} \cdot \frac{\nu \cdot \{(\nu^2 + x^4)^{\frac{1}{2}} \pm x^2\}^{\frac{1}{2}} + \{(\nu^2 + x^4)^{\frac{1}{2}} \mp x^2\}^{\frac{1}{2}}}{\{\nu^2 + x^4\}^{\frac{1}{2}}}. \quad \dots \quad (46)$$

As the forced period tends to coincidence with a free period, ν being supposed to have a definite value, this expression tends to the limit

$$\frac{\rho}{\sqrt{2}} \cdot \frac{c}{8\pi a^2 \{J_0(ka)\}^2} \cdot (\nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}}),$$

whichever of the two periods was originally the greater. Thus the discontinuity which occurred in the case of no friction does not now present itself.

This problem is interesting as furnishing an instance of a "double limit." If in the expression (46) we make first x and then ν tend to zero, that is to say, if we make the periods tend to coincidence and then make the friction vanish, the expression tends to an infinite limit. If, however, we make ν vanish before x , the limit to which the expression tends is zero or infinite, according as the upper or lower sign is taken, that is according as the forced or free period is originally the greater.

The accompanying diagram shows the variation of the last factor in the expression (46) as a function of x^2 . In the lower of the two curves the value of ν has been taken to be 1/100, and in the higher 1/400. The left-hand side of the figure corresponds to the upper sign in (46), that is to say, to the case when the forced period is slightly greater than

the free period. The more abrupt descent on this side of the figure, of the curve for which ν has the smaller value,

shows how the discontinuity which occurs in the case of no friction is approached as the friction diminishes.

It is to be noted that the maximum value of the expression (46) does not occur when the forced period exactly coincides with the free period, but for a slightly smaller value, the difference between the two values diminishing as the friction is diminished.

LXIV. *Note on a Ruler for Drawing Curves.*

By The Earl of BERKELEY, F.R.S.*

IT is well known that by bending a steel ruler so as to cover five experimental points, a fair approximation to the true curve may be obtained, thereby affording a ready means of interpolation. Should high accuracy be required, it is often best to plot the known points on an extended scale, the steel ruler then becomes extremely difficult to manipulate: to obviate this, some two or three years ago I devised the ruler here described. Fig. 1 gives a plan of the apparatus as set up to draw a curve through the five points marked with a cross—the dimensions of the parts are to the scale given on the plan, but it may be mentioned that for the highest accuracy a curve on a larger scale is advisable.

A and B are two metal T squares cut out of sheet "duralumin." The two claws C and D (shown in plan and

* Communicated by the Author.