On certain Quartic Curves having a Tacnode at Infinity at which the line at infinity is the Multiple Tangent. By HENRY M. JEFFERY.

[Read Feb. 9th, 1882.]

1. The quartics, which will be here classified, are comprised in the equation

$$\kappa\alpha^4 = u_3,$$

and may be considered in three divisions, according as the satellite-conic ($u_3 = 0$), which has double contact with the quartic of the third order in each case, is hyperbolic, elliptic, or parabolic. Varieties occur in these three main divisions, if the quadruple asymptote ($\alpha = 0$) pass through the centre of this conic, or is parallel to one of its asymptotes.

Since there is a tacnode at the extremity of the quadruple asymptote ($\alpha = 0$), at which the line at infinity is the multiple tangent, the class is at most the eighth, or the quartic may be reduced by critical values, and otherwise, to the sixth, fifth, and fourth classes.

2. The three main divisions are denoted with two parameters (κ, λ),

$$(1) \dots \kappa\alpha^4 = \beta\gamma + \lambda,$$

$$(2) \dots \kappa\alpha^4 = l\beta^2 + m\gamma^2 + \lambda,$$

$$(3) \dots \kappa\alpha^4 = \beta^2 + n\gamma + \lambda.$$

It is proposed to examine every possible case which can arise for varying values of κ, λ .

With this view, since changes can only occur (1) for critical quartics, (2) when the folium-depressions in an oval shrink to a folium-point or point of undulation, two mutual relations are investigated between κ, λ ,—(1) when the quartics in a group are critical, (2) when they have folium-points.

These mutual relations are (1) a quintic, quartic, or quadric in (κ, λ), and (2) the (λ) or (κ) axes of coordinates or parallels to them; these relations are here designated the first and second discriminating curves. Beside the tacnode, real or imaginary, there may be a crunode or an acnode; and all these quartics are bifolium, unifolium, or non-folium. The tacnode itself may be merged in a tacnode-cusp, a triple-point, or a stapete-point, thereby reducing the class of these quartics from the eighth to the fifth or fourth respectively.

3. To determine the tangential equivalent equation to a group of quartics with quadruple asymptotes, as above defined.

Let the quartic assume the general form

$$\kappa x^4 = ax + 2bxy + cy^2 + 2ex + 2dy + \lambda.$$

Combine it in the usual way with the line $(x\xi + y\eta = 1)$ to make the binary quartic homogeneous.

Then, if (ξ, η) touch the quartic, $S^3 = 27T^3$, where S, T are the quartic and sextic invariants,

$$S = \frac{1}{12}\phi^3 - \kappa(\lambda\eta^4 + 2d\eta^3 + c\eta^2),$$

where ϕ denotes, for brevity, the ends of the asymptotes, $a\eta^3 - 2b\xi\eta + c\xi^2$,

$$T = -\frac{1}{24}\phi^3 - \frac{1}{6}\kappa\phi\eta^3(\lambda\eta^2 - d\eta - \frac{1}{2}c) \\ + \frac{1}{2}\kappa\eta^4(e\eta - d\xi)^2 + \frac{1}{2}\kappa\eta^4[(bd - ce)\xi + (be - ad)\eta] - \frac{1}{2}\kappa\eta^4(ac - b^2).$$

(T is most readily calculated from its determinant-form.) The class is hence seen to be in general the eighth, since η^4 disappears as a factor from the equation to the class-curve

$$S^3 = 27T^3.$$

There is usually a tacnode at the end of the (y) axis, and these quartics, if they have no other singularities, belong to Dr. Salmon's Genus IV.

4. To determine the first discriminating curve, or the mutual relation which subsists between the parameters κ, λ , when there are critical quartics in this group.

Let the group of quartics be thus denoted

$$\kappa x^4 = ax^3 + 2bxy + cy^3 + 2ex + 2dy + \lambda.$$

For critical values, $2\kappa x^3 = ax + by + e,$

$$0 = bx + cy + d,$$

(the locus of nodes, viz., the axis of figure of the quartic, which also bisects the asymptotic triangle,)

$$0 = 2\lambda + 3ex + 3dy + ax^3 + 2bxy + cy^3.$$

The first discriminating curve is the quintic eliminant of

$$2\kappa cx^3 = (ac - b^2)x + ce - bd,$$

$$0 = 2\lambda c - 2d^2 + 3(ce - bd)x + (ac - b^2)x^3.$$

The asymptotes ($c\lambda = d^2$), and ($\kappa = 0$), correspond to the values a, ∞ of x . The point ($\kappa = 0, \lambda + ex + dy = 0$) corresponds to the centre of the conic $x_1 : y_1 : 1 :: bd - ce : be - ad : ac - b^2$.

COR. 1.—If $a=c=e=0$, or one of the satellite-asymptotes is also a quadruple asymptote,—this first curve becomes a line through the origin ($b^4\lambda = d^4\kappa$). See infra, § 13.

COR. 2.—If the satellite-conic be resolved into two lines, this first curve becomes the (λ) axis.

