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“Mechanical Integrators.”

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ALL measurements are made in terms of some fixed unit. The method may consist of a simple comparison of the unit with the quantity to be measured, but when this cannot conveniently be done some indirect means must be employed. Indirect measurements may be made by measuring some physical effect the magnitude of which is known to be a function of the quantity to be measured, as for instance when the length of a rod or wire is estimated by its weight. Where, however, the unit in terms of which the measurement has to be made is what is known as a derived unit, the indirect method generally consists in measuring in terms of the simple units from which the former is derived and performing, with the results, the necessary calculation. An example of this latter method is given in obtaining the contents of an area by taking its length and breadth and multiplying them together, instead of adopting the tedious process of ascertaining, by direct comparison, how many times the unit of area would be contained within it.

Now such calculations, even when of so simple a kind as mere multiplication, often become very inconvenient, and a large number of instruments have been designed for performing them by mechanical means. Such instruments may be divided into two classes: one in which the final result of conditions which vary in an arbitrary manner is found, such as the contents of a surface or the work of a motor, requiring a process of multiplication or addition; the other in which the relation or ratio at any instant of two such quantities is given, such as space and time in the case of velocity, requiring at each instant a process of division. The object of the present Paper is to deal with the theory, design, and

practical applications to engineering problems of the former class alone. It may be briefly stated that very little has yet been practically done in the use of the latter. Quite recently Professor A. W. Harlacher, of Prague, has published an account of the instruments and methods of Harlacher, Henneberg, and Smreker,¹ for gauging the velocity of a river current, the principle of which is the same as that independently adopted by the Author² and others in this country.³

The conditions or data above referred to, from which the required result has to be calculated by instrumental means, are obtained in two ways:—

- (1) By intermittent or separate observations and measurements.
- (2) By the continuous motion of a machine in connection with self-recording apparatus.

The former is the case in measuring an area of country, taking dimensions of a river or embankment section, or obtaining the forces exerted at different times by a machine or body in motion. The latter is generally given in the form of a graphic record, an important example of which is the diagram of energy or work taken from a prime mover. In both cases the result, whatever it be, whether boundary, area, volume, work, &c., can be found by calculation, but only with an approximation to the truth depending upon the extent of the calculation. The reason of this is that the data of calculation, which are taken directly in the first case or selected from the graphic record in the second, only represent actual conditions more or less closely, according as the number of data so taken is greater or less, and the greater the number the greater is the labour of determining the result. The instruments discussed in this Paper perform such work mechanically,⁴ with the great advantages of rapidity of operation, accuracy of results, and without requiring mental effort on the part of the manipulator; and all this, moreover, to a great extent independently of the complexity of the calculation required. All the results, the measurement of which will be considered, can be represented graphically. If the observations have been made separately they can be plotted whether in the form of a diagram of energy, or on the plan or elevation of an area or section, and the boundary can be

¹ Technische Blätter, vol. xvi., part i., 1884.

² Proceedings of the Royal Society, p. 190, No. 232, 1884.

³ Proceedings of the Physical Society of London, vol. vi., p. 147.

⁴ These instruments do not include such appliances as the computer, used in Ordnance survey work, and elsewhere, which is really a slide-rule to simplify the calculations required in the approximate method.

filled in with a tolerably close approximation to accuracy. In the other case the graphic record is, or may be, directly given. The subject, as far as the theory of the calculation goes, can therefore be studied with reference to such diagrams without the necessity of considering in the first case how they were obtained, and it will be convenient to do this, and afterwards to examine separately various examples of their application. Such diagrams may be drawn upon any kind of surface, and an instrument for dealing with measurements upon that of a sphere will be hereafter described. A plane surface may, however, be employed upon which to represent all cases of any practical importance, and the question thus arises—What are the measurements of the nature under consideration which are required, and which can be obtained from either a regular or an irregular plane of figure?

Such measurements are of three kinds:

- (1) The length of its perimeter or boundary.
- (2) The area of its superficial contents.
- (3) Its relation to some point line or other figure on the surface, *e.g.*, its moment of area or moment of inertia about a given line.

All these three kinds of quantities can be ascertained by successive operations of addition. The first requires the addition of elements of length, the second may be obtained by adding up successive elements in the form of strips of area, and the third by adding products obtained by multiplying such strips by some quantity, the magnitude of which depends upon the position of the other point, line, or figure in question. Taking the general case of an irregular figure, it is evident that absolute accuracy can only be obtained when this operation becomes that of integration or summing-up of an infinite series of indefinitely small quantities. Instruments for performing this operation are therefore called “mechanical integrators.” In all such instruments the rolling action of two surfaces in frictional contact is employed, for this, as will be hereafter seen, enables the conditions of motion to be continuously varied in a way which could not be effected by mere trains of wheel-work, such as form the mechanism of some kinds of calculating machines. This fact necessitates something more than a mere discussion of the mathematical principles upon which the calculations are performed, for though the action of an integrator may be absolutely correct as far as its theory of the performance of the calculation is concerned, yet there is always some instrumental error depending upon the rolling, and also, as will be seen, of the slipping of the two surfaces in frictional

contact. This error may be exceedingly small, but it is a matter of great importance to ascertain its exact amount, and the subject will therefore be investigated at length under the heading "Limits of accuracy of Integrators," where an account will be also given of the experimental results of Professor Lorber of Leoben, Dr. William Tinter of Vienna, and Dr. A. Amsler of Schaffhausen. In this investigation it will be shown that when integrators are examined upon the mechanical principles of action, they are all found to belong to one of two classes.

- (1) In which the surfaces in question slip over each other.
- (2) In which only pure rolling motion of the surfaces is assumed to take place.

The significance of this mode of classification is that it not only leads to a clear understanding of the nature of the results to be expected from any particular instrument, and teaches the best method of manipulating it, with regard to its position relatively to the figure to be measured, but it also brings out prominently the mechanical principle upon which the inventor has relied, sometimes as it would appear unconsciously, for the accuracy of the results expected to be obtained.

It may be here remarked that the same principle, by which an integrator is employed to determine a result from an autographic record, may be applied directly to obtain a continuous result from the machine, or body in motion, such for instance as an ordinary dynamometer or dynamo-electric machine, from which the autographic record was obtained. Thus, after discussing the action of integrators for dealing with diagrams, it will only be necessary to consider the mechanical details of the instruments for direct application to a machine, and this will be done under the head of "Continuous Integrators."

Coming now to the consideration of the actual measurement of the three kinds of quantities, it will be found that the first is very simple, and in fact the only reason why it is not convenient to measure it, by comparing it with the unit of length in the ordinary way, is its continuous change of direction.

The only mechanical method of rectifying a curve, as it is called, that is obtaining its length as a right line, is by rolling a wheel along it. This wheel is connected with a suitable train of wheels for recording the total number of revolutions, and as the rolling circle of the first wheel is either a unit in length or contains a known relation to this unit, the length of the curve or boundary is given at once by the reading of the graduated wheels. The use of such instruments is very ancient, and Beck-

mann in his "History of Inventions" describes, amongst various odometers, one mentioned in the Tenth Book of Vitruvius. Such an instrument is made upon a small scale for use by a draughtsman, and in one form it is sometimes termed an "opisometer," in another form the chartometer or Wealemfefna; it is also employed upon a large scale as a road or route measurer. The same principle has been employed in one of the latest forms of anemometer,¹ in which the plane of the wheel is always kept coincident with the direction of the wind, while its edge rolls in contact with the recording surface and measures the total travel of the wind. In this class falls the device suggested in a letter to "Nature,"² by Mr. V. Ventosa of Madrid, for continuously obtaining the N., E., S. and W. components of the wind, a device which was independently arrived at by the Author in connection with certain mathematical principles referred to hereafter.

The object of employing a rolling wheel is merely to enable the direction to be changed so as always to coincide with the curve to be measured, and the principle is therefore that of direct unit measurement or comparison. The foregoing instrument, which evidently belongs to the second class in which only rolling motion without slipping is supposed to take place, forms, however, one kind of mechanical integrator.

The measurement of the other two kinds of quantities is, as in the case of the first, a problem of addition; but instead of being the addition of an infinite number of infinitely short lines, an infinite number of continually changing magnitudes has to be added. This is the same both for instruments required in the second kind of measurement or "Area Planimeters," and those required in the third kind, or "Moment Planimeters." From the fact that the changing magnitudes referred to are simpler to deal with in the case of calculating the contents of areas than in finding their moments or any mathematical results of an equivalent kind, the discussion of the theory of the two above kinds of instruments is not the same, and will therefore be dealt with under two separate headings.

AREA PLANIMETERS.

The area of any plane figure, such as A B D E (Fig. 1), can be obtained in the following manner. Take any straight lines,

¹ "On a new Integrating Anemometer." By the Rev. J. M. Wilson, M.A., and H. S. Hele Shaw. Report of the British Association for the Advancement of Science, 1881, p. 543.

² Vol. xxv., p. 79.

O X and O Y, at right angles to each other, and parallel to O Y; draw a series of straight lines equidistant from each other, dividing the figure into a number of strips or elements of area. A series of rectangles may then be found, as shown at A B, the area of which is equal to that of the corresponding strip, so that by adding the rectangles together the area of the whole figure is obtained, as adopted in the common method of finding the area of an indicator-diagram. The greater the number of strips the more closely will the height of the two sides of each tend to

FIG. 1.

become equal to each other, and to the height of the corresponding rectangle.

Let Δx = the width of any element of area such as A B, at a distance x from O.

y = height of the corresponding rectangle.

Then $y \Delta x$ = area of the element A B,

and the sum of all such elements of area is the area of the figure A B D E. When the number of elements is increased indefinitely, this expression becomes

$$\int_a^b y \, dx = \text{area of the figure A B D E,}$$

a and b being the extreme values of x , i.e., the limits of integration.

It is evident, therefore, that the area will be correctly measured by an instrument in which the recording wheel or measuring roller always turns at a rate proportional to the ordinate y of the

curve, while the body from which it derives its motion moves at a uniform rate along the axis O X.

Area planimeters have been classified according to apparently different modes by which the operation of integration is performed; but since the action of them all can be explained upon the foregoing principle of adding elements of area, and, in fact, by means of the same notation, it is not surprising that such classifications are anything but satisfactory. In fact, in one important kind of planimeters, it becomes doubtful to which class they belong, or whether they should not be placed in two or more classes. It is without doubt very convenient to distinguish different planimeters, and therefore the names which have been given them will be used; but this will not denote any difference of principle, and the classification which will be adopted is that already explained, and depends on mechanical conditions of action. In what follows, one mode of viewing the mathematical operation is adhered to throughout, and it may be stated that the object of the Author has been to make clear the principles of action of integrators rather than to obtain rigid and exhaustive demonstrations of their theory, reference to authorities on the latter point being given in the foot-notes.

PLANIMETERS IN WHICH SLIPPING OF THE MEASURING ROLLER TAKES PLACE.

From a brief account of the subject by Professor Lorber,¹ it appears that the first recorded idea of a planimeter is attributed to Hermann of Munich, who worked it out with Lämmle. This idea of Hermann's, which was published in 1814, seems to have fallen into oblivion, for in 1827 Oppikofer of Berne constructed a planimeter upon similar principles, and it was thenceforth called after his name. On the other hand, Fava gives the priority to Professor T. Gonella of Florence, who in 1828, without any knowledge of what Hermann had done,² invented and described a very similar instrument. The development of the planimeter seems to have grown out of the instrument of Oppikofer, who, in conjunction with a Swiss mechanic, Ernst, finished a planimeter which won a prize in Paris in 1836. Important improvements are due to Wettli of Zurich, who, with Starke in 1849, took out a patent in Austria for the instruments

¹ "Ueber die Genauigkeit der Planimeter," von Franz Lorber. Oesterreichischen Zeitschrift für Berg- und Hüttenwesen, vol. xxxi., Jahrg. 1883.

² Reports by the Juries, Exhibition 1851, vol. ii., p. 664.

