45. To Simplify Expressions of the Form $\frac{a^n(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^n(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^n(x-a)(x-b)}{(c-a)(c-b)}$

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and y , d , a respectively. The general theorem can be simply deduced from this particular case.

For an extension of the theorem to polygons with any number of sides see the *Proceedings of the London Mathematical Society*, vol. xxviii., p. 442.

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45. To simplify expressions of the form

$$\frac{a^n(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^n(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^n(x-a)(x-b)}{(c-a)(c-b)}.$$

Denote the given expression by $f(x)$.

Then $f(x) - x^n$ vanishes for the three values a, b, c of x . It must also be kept in mind that $f(x)$ is a quadratic function of x .

(i.) Let $n=0, 1$ or 2 , then $f(x) - x^n$ must vanish identically, and therefore $f(x) = x^n$.

(ii.) Let $n=3$, then $f(x) - x^3 = -(x-a)(x-b)(x-c)$, and therefore $f(x) = (a+b+c)x^2 - (bc+ca+ab)x + abc$.

(iii.) Let $n=4$, then $f(x) - x^4 = -(x+a+b+c)(x-a)(x-b)(x-c)$ (since the coefficient of $x^3=0$),

therefore $f(x) = (a^2+b^2+c^2+bc+ca+ab)x^2 - \{(a+b+c)(bc+ca+ab) - abc\}x + (a+b+c)abc$.

(iv.) Let $n > 4$, then $f(x) = p_{n-2}x^2 - (s_2p_{n-3} - s_3p_{n-4})x + s_3p_{n-5}$, where $s_1 = a+b+c$; $s_2 = bc+ca+ab$; $s_3 = abc$,

and p_r = sum of homogeneous products of r dimensions of a, b, c .

For $f(x) - x^n = -(x^{n-3} + p_1x^{n-4} + p_2x^{n-5} \dots p_{n-3})(x^3 - s_1x^2 + s_2x - s_3)$, where $p_1, p_2, p_3 \dots p_{n-3}$ are determined by the $n-3$ equations,

$$\begin{aligned} p_1 - s_1 &= 0, \\ p_2 - p_1s_1 + s_2 &= 0, \\ p_3 - p_2s_1 + p_1s_2 - s_3 &= 0, \\ &\dots\dots\dots \\ p_{n-3} - p_{n-4}s_1 + p_{n-5}s_2 - p_{n-6}s_3 &= 0. \end{aligned}$$

Hence $(1 + p_1x + p_2x^2 \dots)(1 - s_1x + s_2x^2 - s_3x^3) = 1$,

and therefore $1 + p_1x + p_2x^2 \dots = \frac{1}{(1-ax)(1-bx)(1-cx)}$

and therefore p_r = sum of homogeneous products of r dimensions of a, b, c .

$$\begin{aligned} \text{Hence } f(x) &= (s_3p_{n-5} - s_2p_{n-4} + s_1p_{n-3})x^2 + (s_3p_{n-4} - s_2p_{n-3})x + s_3p_{n-5} \\ &= p_{n-2}x^2 + (p_{n-1} - s_1p_{n-2})x + s_3p_{n-5}. \end{aligned}$$

Similarly, if $\phi(x)$ denote

$$\begin{aligned} \frac{a^n(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + \frac{b^n(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)} + \frac{c^n(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)} \\ + \frac{d^n(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}; \end{aligned}$$

(i.) if $n=0, 1, 2$, or 3 , $\phi(x) = x^n$;

(ii.) if $n=4$, $\phi(x) = (a+b+c+d)x^3 - (ab+ac+ad+bc+bd+cd)x^2 + (bcd+cda+dab+abc)x - abcd$
 $= s_1x^3 - s_2x^2 + s_3x - s_4$;

(iii.) if $n=5$, $\phi(x) = p_2x^3 - (s_1s_2 - s_3)x^2 + (s_1s_3 - s_4)x - s_1s_4$;

(iv.) if $n > 5$, $\phi(x) = p_{n-3}x^3 + (p_{n-2} - s_1p_{n-3})x^2 + (s_3p_{n-4} - s_4p_{n-5})x - s_4p_{n-6}$, where p_r = sum of homogeneous products of r dimensions of a, b, c, d .

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