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now called the Wettli-Starke planimeter. Later on, in England, other inventors (Sang, Moseley) worked at the subject, but all these instruments depended for their action on the same principle, which is as follows :—

Let M (Fig. 2) be the plan of a disk rolling in contact with a straight guide PQ , which is parallel to $O X$, and at a distance from it equal to the radius of the disk, so that the plan of the centre of the latter always lies in $O X$. Let m be a roller upon the surface of the disk, graduated and connected with wheel-work and an index, so that the distance turned through over the surface of the disk can be read in revolutions or parts of a revolution. The plan of the point of contact (B) of the roller with the disk is always made to coincide with that particular point on the

FIG. 2.

curve, which is in the line drawn at right angles to $O X$, through the centre C of the disk. The plane of rotation of (m) which may be called the measuring roller, is always perpendicular to the disk M , and the plan of its axis as shown in the figure, is always parallel to $O Y$, so that in following the curve it slips backwards or forwards across the surface of the disk, in a direction parallel to $O Y$. Suppose the disk to roll along $P Q$ for a distance Δx , equal to the width of the element $A B$.

Then if y_1 = distance of B from $O X$.

R = radius of disk M .

r = radius of measuring roller m .

n_1 = consequent reading of measuring roller for this travel of disk.

Then $\frac{y_1}{R} \Delta x$ = linear distance turned through by a point on the disk at a distance y_1 from the centre.

$2 \pi r n_1$ = linear distance turned through by a point on the circumference of m ; but since m rolls on M these distances are equal.

Therefore $2 \pi r n_1 = \frac{y_1}{R} \Delta x$,

or $n_1 = y_1 \Delta x \times \frac{1}{2 \pi r R}$;

but $\frac{1}{2 \pi r R}$ is a constant, which by taking r and R in suitable ratio may be made unity.

Then $n_1 = y_1 \Delta x$,

that is, the reading of the roller m measures that part of the area of the element above $O X$.

If the point of contact be made to follow round the curve continuously in one direction, then when the portion of $A B$ below $O X$ is being measured, the disk is moving in the opposite direction along $P Q$, but at the same time the roller is turning in the opposite way relatively to the disk to that which it was doing before, since the point of contact is now below C . The final result of these two opposite motions is to cause the roller to turn as at first, and so add the result given for $C A$ to what was given for $C B$. If the motion of the disk Δx for the width of $A B$ be now regarded as negative,

and $-y_2$ = distance $A C$

also n_2 = reading of roller for this element of area.

Then by similar reasoning to that already used,

$$\begin{aligned} n_2 &= (-y_2) (-\Delta x) \\ &= y_2 \Delta x, \end{aligned}$$

and $n = n_1 + n_2 = (y_1 + y_2) \Delta x = y \Delta x$
= area of element $A B$.

This reasoning holds for any possible position of the roller or of the axis $O X$, which may be altogether outside the figure, as it practically is for the integration of the portion DHE . Then it

will be found that DKH is subtracted, and $DKHE$ is added so as to give the required actual area DHE .

Inasmuch as this reasoning is independent of the actual value of the width of the element, and as the vertical motion of the roller m has no effect in theory upon the distance rolled through by it, therefore in the limit when Δx becomes infinitely small, the actual value of the series of infinitely narrow strips which compose the figure $ABDE$ is given by the final reading of the roller when the traverse of the boundary is completed.

The Wettli-Starke planimeter (Fig. 3) acts directly upon this principle, with the exception that it is the disk that is moved according to the changes in y instead of the measuring roller, and the following is a description of the best form of this instrument.

FIG. 3.

On a base-plate (P), Fig. 3, are three guides SSS, along which a frame carrying the vertical axis of the disk M can be moved to and fro. The disk, which is made of glass and covered with paper, has two motions, one rectilinear along the guides, and one of rotation about its axis. The motions are imparted to it by means of an arm (L), which passes through the roller-guides (gg) in the frame carrying the disk, and rotates the latter by means of a German silver wire (dd) passing round a cylinder w upon its axis, and attached by the two ends to the extremities of the arm. The measuring roller (m) rests upon the surface of the disk, being carried in another frame (B), which is hinged to the base-plate. The action is as follows:—The base-plate being placed in

juxtaposition to the figure to be integrated, any line parallel to the guides, *i.e.*, to the direction of rectilinear motion of the disk, may be taken as the axis OY; and line OX, drawn through the edge of the roller, perpendicular to OY, may be taken as the other axis. Then, as the pointer *p* at the extremity of the arm is made to pass round the boundary of the figure, the disk will be turned through a distance proportional to the travel along OX, while at any instant the roller (*m*) is at the same distance from the centre of the disk as the pointer is from OX.

If $y_1 = \text{C B} = \text{mean height of element}$
 $\Delta x = \text{width of element A B.}$

Then, by the reasoning already given, the reading of the roller which the pointer passes over the upper boundary of the element AB, is

$$n_1 = y_1 \Delta x,$$

and the final reading of the roller is

$$N = \text{area of the figure A B D E.}$$

Hansen in 1850 still further improved this instrument, and, in conjunction with Ausfeld, introduced a different method of reading the result, and of carrying the frame, this instrument being known as the Hansen-Ausfeld planimeter. Various other instruments of the same kind were shown in the Great Exhibition of 1851, but in all, the motion of the arm carrying the pointer was "linear"; that is, the motion, which must be possible in every direction, is obtained by compounding two rectilinear movements at right angles to each other. Such instruments are therefore called "linear planimeters."

Many different forms of linear planimeters have been suggested, but the only modification of the disk and roller which it will be worth while to notice is the cone and roller.

Let MM', Fig. 4 (p. 14), be the cone corresponding to the disk M, and rolling on the edge of its two bases in a direction parallel to OX. Let the roller *m* always be in contact with a circle on the cone, whose centre B' is at a distance CB' from the apex C of the cone, such that

$$\text{C B}' = \text{S B} = y = \text{mean ordinate of element S B.}$$

where the element AB is being at that instant integrated. Adopting the same notation as before, when the cone has rolled over the

surface through a distance Δx , then, whatever be the angle of its apex, the distance rolled through by the roller m is

$$2 \pi r n_1 = \frac{y_1}{R} \Delta x,$$

or

$$n_1 = y_1 \Delta x \times \frac{1}{2 \pi r R}.$$

As might have been anticipated, the expression is the same as was obtained in the case of the disk, the latter being a special case of the cone when the vertical angle is 180° .

Thus the cone may be employed instead of the disk, and such an

FIG. 4.

instrument was invented by Mr. John Sang, who in 1852 published a description¹ of it, according to which the action was extremely accurate, but it does not appear to have come into very extensive use.

No more instruments of the kind will be described, since they have given place to those in which the arm carrying the pointer turns about a centre or pole, and which are therefore called "polar planimeters."

In the year 1856 Professor Amsler-Laffon invented and brought before the world the now well-known polar planimeter bearing his name, and since then no less than twelve thousand four hundred

¹ Transactions of the Royal Scottish Society of Arts, vol. iv., p. 119.

of these instruments have been made and sent out from his works at Schaffhausen. According to authorities,¹ which Professor Lorber quotes, Professor Miller, of Leoben, invented independently a planimeter of this kind in the same year (1856), which being made by Starke, of Vienna, is known as the Miller-Starke planimeter. Previous to this, in 1854, Decher, of Augsburg, as well as Bouniakovsky, of St. Petersburg (1855), had improved upon previously-existing forms of polar planimeter, though it is well to note that the planimeters already mentioned as sent to the

FIG. 5.

Great Exhibition of 1851 from various parts of Europe, as Italy, Switzerland, France, and Prussia, were all linear, and no mention is made of polar planimeters in the jurors' report.

The Amsler planimeter is shown in Fig. 5. It consists of two bars, (a) the radius bar, and (b) the pole-arm, jointed at the point C. The tracing point *p*, which now coincides with the point B of the figure ABDE, is carried round the curve, and the roller *m*,

¹ "Ueber die Genauigkeit der Planimeter," by Professor Lorber, p. 8.

which partly rolls and partly slips, gives the area of the figure; and by means of the graduated dial h , and the vernier v , in connection with the roller m , the result is given correctly in four figures. The sleeve H can be placed in different positions along the pole-arm b , and fixed by a screw s , so as to give readings in different required units. A weight at W is placed upon the bar to keep the needle-point in its place, but in instruments by some other makers T is a pivot in a much larger weight, which

FIG. 6A.

FIG. 6.

rests on the paper. A recent minor improvement has been to fix a locking spring to ¹ the frame, so that the roller can be held when the planimeter is raised for the purpose of reading it.

The theory of the polar planimeter may be simply deduced from that of the disk and roller thus:—

Let Fig. 6 represent the same disposition of the disk M with regard to the figure $A B D E$, as in Fig. 2, but now let the roller m

¹ "The Engineer," vol. lviii., p. 336.

move round the edge of the disk instead of across it, its distance from O X being always the same as before, viz. :—

$$Oq = SB = y_1.$$

The turning of m for a given travel, Δx , of the disk is found thus, draw lq (Fig. 6 A) tangent to the disk at m , so that

$$lq = \Delta x,$$

and draw lk parallel to the axis of rotation of m , then qk is the distance turned through, and lk is that slipped through by the edge of the roller m , when the disk has rolled through a distance Δx ; therefore, using the same notation as before,

$$qk = 2\pi r n_1$$

$$\text{and} \quad \frac{qk}{lq} = \frac{2\pi r n_1}{\Delta x} = \sin \angle qlk;$$

$$\text{but in Fig. 6} \quad \frac{Oq}{Sq} = \frac{y_1}{R} = \sin OSq.$$

But by similar triangles $\angle qlk = \angle OSq = a$;

$$\text{therefore} \quad \frac{2\pi r n_1}{\Delta x} = \frac{y_1}{R},$$

$$\text{or} \quad n_1 = y_1 \Delta x \times \frac{1}{2\pi r R}.$$

This is the same result as previously obtained, and it has been given in this way because there is an important class of planimeters to be hereafter described, combining the polar planimeter with the disk and roller, in which a principle is employed which is thus made obvious. This principle is, that the turning of m is exactly the same as if it were in contact at the point B, no matter in what position it may be along the line through B parallel to O X. The turning of m thus measures the area of the element as long as y does not change. If, however, the value of y changes so that m changes its distance from O X, the measuring roller is likewise turned a certain additional amount from this cause; but this does not affect the correct reading of the area so long as its first and last positions are equally distant from O X. The reason is, that then the roller has turned as much in one direction in moving away from O X as it has in moving towards it, and this is the case for the initial and final positions of the pointer when the complete travel of the closed curve has been made. Now, inasmuch as the velocity of the edge

of the disk is just the velocity at which the centre has been shown to move along $O X$, the disk may be removed altogether. The roller is then moved in contact with the surface of the figure, and with identically the same amount of turning as before, provided that its plane of rotation is turned so as to make the same angle with $O X$ (which is now its direction of translation) as it did before with the direction of motion of the edge of the disk in contact with it. This is the case when it is turned through that angle, and then its axis of revolution coincides with the radius $S q$.

FIG. 7.

In order to keep the direction of the plane of rotation always at right angles, it is only necessary to have a rod or bar $S q$ capable of turning on a pin at the point S . The pin at S is attached to a sleeve, which can freely move along a guide-bar, whose direction coincides with the axis $O X$. By employing the bar $q S q^1$ itself as the axis of rotation on which m turns, the simple planimeter shown in Fig. 7 is obtained, in which the point of contact of the measuring roller is made to pass round the diagram. The turning of the roller m correctly gives the superficial contents.

The roller can be moved to any position on the rod, such as shown in dotted lines, Fig. 7, without in any way affecting its resultant turning, and the former point of contact of the roller is replaced by a pointer, which is made to follow the curve instead of the roller. Professor Burkett Webb has described to the Author a planimeter of this form used in the United States known as "Coffin's" planimeter.

If, finally, the point S, instead of moving along the straight line OX, which may be considered as a portion of a circle with its centre infinitely distant, moves along the arc SZ (Fig. 7), or any other arc, as, for instance, that with radius TC, the instrument becomes the ordinary Amsler planimeter (shown in dotted lines).¹ This explanation so far is based upon that given by Sir Frederick Bramwell, who has further shown that the change from the motion in a straight path to that in the arc of a circle has no effect upon

FIG. 8.

the ultimate reading when the complete travel round the closed curve has been made, and the arm SB has returned to its initial position. The following demonstration of the truth of this appears, however, to have an advantage, in that it follows throughout the operation of integration, especially as recent planimeters are more complicated than that of Amsler.

1st. Let point S (Fig. 8), at the extremity of the radius-rod, move along the broken line STUVWXYZ, and from these points draw arcs with a common radius = R, cutting the curve in points $t_1 t_2, u_1 u_2, v_1 v_2$, &c., s and z being tangent points at the end of the curve from the points S and Z. The proof has already been given that the integration of the complete portion $st_1 t_2$, taken separately, is given by the reading of the measuring-roller; so

¹ "On Amsler's Planimeter." By F. J. Bramwell. Report of the British Association for the Advancement of Science, 1872, p. 401.

also are given the areas of the various other portions, $t_1 u_1 u_2 t_2$, &c. If the separate portions were integrated consecutively, any arc, such as $t_1 t_2$, would be traversed in both directions by the measuring-roller, because it would move one way round in traversing one figure and the opposite way in going round the adjacent one, and the reading due to the arc would be eliminated.

Thus the whole curve may be integrated correctly at once without going round each separate portion formed as above, even if the point S at the end of AB moves upon a broken line instead of along OX. Next, substitute a continuous curve TUVXYZ (Fig. 8A) for the broken line. This curve may be supposed to consist of an infinite number of straight portions. The infinitely small portions contained between an arc, as $u_1 u_2$, and another very close to it, drawn from the beginning and end of these

FIG. 8A.

straight portions, may, just as in the case of the broken line, be supposed to be integrated separately and with a correct result, which is independent even of a possible crossing of the arcs, as $v_1 v_2$ and $u_1 u_2$. In the same way as before, it may be seen that the arcs $u_1 u_2$, &c., need not be traversed, and, so long as the point S returns to its initial position, the area of the figure is given by the simple traverse of its boundary, whatever be the curve on which the point S moves, which, in the case of the Amsler planimeter, is a circular arc.

Various writers¹ have explained the action of the simple polar planimeter in ways more or less different. One of these ways, recently given by Mr. F. Brooks, of Lowell, U.S., may be alluded

¹ Some of these in English are: "Integral Calculus," § 146, p. 203. Williamson. "Mathematical Instruments," vol. i., p. 80. By J. F. Heather. "Mechanics of Engineering," vol. ii., p. 69. By Dr. P. J. Weisbach. Report of the British Association for the Advancement of Science, 1880, p. 371. Journal of the Association of Engineering Societies, vol. iii., p. 294. Brooks.

to. He shows that the area may be treated as the difference between the area swept out by the line Tp (Fig. 9) and the sector of the zero circle, or circle upon whose circumference the pointer (p) being moved, the measuring roller is not in consequence turned. This is true, both for the outside or inside, if proper signs be taken. Let the element of area pq be passed over, the curve at p being for a small distance considered concentric

FIG. 9.

with the zero circle, this small area subtending an angle w at the centre; then if values be taken as shown upon Fig. 9, in which

$$CT = \text{radius-bar} = a;$$

Cp = one portion of pole-arm = b ;

C_m = the other portion of pole-arm = c ;

$$\begin{aligned}\text{Area } p q &= \frac{1}{2} w (a^2 + b^2 + 2 a b \cos \theta) - \frac{1}{2} w (a^2 + b^2 + 2 c b) \\ &= w b (a \cos \theta - c),\end{aligned}$$

and by a geometrical construction the travel of the measuring roller is easily shown to be equal to the same expression. Mr. Brooks also explains why the area of the zero circle must be added to the reading if the figure to be integrated contains the centre T . The following appears, however, to be a still simpler explanation. Referring to Fig. 7, it is evident that going round the outside of the zero circle corresponds to a movement taken continuously above the zero line OX when only the portion above OX is measured. In this latter case the curve could never be completely traversed as long as the pointer moves in one direction. Suppose, however, that the ends of OX are bent round and brought within the figure, then the motion in one direction will enable the complete circuit to be made; but only the portion outside the line, *i.e.*, corresponding to that originally above OX , will be measured by the roller, and that within must consequently be added to the recorded result. This quantity is evidently the area of the zero circle in the

FIG. 10.

case of the Amsler planimeter, which must therefore always be added to the result when the centre of the radius-arm is within the diagram to be measured.

As the Amsler planimeter alone, so far as the Author is aware, has been modified to measure the area of any non-developable surface, this modification may be here noticed. The only surface of the kind to which it has been adapted is a spherical one. Fig. 10 shows the instrument, and from that figure it will be seen that the chief alteration is the placing of two joints j, j' , one upon the radius bar (a), and the other upon the pole arm b , so as to allow the employment of the integrator for surfaces of varying curvature. The joints are equidistant from the end of each bar, and exactly opposite to each other—the radius-bar and pole-arm being now of equal length, and a pin f is placed on (a), which fits into a corresponding recess in (b), so that when the two arms are closed, they can be together bent at the joints to the required amount, and the joints being purposely made stiff, they will remain

at the proper angle when the instrument is used. The joint (*j*) acts so that the tracing point (*p*) is always in the plane of the axis of rotation of the measuring roller. The theory of the action of this instrument has been fully explained by Professor Amsler,¹ in an article in which the theory of the relations between measurement upon a spherical surface and upon a plane surface is discussed.

The various applications of the simple planimeter for finding areas are well known, and need not be explained; but there are some slight modifications of the instrument for special purposes, and one of these recently applied by Professor Amsler to his planimeter is worth noticing. This is illustrated at Fig. 11, and is a device for reading at once the mean pressure given from an indicator diagram. Two points, U and V, are seen, one (U) being upon the upper side of the bar A, which slides in the tube H, and one (V) upon the tube H itself. These points can be adjusted to

Fig. 11.

the length of the diagram by inverting the instrument in the way shown in the figure, and the sliding-bar is then clamped by the screw S. This setting causes the reading of the instrument, when the diagram is traversed by the pointer in the usual way, to give at once the mean height of the diagram in fortieths of an inch. The simple relation is as follows:—

$$\frac{\text{Reading of measuring-roller}}{40} = \frac{\text{Mean height of diagram}}{\text{in inches.}}$$

$$\text{Mean pressure} = \text{Mean height} \times \text{vertical scale of diagram.}$$

As an instance of the great saving of labour by the use of the Amsler planimeter, the Author happens to know a civil engineer's

¹ "Zeitschrift für Instrumentenkunde," 1884.

office where a large amount of earthwork quantities had to be taken out. The calculations proceeded slowly and with many repetitions, until one of the draughtsmen procured a planimeter, and then the other; with the result of a great expedition of the work, and almost the complete absence of errors—and even then only in decimal places—where previously the divergence had been as much as by units.

Although the connection between the disk and roller or linear planimeter and the polar planimeter has been shown, it is possible to regard them as acting upon different principles. The former may be considered as measuring the variation in the ordinate (y) by a change of effective radius of the circle on which the measuring-roller works, the latter measuring the same thing by a corresponding change in the sine of the angle which its plane of rotation makes, with its direction of motion over the surface on which it rolls. They have, in fact, been classified in this way as radius machines, and sine or cosine machines, for the slipping, although occurring in both, appears in the ordinary way of viewing the subject to affect the result in different ways.¹ In the former, slipping is supposed to be entirely due to the variation in the value of (y), and only takes place when the ordinate changes in value; in the latter the change is supposed to be effected by turning the pole-arm about its centre, without any slipping at all. This distinction is however quite an imaginary one, for it will be seen that if the curve be obliquely inclined in either case to the axis OX , the action of the measuring roller is precisely the same. Recently, a large number of what are called "Precision Polar Planimeters" have been designed and constructed, which combine in an obvious manner the above two principles of action, the disk giving motion to the roller, while the pole-arm carries it across the disk in an oblique direction. Thus the advantages of a uniform and invariable surface of contact for the roller, and the convenience of the polar planimeter are combined, with the still more important advantage of a large relative turning of the measuring roller. Before describing a few different forms of the best of these instruments, the general theory upon which they work will be given; it will then not be difficult to understand the action of the several instruments without repeating the explanation in each case. It will be found that both the linear and polar planimeter are only special cases of application of the general principle upon which the correctness of action of precision planimeters depends.

¹ "Philosophical Magazine," Feb. 1882, p. 77.

It will be well to approach the matter from the same point of view as in explaining the linear planimeter. Let the disk *M*, Fig. 12, rotate about an axis *C* as it rolls along the line *PQ*, parallel to *OX*, the pivot on the axle at *C* being attached to a frame which also carries another pivot *S*. This latter pivot always lies upon *OX*, and about it rotates a pole-arm *b*, carrying a pointer *p* at one end, and the measuring roller *m* at the other end. The plane of rotation of the measuring-roller coincides with the direction of the pole-arm, and is carried over the disk in contact with it, along the arc *mm'*. Then from what was proved, p. 17, the motion of the roller *m* is exactly the same as if it were moved, so that its axis always coincides with *Cj*, the perpendicular upon the pole-arm from the centre of the disk—provided only that its axis is always

FIG. 12.

parallel to this line. Thus, adopting the previous notation, and taking

$S p$ = length of upper portion of pole-arm = R .

Then when the disk rolled through a distance Δx ,

n_1 = reading of roller m

y_1 = ordinate $S B$.

$$\frac{\text{turning of roller}}{\text{distance turned by edge}} = \frac{2 \pi r n_1}{\Delta x} = \frac{C j}{C S} = \sin \alpha;$$

but

$$\frac{S' B}{S B} = \frac{y}{R} = \sin \alpha,$$

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therefore $\frac{2 \pi r n_1}{\Delta x} = \frac{y}{R},$

or $n_1 = y \Delta x \times \frac{1}{2 \pi r R},$

which is the same result as in the case of both the linear and polar planimeters. In practice, the portion of the pole-arm which

FIG. 13.

carries the pointer is usually perpendicular to the other portion, as shown by the dotted lines, Fig. 12. In this case, the direction of motion of the disk and frame carrying the centre of the pole-arm S must be taken parallel to the guide P' Q', that is, perpendicular to the former direction. It has already been shown in the case of the Amsler planimeter, that it does not matter in what path the centre S of the pole-arm is carried, so long as the foregoing con-

ditions are observed, and thus there are several forms of precision polar planimeters in which the point S is carried in the arc of a circle instead of along a straight line. It may now be made clear, from Fig. 13, that the two first kinds of planimeters are special cases of the last.

(A) Fig. 13. Let R be the radius of the disk, R_1 the radius about which the roller m is carried. Then the area of the diagram as already explained can be measured by either pole-arm Sp or Sp' .

(B) Fig. 13. Let the radius R_1 of the pole-arm become infinitely great, while R remains finite; thence m moves across the

FIG. 14.

disk M in a straight line usually, but not necessarily, through the centre, and the linear planimeter is the result.

(C) Fig. 13. Let the radius of the disk become infinitely great, and any motion of such an imaginary disk under these conditions would make the result equivalent to moving the roller over the surface of the paper. Therefore the disk may be removed, and the elementary form of polar planimeter is obtained, the roller being placed in either position as shown at m or m' , without affecting the result.

The following is a simple explanation of the action of the precision polar planimeter.¹

¹ This proof is a modified form of one given by Czuber. "Technische Blätter," vol. xvi., 1884, p. 58.

Let M (Fig. 14) be the disk, which can be turned by any suitable means through a distance corresponding to the linear travel of its centre about C.

Let r_0 = radius of zero circle (E' E S').

r = radius of any circle FF'.

α = $\angle$ turned through by pole-arm, when the pointer moves from the zero circle to the circle FF'.

ψ = $\angle$ turned through by radius arm, C S, when an element EE' F' F is being described.

a = radius arm = C S'.

b = pole-arm = F S'.

d = S S'.

Then from the figure—

$$\begin{aligned}\text{and} \quad r_0^2 &= a^2 + b^2 \\ r^2 &= a^2 + b^2 - 2 a b \cos (90 + \alpha) \\ &= a^2 + b^2 + 2 a b \sin \alpha.\end{aligned}$$

$$\text{Therefore} \quad r^2 = r_0^2 + 2 a b \sin \alpha,$$

$$\text{or} \quad \sin \alpha = \frac{r^2 - r_0^2}{2 a b}$$

Now the turning of the plate is proportional to ψ , and may, for the arc FF', be taken as equal to $r_0 \psi c$.

$$y = S K = S S' \sin \angle S S' K = d \sin \alpha,$$

where c and d are constant quantities; also if ρ = the radius of the roller,

$$\text{then} \quad \frac{y}{R} = \frac{\text{linear distance by edge of roller}}{\text{distance travelled by edge of disk}} = \frac{2 \pi \rho n_1}{r_0 \psi c};$$

$$\therefore y = \frac{2 \pi R \rho n_1}{r_0 \psi c}.$$

$$\text{Therefore} \quad \frac{2 \pi R \rho n_1}{r_0 \psi c} = d \sin \alpha = \frac{d \times (r^2 - r_0^2)}{2 a b}$$

$$\text{or} \quad n_1 = \frac{r^2 - r_0^2}{2} \psi \left(\frac{r_0 c d}{2 \pi R \rho a b} \right);$$

but $\frac{r_0 c d}{2 \pi R \rho a b}$ is a constant quantity, and may be made equal to unity.

$$\text{Therefore} \quad n_1 = \frac{r^2 - r_0^2}{2} \psi = \text{area of element EE' F' F'}.$$

Thus n_1 is a measure of an element of area, and as the motion of m due to turning of the pole-arm in moving to a larger or smaller circle does not affect the reading when the pointer at F has passed round a closed curve, the final reading of the roller gives the area of any figure.

The actual construction of the precision polar planimeter appears to have been first carried out by Mr. Hohmann, Bauamt-mann of Bamberg, in 1882,¹ in conjunction with the well-known mechanician Mr. Coradi, of Zurich. A plan of the first instrument

FIG. 15.

FIG. 16.

is given in Fig. 16; but it will be more easily understood by reference to the diagram, Fig. 15, which shows a frame (a) pivoted at one end (c), to a weight G , about which it turns. This frame carries a small disk (w), which rolls in contact with the surface of the diagram, and gives motion to the disk M . The roller m is moved across the disk in the horizontal direction by a pole-arm centred on S as axis. Referring to Fig. 16, which is lettered in a

¹ "Hohmann's Präcisions Planimeter, Beschreibung, &c." Dr. W. Tinter. Vienna, 1882.

similar way to Fig. 15, it will be seen how the pole-arm, in turning about the centre S , effects this motion. A plan and elevation of the frame F , which carries the roller m , is shown on a larger scale, and this frame is moved backwards or forwards through a slot in the supporting frame. The roller m has the arrangement of the screw and worm for obtaining the readings of the dial h , as in the Amsler planimeter, and also the vernier in conjunction with the measuring roller. Two rollers $j j'$ serve to balance the instrument. The details of the arrangement by which the length of the pole-arm b is adjusted are also shown on a larger scale.

In this instrument, the fact that the disk is inclined at an angle makes no difference in the theory of its action, and as the roller W obviously drives the disk so that the angular motion corresponds with the angular motion of the radius bar a , the explanation already given makes its mode of operation clear. The case is rather simplified by the fact that the roller m is moved radially across the disk.¹

An instrument of similar kind has been designed and recently described by Professor Amsler-Laffon.² This is shown in Figs. 17, 18, and 19, where it will be seen that this disk M which is now horizontal, is turned by means of bevel-wheels $z_1 z_2$, the back of one of which forms a portion of a frustum of a cone rolling about the centre C of the radius-arm $a a$. The centre is itself a sphere, which allows any side motion of the instrument, due to the inequality of the surface, to take place without affecting the accuracy of the result. The necessary pressure of the roller upon the disk is obtained by allowing the weight of the portion of the frame b , which carries the roller m , to rest upon the disk by being pivoted by the centres $K K$ (Fig. 18). A peculiar feature of the instrument is that the pole-arm frame can be centred either within or without the frame. If placed in the former position, the reading is twice as great as in the latter, the positions of the centres being purposely adjusted to effect this. The frame can be taken off one centre s' (Fig. 17) by unscrewing a set screw at x , and at once placed upon the other. The weight w can be adjusted in any position by means of the nut and screw t (Fig. 17), and so the pressure of the pointer p upon the surface of the diagram may be regulated.

In both the above instruments the disks derive their motion from

¹ A complete analysis of the action of this instrument is given by Dr. W. Tinter, *Zeitschrift des Oesterreichischen Ingenieur- und Architekten-Vereins*, vol. xxxiv., p. 90, 1882.

² "Planimeter Constructionen," Jan. 1884, p. 22.

a roller in contact with the surface of the diagram, but in the two next instruments to be described, Messrs. Hohmann and Coradi have caused the disk to be turned in a manner which prevents any such error as from the possible slipping of the above roller. The first instrument of the kind is shown (Figs. 20, 21) in plan and elevation. The disk *M* is carried by a frame (*a a*) as before, but the frame now swings about a circular stand, the edge of which is toothed, so that the pinion (*i*), which is upon the axis

FIG. 17.

FIG. 18.

FIG. 19.

of the disk, is turned, and therefore the disk itself, with the same angular velocity. The weight of the frame and disk is, to a great extent, taken off by means of the light rod (*l*) which swings about a central pillar *P*. A side view of the pole-arm is shown, and the mode of adjusting it and supporting the portion which carries the roller (*m*), so that by means of centres *K K* the weight of that portion of the frame is allowed to rest upon the disk.

It is evident that this instrument works upon identically the same principles as the foregoing ones. This "Freely-swinging" precision-planimeter was followed by another, called the "Plate" planimeter (Fig. 22),¹ which is of still simpler construction. In this form advantage is taken of the fact that the measuring roller need not have its path through the centre of the disk, and a support (*v*) is obtained above the disk, so that its pivot (*q*) can work between centres, the weight of the frame being supported by rollers (*j*). The portion of the pole-arm which carries the roller

FIGS. 20 and 21.

(*m*) is (as in the last case) pivoted between the centres K K. The dial for higher readings is as in the case of the previous instruments denoted by *h*.

The last and most recent modification is the "Rolling planimeter" of Coradi. This approaches nearest to the diagram (Fig. 12), which completely explains its action. Here the centre of the

¹ A full account of the theory and action of these are given in "Das freischwebende Präzisions Planimeter und dessen Modificationen." Friedrich Hohmann, Bamberg.

radius-arm is removed to an infinite distance, and the centre of the disk and that of the pole-arm are carried along straight lines

FIG. 22.

FIG. 23.

parallel to the axis O Y in that figure. The way in which this is effected is seen from Figs. 23 and 24, which show Coradi's rolling planimeter in plan and elevation. Two rollers ($c\ c'$) are in contact

with the surface of the diagram, and on their axis is a bevel wheel (z_1) (Fig. 24), which gears with another bevel-wheel (z_2), which is upon the axis of the disk. Thus the wheels z_1 z_2 are turned as the frame is rolled along, and consequently the disk itself. The axis of the rollers c c works upon the centres e e , which are set screws in the frame (a a). The disk M is also carried between centres (q q), as in the instrument last described, and also, as in that case, the path of the roller does not pass through the centre of the disk. This instrument, which has many advantages,

FIG. 24.

FIGS. 25, 26, and 27.

and, notwithstanding that it rolls on the diagram surface, gives results of great accuracy, has been examined with great care by Professor Lorber, who has given a lengthy description of it and a full account of its theory.¹

The last planimeter of this kind to be examined is one by Professor Amsler.² This instrument, shown in Figs. 25, 26, 27, differs

¹ "Ueber das Roll planimeter von Coradi, von Franz Lorber." Zeitschrift des Oesterreichischen Ingenieur- und Architekten-Vereins, vol. xxxvi, p. 135, from whence the figures (23, 24) are taken.

² Zeitschrift für Instrumentenkunde, Jan. 1884.

from the last in that the tooth-wheel z_2 works in gear with a rack $z_1 z_1$, which is cut upon a fixed frame D D. Thus, although it is supported by the rollers cc , there is no possibility of slipping as far as the turning of the disk is concerned. The rollers run in a groove cut in the frame D D, and the action of the instrument is easy and smooth. The theory of its action is identical with that of the foregoing one, as explained by means of the diagram Fig. 12. The various parts are lettered in the figures to correspond with the explanations of that instrument previously given.

In the instruments hitherto described the surfaces of revolution are limited to the disk and cone, but various other surfaces may be made to replace these. The only one that has been so employed is that of the sphere; and in the present class of instruments, in which slipping takes place, the following property of the sphere

FIG. 28.

is made use of: Let a sphere M (Fig. 28), which replaces the disk (Fig. 2), roll along the axis $O X$. Then suppose the roller m can by suitable means be moved round the surface so that its plane of rotation shall always contain the centre of the sphere and be perpendicular to the arm $C B$, which corresponds to the pole-arm of the former instruments; it is evident that if the perpendicular be drawn from q , the point of contact of m with the sphere, to $C Z$ the axis of rotation of the sphere, meeting it in the point u , the line qu is the radius of the rolling circle of contact of the measuring roller.

Therefore $\frac{\text{motion of measuring roller}}{\text{motion of sphere along O X}} = \frac{q u}{C V} = \frac{q u}{q C} = \sin a.$

But from the figure $\frac{SB}{CB} = \frac{y}{R} = \sin a$.

Therefore, adopting the same notation as hitherto used,

$$\frac{\text{motion of } m}{\text{motion of } M} = \frac{2 \pi r n_1}{\Delta x} = \frac{y}{R}$$

$$n_1 = y \Delta x \times \frac{1}{2 \pi r R},$$

which proves that the area of the curve may be measured by any device, on the principle of Fig. 28. It may be shown in the same

FIG. 29.

way as on p. 18, that the result is similar if the sphere rolls upon the arc of a circle, about any centre as T, instead of along the straight line O X. Planimeters of this kind have been constructed by Mr. Hohmann and Professor Amsler. In both cases only portions of the whole surface of a sphere have been employed, and the motion is given by means of an axis, instead of by rolling the

spherical surface upon the diagram. In Mr. Hohmann's planimeter, shown in plan and elevation, Fig. 29, the concave surface M is used. Rotation is given to this by means of an axis gg , an enlarged portion of which (r) rolls upon a circular metal path R . The pole-arm P turns about a centre c , and so causes the rolling circle of the measuring roller m to vary according to the foregoing principles. This instrument has not come into use. Professor Amsler has employed the convex surface in an instrument somewhat similar to the one described, except that better provision is made for obtaining the required pressure between the surface of the roller and sphere and for giving rotation from the roller-path.¹

PLANIMETERS IN WHICH ONLY PURE ROLLING MOTION IS
ASSUMED TO TAKE PLACE.

There have been many efforts to design instruments in which no slipping shall take place. These efforts have resulted in the production of various instruments which, though they differ in external form and mechanical action, yet rely upon the same mathematical principle of action as the planimeters already dealt with, the particular form of disk and roller or sphere and roller being taken. Thus, in every case there is a measuring-roller or its equivalent, the rate of motion of which has to be varied by some means or other. It is in the method by which this is done that this class of planimeters differs from the other. Instead of obtaining the variation of the measuring-roller in bringing it into contact with circles of different linear velocity by sliding it over the surface of the disk or sphere, one or other of the two following principles are employed. A device equivalent either to (1) bringing in succession a series of measuring rollers into contact with the different imaginary circles; or (2) bringing circles of different linear velocity into contact with a single fixed measuring roller.

The disk-globe and cylinder-integrator of Professor James Thomson belongs to the former class. In this a sphere G (Fig. 30) rolls over the surface of the disk M , but is also in contact with a cylinder $m m'$. The motion of G in direction $O Y$ is that in which the roller would slip in the ordinary disk and roller, and does not affect the motion of rotation of $m m'$. On the other hand the motion in direction $O X$, which is due to the turning of the disk, is entirely imparted to $m m'$. Thus as G rolls along $m m'$, the same effect is in theory produced as if a series of rollers m, m_1, m_2 , &c., upon

¹ Zeitschrift für Instrumentenkunde, heft i., Jan. 1884, p. 19.

the same axis as the cylinder, were successively applied to the surface of the disk, and all slipping, at any rate from this cause, is avoided. The actual mechanism which has not been employed for a planimeter takes a slightly different external form in the harmonic analyser of Sir W. Thomson's tide-calculating machine.¹

FIG. 30.

FIG. 31.

The devices which have now to be considered as solutions of the problem under consideration by the first method, are used in connection with the geometrical property of the sphere already discussed, and upon one similar to it.

Let M (Fig. 31) be the plan of a sphere rolling along the line

¹ Minutes of Proceedings Inst. C.E. vol. lxx. p. 2.

O X, carrying with it, by a frame not shown, a cylinder (m) which can roll about it so as to come into contact at any point q upon its horizontal great circle. Then the rotation of the cylinder may be employed exactly in the same way as the rotation of the roller on the integrator described on page 35, and shown by Fig. 28; but in the present case, instead of causing the roller to slip over the surface, the rolling of the cylinder is practically equivalent to bringing in succession a series of rollers $m, m_1, m_2, \&c.$, upon one axis in contact with it.

This principle has been employed both by Professor Mitchelson

FIG. 32.

FIG. 32A.

of Cleveland, U.S., and Professor Amsler. The mechanism of Professor Mitchelson's instrument is shown in Fig. 32. In this a flexible steel band or chain F passing round a semi-circular arc D forces the cylinder C to roll on the sphere G . The cylinder is carried by a frame E , which slides along the bar A by which it is supported. The mode in which it is proposed to apply it to the ordinary Amsler planimeter is shown on a smaller scale Fig. 32A, where b is the pole-arm, a the radius bar, and t the centre of rotation of the latter.

Professor Amsler's planimeter on this principle is similar to the

foregoing, except that instead of being carried by two guides as sleeves by a bar, the cylinder-frame is supported on rollers from a frame above, the rolling friction on the latter being less than that of the cylinder on the sphere. Thus the cylinder always moves to its required position. The motion of the spherical surface is obtained from a bevel-wheel upon its axis, which gears with a larger one formed upon the edge of a circular stand or support of the instrument.¹

A similar principle of the geometry of the sphere has also been employed in an instrument suggested in a Paper in 1855,² by the late Professor Clerk Maxwell, when an undergraduate at Cam-

FIG. 33.

bridge. Instead of the cylinder in Fig. 31, let a sphere m' roll round on the sphere (M), as shown in Fig. 33. Then from the property of the sphere, which is proved at length in the above Paper, the turning of the sphere m' about its axis of rotation x , relatively to the turning of M along $O X$, is proportional to the tangent of the angle α in the figure. In the other case it will be remembered that the turning of the cylinder or disk was proportional to the sine of the same angle. By suitable means the principle can be employed in the construction of a planimeter. Two forms of such planimeter are shown in the Paper, and though they

¹ Zeitschrift für Instrumentenkunde, Jan. 1884.

² Transactions of the Scottish Society of Arts, 1855.

are both in the form of the linear planimeter, and are scarcely suitable for practical application, yet the matter is dealt with in a way worthy of the inventor. It is evident that this is another case of bringing the equivalent of a series of rollers m, m_1, m_2 , into contact with the sphere, though these are no longer of one size, but vary from a diameter zero to a diameter of the size of that of the sphere m .

Coming now to the instruments in which the alternative device adopted for the avoidance of slipping is by bringing into contact with one roller different circles of the disk M , or of its equivalent.

This may be done in the following way. Instead of allowing the cylinder (m) to roll on the sphere M (Fig. 31), and so to change the radius of the imaginary rolling circle (whose diameter is qq') on which it rolls, suppose that the cylinder is kept in contact as shown by the dotted lines, and the axis of rotation zz' of the sphere is turned, as, for instance, would happen if a sphere in combination with rollers were used as suggested for an anemometer by Mr. Ventosa, through an equivalent amount, *i.e.*, through the angle α . This will give the same result as far as the rotation of the cylinder is concerned, but with an important difference. The cylinder (in Fig. 31) or sphere (in Fig. 33) is no longer needed, and may be replaced by the original measuring roller, whose axis has a fixed position parallel to OX . It will be seen that this device practically amounts to bringing different circles on the sphere M into contact with the measuring roller (m), with the great advantage that exactly the same circle on the sphere M is scarcely likely to again roll in contact with the roller (m), though of course the radius may be the same. This method has been recently proposed by the Author, and the mode of carrying it out without involving slipping, by what is called the "sphere and roller mechanism," which mechanism has been explained and developed at length in a Paper before the Royal Society.¹ It need here be only remarked that the planimeter there described, and afterwards exhibited to the British Association at Montreal, was of the linear form, and of little practical use; but the Author has since completed a polar planimeter and exhibited it before the Royal Society.

Only one more area planimeter remains to be mentioned, and this is the one invented and brought before the Physical Society by Mr. C. V. Boys.² The principle of action is briefly this:—A

¹ Proceedings of the Royal Society, No. 232, June 1884, now being published *in extenso* in the Phil. Trans.

² "Philosophical Magazine." Mr. Boys has since shown the Author a Paper which came into his hands after his own communication in the Philosophical

wheel, or roller which is not supposed to slip sideways on the diagram, has its plane of rotation kept always at an angle α to the axis $O X$ of the figure to be integrated, such that

$$\begin{aligned} y &= \text{ordinate of the curve} \\ &= \tan \alpha \times K \end{aligned}$$

where K is a constant, and y is the ordinate with respect to $O X$ of that point on the curve which the pointer of the instrument is at the same instant tracing. If the component of a small motion of the wheel parallel to $O X$ is Δx , and the component of the same movement parallel to $O Y$ is Δt

$$\begin{aligned} \text{Then} \quad \frac{\Delta t}{\Delta x} &= \tan \alpha = \frac{y}{K} \\ \Delta t &= y \Delta x \times \frac{1}{K}, \end{aligned}$$

or the distance moved by the wheel parallel to the axis $O Y$ becomes the measure of an element of area. It is easy to see that the height moved by the wheel, becomes a direct measure of the area of the figure. Various examples of the action of this planimeter, called by the inventor the tangent integrator, are given by Mr. Boys; but the action is obviously limited, and an investigation of the theory reveals the fact that it is only a special case of the general problem, not only of the method of applying circles of varying diameter to one roller, but of the sphere and roller mechanism itself. This will be rendered clearer by stating that, in order to employ the component parallel to $O Y$, the roller was made to work against a cylinder, which, by its turning acted as the measuring roller. Evidently the length of the cylinder limited the travel in that direction. The cylinder was carried bodily along in the direction of its axis (corresponding to $O X$), and made to effect its own turning, the amount of turning varying with the tangent of inclination of the wheel, and this was sufficient in the application to the steam-engine integrator to be hereafter described, where the longitudinal motion of the cylinder could be made proportional to the stroke.¹ Mr. Boys endeavoured, by various means, to obtain continuous motion in both directions—one being equivalent to bending the

Magazine was made. This Paper, "Integrator Napisal Bruno Abakanowicz," published at Warsaw in 1864, evidently from the diagrams sets forth the main principles of Mr. Boys' integrator, though the instrument is of rather different form.

¹ "Philosophical Magazine."

ends of the cylinder round, and so attempting to solve the difficulty by what he has termed a "mechanical smoke ring." The Author, however, by approaching the matter from a different point of view, designed the sphere and roller integrator, which is nothing more or less than the inversion of the mechanism of Mr. Boys. In this the roller of Mr. Boys is replaced by the sphere; and instead of the two motions, one of the cylinder about its axis, and one of the cylinder longitudinally, the two rollers are used. It may be easily shown that the turning of the plane of rotation of the roller of the tangent integrator is equivalent to changing the axis of the sphere in the sphere and roller integrator.

MOMENT PLANIMETERS.

The moment of an area, and its moment of inertia about a given line, may be obtained mechanically upon similar principles to those by which a simple area was obtained. If A B C D E, Fig. 1, be the figure whose moment of area and moment of inertia are required about any line O X; then, taking any element of area A B, if y = height of upper portion S B, then the moment of area of the element S B about O X

$$\begin{aligned} \text{is} \quad m &= \text{area of S B} \times \frac{y_1}{2} \\ &= \frac{1}{2} y_1^2 \Delta x. \end{aligned}$$

Similarly the moment of inertia of the element is

$$i = \frac{1}{3} y_1^3 \Delta x.$$

The sum of an infinite number of such expressions as these, when Δx become infinitely small, gives respectively the moment of area and the moment of inertia of the whole figure according to the expressions.

$$\begin{aligned} \text{Moment of area} &= M = \frac{1}{2} \int y^2 dx, \\ \text{Moment of inertia} &= I = \frac{1}{3} \int y^3 dx. \end{aligned}$$

Now there are two possible ways of obtaining these results mechanically. One of these ways is by applying for the purpose the suggestion made by Sir William Thomson in connection with the disk-globe-and-cylinder integrator of Professor James Thomson¹ of using a train of such mechanisms to obtain the integration of

¹ Proceedings of the Royal Society, vol. xxiv., p. 262, and vol. xxvii., p. 371.

a simple linear differential equation. By certain simple arrangements

The first mechanism would give $\int y \, dx$,
 „ second „ „ „ $\int y^2 \, dx$,
 „ third „ „ „ $\int y^3 \, dx$.

This method need not be further considered here, since, so far as the Author is aware, it has never been carried into actual practice. It may be, however, said that the mechanical difficulties in the way of causing the measuring wheel or roller of the first mechanism to actuate the second, and the roller of the second to actuate the third without introducing serious error, are not easy to overcome, and require a very easily working piece of apparatus. The Author has discussed the applications of the sphere and integrator for the purpose in a Paper to the Royal Society.¹

FIG. 34.

The other principle is to cause the measuring roller to be directly turned at a rate which is made to vary, not as in the simple planimeter with the value of the ordinate (y), but with its second or third power. Though no method of directly doing this has apparently yet been suggested, yet the same result is practically effected by the beautiful application of a mathematical principle in the "moment integrator" of Professor Amsler.

Let the pole-arm CB (Fig. 34) be attached to a toothed segment (z_1) one portion of which gears with a toothed wheel z_2 ; the radius being as 2 to 1. Let the centre C of z be carried along OX, while the centre C_2 of z_2 is carried along a line ox parallel to OX. Let

¹ Proceedings of the Royal Society, No. 232, 1884. The figures and diagrams will shortly be published in the Philosophical Transactions of the Royal Society.

m_2 be a roller acting in every way as the measuring roller of the Amsler planimeter, whose axis is carried in the plane of the wheel z_2 , its direction passing through the centre C_2 . When the pole arm coincides with $O X$, let the plane of rotation of the roller m_2 be parallel to $O X$, and its axis parallel to $O Y$. When the pole arm is turned through an angle $S C B = \alpha$, the angular motion of the wheel z_2 is twice that of the arm; thus the roller m_2 takes the position shown in the figure.

$$\begin{aligned} \text{This is so because} \quad \frac{\angle^r \text{ motion of } z_2}{\angle^r \text{ motion of } z_1} &= \frac{\text{radius } z_1}{\text{radius } z_2} = \frac{2}{1}, \\ \therefore \angle K C_2 l &= 2 \alpha. \end{aligned}$$

Suppose the pointer p to move through the width of the element $S B$ at a height $= y$, and with it z , and z_2 , the roller m_2 being in contact with the diagram surface. Then, by what was proved in the case of the Amsler planimeter, and adopting the same notation.

$$\begin{aligned} \frac{\text{Turning of } m_2}{\text{Motion of translation of } m_2} &= \frac{2 \pi r n_2}{\Delta x} = \frac{l c_2}{c_2 K} \\ &= \cos 2 \alpha \\ &= 1 - 2 \sin^2 \alpha, \end{aligned}$$

$$\text{but} \quad \frac{S B}{C B} = \frac{y_1}{R_1} = \sin \alpha \quad (\text{where } C B = R_1),$$

$$\therefore \frac{2 \pi r n_2}{\Delta x} = 1 - 2 \sin^2 \alpha = 1 - \frac{2}{R_1^2} y_1^2,$$

$$\text{or} \quad n_2 = \left(\frac{1}{2 \pi r} \right) \Delta x - \left(\frac{2}{2 \pi r R_1^2} \right) y^2 \Delta x.$$

When the complete travel of the curve has been made, the sum of a series of quantities similar to the first becomes zero; so that, by making the constant $\left(\frac{1}{\pi r R_1^2} \right)$ equal $\frac{1}{2}$, the reading of the roller gives the value—

$$M = \frac{1}{2} \int y \, d x,$$

or the moment of area of the figure $B D E A$.

For the moment of inertia the segment of z_1 is used, the radius of which is three times that of another wheel, z_3 , with which it gears. The action of a roller m_3 , carried by the wheel z_3 , is exactly the same as that of m_2 , except that its angular motion is three times as great as the pole arm $C B$, instead of twice as great, as in the case of the other roller.

By reasoning similar to that already adopted, and taking the plane of rotation of m_3 perpendicular to O X in its initial position, instead of, as in the former case, parallel to it

$$\begin{aligned}\frac{\text{travel of } m_3}{\text{motion of translation of } m_3} &= \frac{2 \pi r n_3}{\Delta x} = \frac{l' c_3}{c_3 k^1} \\ &= \sin 3 \alpha \\ &= 3 \sin \alpha - 4 \sin^3 \alpha.\end{aligned}$$

$$\frac{S B}{O B} = \frac{y_1}{R} = \sin \alpha.$$

Therefore $\frac{2 \pi r n_3}{\Delta x} = 3 \sin \alpha - 4 \sin^3 \alpha = 3 \frac{y_1}{R_1} - 4 \frac{y_1^3}{R_1^3}.$

Or $n_3 = \left(\frac{3}{2 \pi r R_1} \right) y_1 \Delta x - \left(\frac{4}{2 \pi r R_1^3} \right) y_1^3 \Delta x,$

FIGS. 35 and 36.

which, when the pointer is taken round the curve, gives, with suitable values of the constants,

$$\begin{aligned}n_3 &= \int y dx - \frac{1}{3} \int y^3 dx \\ &= \text{area of B D E A} - \text{moment of inertia of B D E A} \\ &= A - I. \\ \text{or } I &= A - n_3.\end{aligned}$$

The instrument, Figs. 35 and 36, has an area planimeter attached to it, so that by reading the rollers m_1 and m_3 , and subtracting the results, the moment of inertia is obtained.

The details of the moment planimeter shown (Figs. 35 and 36) are easily explained. A guide P Q of steel has a groove g g,

which is placed parallel to the axis $O X$ by means of the gauges G , one, as shown, being at each end, which are adjusted with their points q upon the line $O X$. The rollers $R R$ run in the grooves $g g$, and support a frame $F F$, which carries by means of an axle $J J$, the frame $E E$. This frame supports the toothed segment z_1 , and the two toothed wheels, z_2, z_3 , upon vertical axes. The former between centres, one of which is shown, Fig. 36, i , the latter by steel axles within the column $i_2 i_3$. The pole-arm carries, in addition to the pointer p at the end, the roller m_1 with its dial h_1 , forming an ordinary planimeter, and is itself carried on the centres $s_1 s_2$. The two other rollers and dials are shown as $m_2 m_3$ and $h_2 h_3$ respectively. The weight W serves to balance the instrument, so as to avoid undue pressure on the paper, and the motion is so smooth as to enable a curve to be traced with the greatest ease and accuracy.

FIG. 37.

Attention has been called by the late Dr. Merrifield and others to the valuable applications of this instrument for purposes of naval architecture, but so far as the Author is aware, no account has been given in this country of its applications in civil engineering, as proposed by Professor Amsler.¹ The following brief account of the methods in the case of calculating the contents of embankments, cuttings, &c., is therefore given from an abstract for which the Author is indebted to the kindness of Dr. A. Amsler:—

Let Fig. 37 be the plan of a portion of an embankment or cutting, the character of which is supposed to be the same throughout, viz., of uniform width of roadway, and uniform side-slopes, the surface of the ground, the gradient, and the horizontal curvature of the roadway, being restricted in no way. $A A'$

¹ "Anwendung des Integrators zur Berechnung des Auf- und Abtrages bei Anlage von Eisenbahnen Strassen und Kanälen." Von I. Amsler-Laffon.

represents the centre line of the railway; B_1B_1' and B_2B_2' its two borders; C_1C_1' and C_2C_2' the intersections of the side-slopes with the surface of the ground. Suppose now the embankment or cutting to be divided into thin layers by vertical planes, perpendicular to the centre line AA' of the roadway; PQ and P_1Q_1 may be the intersections of two adjacent planes with the plane of the diagram.

Then if p = area of section PQ ,

Δs = interval between PQ and P_1Q_1 measured upon the centre of gravity of the section.

Total volume of embankment is (from one of the properties of Guldinus)—

$$V = \int p \, ds,$$

the integral extending over the whole length of the embankment under consideration.

FIG. 38.

There are three cases dealt with in the Paper of Professor Amsler, corresponding to the three forms of sections, I, II, or III, Fig. 38.

The first of these, I, is simple enough, since the centre of gravity of the section always coincides in plan with the centre line of the roadway, and the plan of operation is as follows:—

Let Fig. 39 represent a longitudinal section of a portion of the embankment of uniform gradient, developed into a plane; the straight line $E'E'E''$ represents the top of the embankment; $G'G'G''$ the profile of the ground; the straight line $A'A'A''$, which is parallel to $E'E'E''$, is the locus of the imaginary vertex of the trapezoidal cross-sections. The level line MN is the line to which the offsets of the profile of the surface of the ground refer. BG shows the intersection of a vertical cross-section with the figure, and AG the intersection of a plane perpendicular to the

top of the embankment (and also to the line $A' A''$) with the figure.

Let $i = \angle A G B = \angle M N A = \text{gradient};$
 $y = A G = \text{distance of vertex to bottom of embankment};$
 $y_0 = A E = \text{distance of vertex to top of embankment};$
 $2 \beta = \text{angle at vertex at A.}$

It may be easily proved the area of the section made by the plane $A E G$ is—

$$p = (A G^2 - A E^2) \tan \beta = (y^2 - y_0^2) \tan \beta$$

but since $V = \int p \, d x$

Therefore volume $= \tan \beta \int (y^2 - y_0^2) \, d x.$

FIG. 39.

And thus, if β is known, the volume of the portion $E'E'' G''G'$ (Fig. 39) is easily found with the mechanical integrator thus—

Take $A' A A''$ as the axis of moments, and adjust the rail of the instrument so as to be parallel to it. Start the pointer anywhere on the shaded figure, and trace round it; the travel of the roller m_2 being denoted by M , the scale of the drawing longitudinally being—

$$1'' = m \text{ feet,}$$

and vertically $1'' = n \text{ feet;}$

then $\text{volume} = V = 20 \, m \, n^2 \tan \beta \times M \text{ cubic feet.}$

It only remains to insert a known value for $\tan \beta$, which is easily done thus—

Let Fig. 40 be a perspective view of the sections A E G and B G (Fig. 39), where—

$$\angle C_1 B C_2 = 2 \alpha.$$

Then from the diagram $\frac{C_1 G}{A G} = \tan \beta$;

also $\frac{C_1 G}{B G} = \tan \alpha$ and $\frac{A G}{B G} = \cos i$.

Therefore $\tan \beta = \frac{\tan \alpha}{\cos i}$,

or $V = \left(20 m n^2 \frac{\tan \alpha}{\cos i} \right) M$,

where $\angle \alpha$ and $\angle i$ are known constants.

FIG. 40.

To complete the calculations for the whole route separate portions are taken, with the various proposed gradients.

The above formula is exact for the integrator shown in Fig. 36, as arranged for English measures, a complete revolution of the measuring roller being taken as a unit of reading.

It is to be noted that nothing is supposed as to the curvature of the centre line of the roadway horizontally, as it is supposed to be developed in the figure. Also, that the aggregate error arising from the assumptions that the cross-sections are exact trapezoids will in most cases be very slight, on account of the errors in cuttings and those in embankments partly compensating for each other, in addition to the cutting and filling in each section, as shown in Fig. 41, where the small triangular portion in dotted lines C' D H represents the amount taken off the former, and added to the latter.

Alterations of the proposed roadway, otherwise involving tedious calculations, simply necessitate an alteration in the line A' A A'', and a repetition of the mechanical work of the integrator, but need no fresh diagram. In preparing the drawing, allowance should be made for ditches along the roadway in cuttings, which is easily done, as shown in Fig. 42, where B₁ B₂, which equalizes

the amounts taken and left, must be considered as the roadway line. In the case shown in Fig. 43, the excess of the embanking over the cutting is approximately equal to the layer above the dotted line C_1C_2 . The contents of this layer could be measured either by considering it as an embankment, and treating it as such, or by the simpler—and for a first estimate sufficiently accurate method—of

FIG. 41.

FIG. 42.

assuming its section to be a parallelogram. The area of the shaded portion (Fig. 43) is then simply to be measured, and the result, multiplied by the length of the road, gives the required contents. The supposition that the slopes CD and $C'D'$ are the same is also sufficiently accurate.

The foregoing is the first method described by Professor Amsler, and is extremely simple, but obviously only approximately accurate. The two other methods are capable of giving very accurate results, and are dealt with by him at considerable length. Only a short account of them will be given here.

FIG. 43.

FIG. 44.

The first thing to be noted is that, as a rule, the centre of gravity of the section will not really coincide in plan with the centre line of the roadway, but will curve at the line S P P' S', Fig. 44, A A' being the true centre line. Thus, in the expres-

$$\therefore V = \tan \beta \int (y_1 y_2 - y_0^2) dx + \frac{\tan^2 \beta}{6r} \int y_1 y_2 (y_1 - y_2) dx.$$

By a simple transformation this expression is brought into such a form as to allow of mechanical integration. The final formula being—

$$V = \frac{\tan \beta}{2} U + \frac{\tan^2 \beta}{6r} W,$$

where

$$U = \frac{1}{2} \int (y_1^2 - y_0^2) dx + \frac{1}{2} \int (y_2^2 - y_0^2) dx - \frac{1}{2} \int (y_1 - y_2)^2 dx$$

$$V = \frac{1}{2} \int (y_1^3 - y_0^3) dx - \frac{1}{3} \int (y_2^3 - y_0^3) dx - \frac{1}{3} \int (y_1 - y_2)^3 dx.$$

Another simple diagram has to be prepared, and by means of three operations of the integrator, the values of U and V are given thus—

$$U = 20 \times m n^2 (v_1 + v_2 - v_3)$$

$$W = m n^3 \{320 (u_1 - u_2 - u_3) - 100 (w_1 - w_2 - w_3)\}$$

where m and n have the significations formerly explained, and u , v , and w are the respective readings of the area, moment of area, and moment of inertia rollers in each of the three operations. Considering the great amount of calculation thus saved, and the accurate nature of the results, this second method, although involving rather more labour than the first, is a very important one.

The third method, which deals with the broken base, is much the same in principle as the second, but the expressions become more complicated, whilst six readings of the measuring rollers are involved. The case of an embankment, consisting partly of a cutting, is completely and accurately worked out by this method.

The Paper of Professor Amsler concludes with an example of the application of his integrator to the problem of the strength of a girder.

CONTINUOUS INTEGRATORS.

Any piece of mechanism which continuously adds up results may be regarded as a continuous integrator. Of such instruments, revolution counters as employed in meters of various kinds, form the simplest example, and correspond in action to the devices already described, by which the linear measurement of a boundary is performed. These will not be further referred to, and it is only necessary to consider those computing mechanisms which, dealing

with the result of two simple unit measurements, correspond in principle to area planimeters.

It appears that Poncelet before the year 1838 suggested the employment of a continuous integrator for computing the two factors in dynamometrical measurements. This was described by Morin in 1838 as applied in his "compteur" for registering the work done by a team of horses, dragging a loaded carriage at any given velocity over any length of road. The principle employed was that of the disk and roller, the use of which as already shown had been suggested for application in a planimeter more than twenty years before. In the case in question, the disk was turned by an endless cord or band from one of the wheels of the carriage, while the position of the roller on the disk was caused to vary with the tractive force, and its reading thus gave the product of force and space, or the actual work done.

In 1840 a Committee of the British Association, consisting of Professor Moseley, Mr. Enys, and Mr. Hodgkinson, was appointed to procure the dynamometrical apparatus of Mr. Poncelet, and to obtain a series of experiments on the duty of steam-engines by means of that apparatus, the sum of £100 being placed at their disposal for the purpose. The report of this committee in 1841 describes at length the "constant indicator" of Professor Moseley, which was in reality a continuous steam-engine integrator. It was entirely a new instrument, except that the principle of the disk and roller was employed, and also the traction springs of General Morin. The pressure of steam was allowed to act upon a piston so as to vary the position of the measuring roller, while motion was given to the surface of revolution by means of the stroke of the engine. The surface of revolution was a cone, which was substituted for a disk, as by this arrangement the rapidity of the changes of velocity due to corresponding changes in the position of the integrating wheel is diminished in the same proportion in which the sine of one-half the angle of the cone is less than unity. The force necessary to drive the integrating wheel is diminished in the same proportion, and therefore the chance of an error arising from the slipping of the edge of the integrating wheel on the surface which gives it motion. The reports of the committee in 1842, 1843, and 1844 (which was joined in the first of these years by Dr. Pole), show that the action of the above instrument was, as far as could be determined in its application to a single-acting Cornish engine, very satisfactory, but apparently no mention has been made of it, or results obtained from it, in any succeeding report.

Various other steam-engine integrators have been brought forward since then, most of them acting upon the same principles; amongst these is the recent power-meter of Messrs. Ashton and Storey, which is described at considerable length in the American edition of Weisbach's *Mechanics* as apparently something new. It is, however, the same instrument as Moseley's integrator, except that it employs a spiral instead of a straight spring, and returns to the use of the flat disk.

The disk and roller has also been since the time of Morin applied in many dynamometers, which thus become really "ergometers" or "work" measurers. In a series of articles which recently appeared in "*La Lumière Electrique*" those of Hirn, Megy, Bourry, and Darwin, are described as having a "totaliser"¹ or integrator of this kind attached to them. The position of the measuring roller varies with the force exerted or transmitted,

FIG. 46.

and the motion of the disk with the revolutions of the motor or machine. Thus by suitable counting apparatus the continuous product is given of force and space, or work done.

A cone has sometimes been used instead of the disk with dynamometers, as in that of Baldwin and Eickemeyer, used at the Centennial Exhibition in 1876 for testing mowing-machines, in which, instead of a measuring-roller, a cylinder is placed with its axis parallel to one side of the cone, while an endless band of round cord is rolled along between the two surfaces, so as to transmit the varying motion of circles of different radius on the cone.² Fig. 46 shows the apparatus of Mr. Roury applied to a

¹ An account of these, with several additional ones, will be found in Dredge's "Electric Illumination," vol. ii., section iii., p. 192.

² "The Practical Dictionary of Mechanics." E. H. Knight. Vol. iv., p. 287.

transmission-dynamometer.¹ In this the force is transmitted through a differential train of three bevil wheels, the middle one of which is attached to the spindle (S), which supports the weight (W), and is suspended from the main shaft by the joint at (k), about which the whole can turn. Thus the deviation of the weight from the vertical (as shown by the dotted lines) changes with the force. The change of position of the spindle (S) causes a band (b) to move along the surfaces of revolution R R₁, the upper one R being turned from the shaft by the spindle (Z Z). It is to be noted that the distance of the band (b) from its zero position is not directly proportional to the force represented by the change of position of the weight, and therefore the surfaces must be formed with a certain curve, found by construction, in order that the dial and counting apparatus at D may correctly give the product of the two variables, force and space, and so the work transmitted through the dynamometer.

It cannot be said that continuous integrators of this kind are at present practically employed to any great extent. There are probably two reasons for this. One is the want of durable and reliable instruments. The other, the question as to how much, and to what degree they are really needed.

With regard to the first of these it is evident that in all the arrangements hitherto considered (with the exception of Baldwin and Eickemeyer device) there is that slipping of surfaces in contact, which, though of little effect as far as wear goes in the limited operations of a planimeter, becomes a very serious consideration when continuous action is required to be maintained. The only integrator of the second, or non-slipping class, which, as far as the Author is aware, has yet been practically applied, is the "power-meter" of Mr. Vernon Boys. This instrument is shown in Figs. 47. and 48, and acts upon the same principle as Mr. Boys' integrator. The piston C, subject to the varying pressure in the engine-cylinders, with which the barrel A is connected by the connections at B and B¹, is moved up and down against or with the tension of the spring D; its rod acting on the arm g causes the plane of rotation of the roller G to take positions more or less inclined to the axis of the cylinder H. This cylinder H is moved to and fro with the stroke of the engine by means of the cord L, Fig. 48, and the roller G being in frictional contact with it causes it to turn round to a

¹ "Registering Apparatus." M. Roury. Civil Ingénieur, vol. viii., 1862, p. 367.

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greater or less extent, according as the plane of G is more or less inclined to the axis of H. The amount of its revolution is registered by the counting apparatus in I (Fig. 48), to which the axis of H is geared, and is thus a measure of the power of the engine, for it gives the product of the tangent of the angle to which G is inclined and the distance moved through by H, that is the product of pressure of steam into the stroke of the engine. The steam being (as originally in Moseley's and also in subsequent integrators) supplied both above and below the small

FIG. 47.

FIG. 48.

piston, the absolute pressure is given. Thus, in the present case, as the change of pressure on C at the beginning and end of each stroke causes the rod of g to be alternately above or below the axis of H, so the motion of the cylinder to and fro will always cause the cylinder H to turn in one direction, and thus to continuously integrate the work done. This device only enables a reciprocating movement of the cylinder H to be made, and the Author has already mentioned the device of the sphere and rollers, which by the inversion of the higher pair of Mr. Boys, enables

continuous motion to be obtained, and is suitable for application in dynamometers, electric-motors, and other purposes.

With regard to the want of such instruments a very strong case was made out by the committee, already mentioned, in their report in 1841, where the application of a continuous integrator to steam-engines was alone discussed. The application has been made to electric-motors, and in trials of motors and machines generally, and there is little doubt if continuous integrators combining the three qualities of durability, accuracy and cheapness could be produced, that in these days of increased regard for measurement of all kinds, there would be a much larger and increasing application of them.

LIMITS OF ACCURACY OF INTEGRATORS.

In all calculating machines accuracy of the result must be the question of first importance. Assuming the theory relied on in the various instruments for the mathematical operation to be correct, the accuracy depends primarily upon the mechanical arrangements, though in the case of planimeters it also depends upon the skill and care of the manipulator, and involves the question of a personal error. This latter point need not be considered, partly because this occurs more or less in all results obtained by observers, but also because it is less than might be at first anticipated, from the fact that in tracing the pointer round the curve there is no reason why the error due to moving it on one side should exceed that due to moving it on the other side, that is why equal errors of opposite effect upon the final reading should not be made.

It has been seen that the action of all integrators, except mere revolution-counters, depends upon the motion of the measuring roller, or its equivalent, over surfaces of various forms, therefore the above-mentioned mechanical arrangements resolve themselves into an examination of the nature of the frictional contact of two surfaces. It was for this reason that integrators have been classified according to the nature of this frictional contact, and it now remains to investigate the nature of this, to show to what the classification leads, to give the direct results of experiments upon the subject, and also the indirect results obtained from the instruments themselves.

Planimeters and integrators generally have been divided into —

I. Those in which the frictional surfaces slip as well as roll over each other.

II. Those in which slipping of the surfaces is supposed not to take place.

The order of this arrangement was adopted upon historical grounds, and also because the former class is at present by far the most important; but it would be more convenient, upon mere grounds of mechanical simplicity, to invert the order.

Let AB (Fig. 49) be the plan of the measuring-roller. Suppose a force applied in the direction OX , making an angle (α) with the plan of the axis of AB .

Let $d = OX =$ distance through which the force acts.

1st. Suppose that frame which carries the measuring-roller is free to move in any direction horizontally, but maintains the plane of rotation of the roller vertical, then the application of a force

FIG. 49.

along OX , at the centre of AB , will cause it to roll along the line coinciding in direction with the plan of the centre line AB of the roller, that is along the line OY . This will always be the case except when this force is applied in the limiting case in the direction perpendicular to the plane of AB (*i.e.* when $\alpha = 0$).

Thus the distance in this case travelled by the centre of AB , which is the same as the path rolled by it, is

$$OY = \frac{d}{\sin \alpha},$$

and the distance moved through by the centre at right angles to OX is

$$XY = d \cot \alpha.$$

The latter value is the one usually taken or recorded by the

instruments at present in use, but depends directly upon the former.

Next, suppose the frame carrying the roller is constrained either by guides, as in the linear planimeter, or by the radius-bar of the polar planimeter, or otherwise, to move in the direction of $O X$, that is, in the direction in which the force acts. When the centre of the roller has reached the point X , that is, when the force has been exerted through a distance $O X$,

Then $O Z = \text{distance slipped by } A B = d \cos \alpha,$
 $X Z = \text{distance rolled by } O B = d \sin \alpha.$

Upon the degree of accuracy with which the above conditions are fulfilled depends the correctness of the working of all integrators; for not only do these two cases entirely cover the action of the two classes of planimeters and the corresponding continuous integrators, but one of the limiting cases in each, viz., that in which the force acts in the plane of rotation of the wheel (when $\alpha = 90^\circ$), represents the conditions under which the wheel of the boundary measurer or opisometer is employed. It may be therefore said that the theory of mechanical action of integrators is based upon one or other of the following assumptions, in which the limiting case, (namely, when $\alpha = 90$), is included.

Class I.—That the rolling of the planimeter when slipping is allowed is

$$N_1 = k_1 d \sin \alpha.$$

Class II.—That no slipping takes place, which amounts to the assertion that

$$N_2 = k_2 \frac{d}{\sin \alpha},$$

N_1 and N_2 being the readings in each case, and k_1 and k_2 suitable constants for the instruments.

It is easy to see that the first of these is really the assumption made for all instruments in Class I.; but in the various instruments in Class II., it is only with the planimeter of Mr. Boys that it becomes directly obvious that the above assumption is made. With the others though it is less evident, nevertheless, it will be found, on examination, to be equally true that the second supposition is really made, and that upon its truth the correct action of all instruments in the second class depends. The forces acting in each of the two cases must therefore be taken into consideration and the mechanics of the problem examined.

Proceeding in order of simplicity, Class II. will be examined first.

Let AB in both cases (Fig. 50) be the plan of the measuring-roller.

Let S = re-action of surface upon which AB rolls, that is, the force with which it is kept in contact with it ;

μ = coefficient of friction between roller and surface ;

$P = OC$ = re-action of surface, which must be brought into action in a horizontal direction to cause the roller to turn on its axis.

FIG. 50.

CLASS 2.

Class II. (Fig. 50).—Suppose the frame in which the roller is carried to be free to move in any direction horizontally, let a force be gradually applied at the centre of the roller AB in the direction perpendicular to the plane of rotation. This will produce no effect as long as it is less than the maximum resistance which can be opposed by friction between the edge of the roller and the surface upon which it rests, that is, as long as

$$R = OD \text{ (Fig. 50)} < S \mu.$$

When the force R is equal to $S\mu$, and acts within the angle θ , the roller AB will move with uniform motion along the line of action of the force without turning. The same thing will hold if instead of the force acting perpendicularly to the plane of rotation it acts at some oblique angle to it not exceeding a certain value measured from the normal. The limiting value of this angle depends on the resistance of the friction of the axle of the roller AB to turning. Let this angle be (θ) and draw OE perpendicular to OC , meeting the circle drawn with O as centre and radius OD in E , and let

$$\theta = \text{angle } EOD.$$

When the line of action of the force falls without the angle θ , as, for instance, when it takes place in direction OF , the roller will still slip along the line of the force, but the roller will now also turn. The component in the plane of rotation will now, however, be of a magnitude such that the motion of rotation of the roller is no longer uniform. Since only uniform motion is being considered to take place, the conclusion is, that when the force acts at an angle greater than θ to the axis of rotation (*i.e.* when $\alpha > \theta$) it must never be so great as to cause the roller AB to slip, and therefore only a motion of pure rolling can take place. By proper mechanical devices the roller can be made to turn very easily, and angle θ be kept very small.

The magnitude of the force which must be applied in any position of the roller to effect this motion is

$$\begin{aligned} P_1 &= OK = \frac{OC}{\sin \alpha} \\ &= F \operatorname{cosec} \alpha, \end{aligned}$$

and is at once given by the intercepts drawn from O to CE in the construction, shown in Fig. 50, for any other value of α .

Class I. (Fig. 51).—Suppose that the frame does oppose restraint, and that this restraint is such as to always cause the centre of AB to move in the direction in which the force acts. Let OF (Fig. 51) lie in this direction, making the angle α with the axis of AB , draw EF perpendicular to OF from the point E , then by the triangle of forces. The force required to move AB is

$$P_2 = OF = S\mu \cos (\alpha - \theta)$$

The reaction which is supplied by the frame is

$$Q = EF = S\mu \sin (\alpha - \theta).$$

I. Case of boundary measures and limiting case of both classes of area and moment integrators.—The conditions of rolling of two surfaces when the force acts in direction of the plane of rotation.

II. Class I. of area and moment planimeters.—Whether the rolling is proportional to the slipping as assumed.

III. Class II. of area and moment planimeters.—Whether there is any slipping in this case, and if so, to what extent.

Case I. There are two separate problems under this head; one where the planes of revolution of the two rolling surfaces coincide; and the other when they do not coincide.

Dr. A. Amsler¹ has experimentally investigated the former thus: Two accurately turned disks, SS_1 , (Fig. 52), each 200 millimetres in diameter, were placed with their edges in frictional contact, so that the mark shown on each coincided at the point of contact. By means of the apparatus (Fig. 53) the lower one S was turned by

FIG. 52.

means of a band (BB_1) through 1700 revolutions one way (thus turning the upper one S_1), and then 1700 revolutions in the opposite direction. The marks which should have coincided if no slipping had taken place, were now found to be as in Fig. 52, the marks being a distance $= \epsilon_n = 0.05$ millimetre apart. The relative error

is thus $= \frac{\epsilon_n}{2n\pi 200} = (\text{about}) \frac{1}{40,000,000}$. This experiment is only

of value to show the error when the wheel travels back by turning in the opposite direction, and at the most shows that the error is nearly the same in both directions, and does not prove anything with regard to the action of the measuring roller of the boundary measurer; concerning this point observations were wanted upon the

¹ "Ueber den Flächeninhalt und das Volumen durch Bewegung erzeugter Curven und Flächen," &c., p. 64. Alfred Amsler, Schaffhausen, 1880.

results of the first 1700 revolutions. A very good way of examining the point would be to note the divergence of the marks in x revolutions when S drives S_1 , then cause S_1 to drive S back through the same number of revolutions, and it would be seen whether the divergence was due to a difference in the periphery of the wheels, or to a slipping of the surface of contact. During Dr. Amsler's experiments no one was allowed to enter the room, in order to avoid

FIG. 53.

alteration of temperature. He found that the heat radiated from the human body, or even from a lighted candle placed at some distance, had a perceptible influence on the result.

Fig. 54 shows a most important case in which the plane of rotation of the surfaces in contact do not coincide with that of the measuring roller (m) being actuated by the disk M . Fig. 55 shows Dr. Amsler's apparatus for examining this case, in which, if the edge of the roller has any width, there must be slipping action, even

though the force always acts in the direction of the plane of rotation.¹

FIG. 54.

FIG. 55.

The roller m which rests on the upper surface of the disk, which latter

¹ "Ueber den Flächeninhalt und das Volumen durch, Bewegung erzeugter Curven und Flächen," &c., p. 67. Alfred Amsler, Schaffhausen, 1880.

has its edge divided, and is in juxtaposition with a vernier (v). The axis of the roller is fixed, and its edge is thus kept always vertically under a microscope (K). The position of the disk is noted, and it is then moved forward about 8 revolutions (or exactly 2900°), which gives the roller about 130 revolutions, and a mark is observed on the latter. Then in theory the result of giving 8 more revolutions to the disk in the same direction should be to bring the same mark of the roller under the microscope. Practically the successive motions of the disk will be a little different, so that the second advance of the disk will not be exactly the same as in the first case. The same mark on the roller is, however, always brought under the microscope, and the difference in turning of the disk is what is noted.

In the following table—

i = number of experiment,

ϕ = angle by which the disk differs from last reading,

so that the second column gives the positions of the disk at the end of successive advances in which the roller is made to take 130 complete revolutions, the third column shows the travel of disk in minutes (2900° having, of course, to be added to the readings). The fourth gives the difference between these and a mean value. The last gives the ratio of these differences to the travel.

i .	$\phi_i - i 2,900^\circ$.	$\phi_i - \phi_{(i-1)} - 2,900^\circ$.	$m - (\phi_i - \phi_{i-1})$.	$\frac{m - (\phi_i - \phi_{(i-1)})}{2,900^\circ}$.
0	2.58	47.22	-0.06	-0.0000003
1	49.80	47.28	-0.12	-0.0003007
2	37.08	47.34	-0.18	-0.0000010
3	24.42	46.98	+0.18	-0.0000010
4	11.40	47.22	-0.06	-0.0000003
5	58.62	47.04	+0.12	+0.0000007
6	45.66	46.98	+0.18	-0.0000010
7	32.64	47.34	-0.18	+0.0000010
8	19.98	46.86	+0.30	+0.0000017
9	06.84	47.34	-0.18	-0.0000010
10	54.18			

Case II. To test the results when a roller partly rolls and partly slips, Dr. Amsler used the apparatus shown (Fig. 56) in plan and elevation. In this C is a carriage running upon four wheels

on the base (B) which has parallel grooves planed in it; the travel of the carriage being limited by two stops at the ends. Upon the surface of C the measuring roller (m) rests, being attached to a plate A. By means of the graduations on A the axis of (m) can be set at any required angle with reference to the direction of motion of the carriage. The frame supporting the roller is carried on the disk by means of pivots, so as to allow (m) to rest on the surface of C with the constant pressure of its weight.

If α = angle of axis of (m) in the direction of the motion
the carriage;
 s = motion of the carriage;
 u = turning of the roller;

$$\text{Then } u = s \sin \alpha = s \cos \left(\frac{\pi}{2} - \alpha \right) = s \cos \beta.$$

If ϕ = actual reading of vernier v ;
and $\phi = \phi_0$ when $\alpha = 90$ or $\beta = 0$;
then $u = s \cos (\phi - \phi_0)$,
 $u_1 = s \cos (\phi_1 - \phi_0)$,
 $u_2 = s \cos (\phi_2 - \phi_0)$, etc.;

$$\text{Then } \tan \phi_0 = \frac{u_1 \cos \phi_2 - u_2 \cos \phi_1}{u_2 \sin \phi_2 - u_1 \sin \phi_1},$$

$$s = \frac{u_1}{\cos (\phi_1 - \phi_0)} = \frac{u_2}{\cos (\phi_2 - \phi_0)}.$$

In the following Table, which represents the results of experiments when the disk was covered with a surface of pear-tree wood, carefully polished (paper being, however, found to afford almost as good results)

- i = as before, the number of the experiment;
 B_i = angle of inclination of roller for experiment (i);
 u'_i = motion of roller as observed for experiment (i);
 u_i = motion of roller as calculated.

i .	$\beta_i = \phi_i - \phi_0$.	Experiment u'_i .	Calculated u_i .	$u_i - u'_i$.
	$\begin{matrix} \circ & ' & '' \\ 0 & 0 & 0 \end{matrix}$			
0	0 0 0	9,649	9,649	0
1	8 9 36	9,552	9,551	-1
2	9 36 0	9,511	9,514	+3
3	26 9 36	8,657	8,661	+4
4	27 36 0	8,550	8,551	+1
5	44 9 36	6,916	6,922	+6
6	45 36 0	6,750	6,751	+1
7	62 9 36	4,505	4,506	+1
8	63 36 0	4,290	4,290	0
9	80 9 3	1,650	1,649	-1
10	81 36 0	1,415	1,410	-5

Case III. The actual conditions of motion when a force smaller than the component of $S\mu$, acts obliquely to the plane of rotation of the measuring roller, do not appear to have been made the subject of direct experiment. It is apparently always tacitly assumed that no slipping takes place. But this crucial point cannot be thus left to mere conjecture, and the Author has designed a method of carefully testing this which he has not yet been able to properly carry out. From a few rough observations there seems little doubt, however, that some slipping always does take place, and that its amount is in the limiting cases by no means inconsiderable.

Lastly, a few words may be said concerning the work of Professor W. Tinter, and of Professor Lorber. The former has examined¹ most carefully no less than nine different planimeters, from which he concludes that the different angles at which the measuring roller of

¹ "Zeitschrift des Oesterreichischen Ingenieur- und Architekten-Vereins," vol. xxix., 1877, p. 154.

the polar planimeters turn has little effect upon the result, and that taking one turn of the measuring roller as $f = 100$ square cm., the average error in the reading was only from $= 0.00075$ to 0.0013 , according as the centre of rotation was without or within the area to be measured. The work of Professor Lorber is so extensive and elaborate that it is impossible to do more than give in the most brief form the results at which he has arrived after many thousands of experiments. He concludes that error in the reading is always represented by an equation of the form—¹

$$dn = K \cdot \pm \mu \sqrt{n},$$

dn = the error in the reading, K and μ being constants,

where n = the reading of the measuring roller;

the above equation gives rise to one of the following form:—

$$dF_n = Kf + \mu \sqrt{Ff},$$

where F = actual area to be measured,

and dF_n = the error in the result expressed in terms of the area.

The following are the results given in his latest paper—²

Linear planimeter. . . .	$dF = 0.00081f + 0.00087 \sqrt{Ff}$
Polar planimeter	$= 0.00126f + 0.00022 \sqrt{Ff}$
Precision polar planimeter	$= 0.00069f + 0.00018 \sqrt{Ff}$
Freely swinging planimeter	$= 0.00060f + 0.00026 \sqrt{Ff}$
Simple plate planimeter .	$= 0.00056f + 0.00084 \sqrt{Ff}$
Rolling (Coradi) planimeter	$= 0.0009f + 0.0006 \sqrt{Ff}$

The degree of accuracy represented by these results may be inferred from the fact that in one case of the last planimeter when

$$f = 100 \text{ the relative error} = \frac{dF}{F} = \frac{1}{13,330}.$$

¹ "Ueber die Genauigkeit der Planimeter." Oesterreichische Zeitschrift für Berg- und Hüttenwesen, vol. xxxi., 1883, p. 22.

² "Ueber das Roll-planimeter von Coradi, von Franz Lorber." Zeitschrift des Oesterreichischen Ingenieur- und Architekten-Vereins, vol. xxxvi., p. 135.