COR. 3.—If the satellite-conic be a parabola, $a = 0, b = 0$, this first

discriminating curve is a quartic with folium- and stapete-points

$$16\kappa(\lambda c - d^2)^2 + 27c^3e^4 = 0.$$

COR. 4.—If $b = c = 0$, or the quadruple asymptotes are diameters of the parabola, there are no critical quartics. See § 15.

COR. 5.—If the quadruple asymptote ($a = 0$) pass through the centre of the conic, $e = 0 = d$, and the locus is a hyperbola (see Fig. 1, § 8),

$$\kappa\lambda \{4c^2\lambda + (ac - b^2)^2\} = 0.$$

5. The first discriminating curve may also be usually obtained as the discriminant of a binary quantic.

Taking an axis of the conic as the (x) axis, the transformed equation becomes

$$-\kappa x^4 + ax^3 + cy^2 + 2ex + \lambda = 0 \equiv f(x).$$

At a double point, $\frac{df}{dx} = 0$, and $y = 0$.

The discriminant is $S^2 = 27T^2$,

where $S = -\kappa\lambda + \frac{1}{12}a^2$,

$$T = -\frac{1}{2}\kappa\lambda a - \frac{1}{24}a^3 + \frac{1}{2}\kappa e^2.$$

When expanded, besides the factor $\kappa = 0$, for the (λ) axis, this equation is

$$\kappa^2\lambda^2 + \frac{1}{2}\kappa\lambda^2a^2 - \frac{1}{2}\kappa\lambda a e^2 + \frac{1}{12}\kappa e^4 + \frac{1}{12}\lambda a^4 - \frac{1}{12}a^2e^2 = 0.$$

But this process may merely obtain points of contact.

6. To determine the second discriminating curve, or the mutual relation between the parameters, when quartics of this group have points of undulation.

At a folium-point, or point of undulation, two stationary points are consecutive. Hence $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$.

If these tests be applied to quartics of this group,

$$\kappa x^4 = ax^3 + cy^2 + 2ex + \lambda,$$

$$2\kappa x^3 = ax + e + cy \frac{dy}{dx},$$

$$6\kappa x^2 = a + c \left(\frac{dy}{dx}\right)^2,$$

$$\kappa x = 0.$$

(1) If $\kappa = 0$, the (λ) axis discriminates between the parameters for bifolium and non-folium quartics.

(2) If $x = 0$, $a\lambda = e^2$; and this parallel to the (κ) axis also discriminates the bifolium from the non-folium quartics, if a and c are of

opposite signs, or the quartics are hyperbolic. This appears also from the quartic equation

$$kx^4 = (lx + my + n)(lx - my + n).$$

To each point in the line ($a\lambda = e^2$), there corresponds a quartic with two folium-points.

In the hyperbolic division of quartics, if one of the satellite-asymptotes coincides with the line at infinity, each has a folium-point, as is evident from their equation

$$kx^4 = 2cx + 2dy + \lambda.$$

7. When folium-points occur on quartics of this group, they lie by pairs on quadruple asymptotes.

Let the quartic assume its most general form

$$kx^4 = ax^3 + 2bxy + cy^2 + 2ex + 2dy + \lambda.$$

If it be solved as a quadratic in y ,

$$bx + cy + d = \pm \sqrt{\phi(x)} \dots\dots\dots (A).$$

At a point of inflexion $\frac{d^2y}{dx^2} = 0$.

Hence $2\phi \frac{d^2\phi}{dx^4} = \left(\frac{d\phi}{dx}\right)^3$.

This sextic determines the abscissæ of the points of inflexion in each folium.

It may also be written to show, that this sextic has the same invariant as the binary quartic

$$\phi \frac{d^2\phi}{dx^2} = H(\phi),$$

where $H(\phi)$ denotes the Hessian, which contains the same square factor as its primitive, if it has one.

Its geometrical meaning is thus explained in this case. When the binary quartic $\phi(x) = 0$ has equal roots, the quartic oval shrinks to a point, and two equal values of the sextic unite at the same point.

Resuming the thesis, it appears from (A) that for every value of (y), there are two of (x) equi-distant from the polar of the tacnode

$$ax + by + e = 0.$$

Points of inflexion, and consequently, in their limiting form of union, folium-points lie on quadruple asymptotes.

To apply these processes, we will commence with simple cases.

8. To determine all the quartics with quadruple asymptotes, whose satellite-hyperbola has its asymptotes concurrent with one of the former.

DIVISION I.

Let a group of these central quartics be thus denoted—

$$y^2 = \kappa x^4 + x^2 - \lambda \dots \dots \dots (1).$$

The following sextic determines the abscissæ of its stationary points (§ 7) for one of the group,

$$(2\kappa x^2 + 1)(\kappa x^4 + x^2 - \lambda) = (6\kappa\lambda + 1)x^2 \dots (2).$$

The first discriminating curve (§ 4) is

$$\kappa\lambda(4\kappa\lambda + 1)^2 = 0.$$

The corresponding quartics are resolved into two coaxal parabolæ

$$\kappa y^2 = (\kappa x^2 + \frac{1}{4})^2.$$

If the axes of κ, λ divide plane space, for critical points (κ, λ) in the second quadrant the quartics are resolved into coaxal parabolæ, and for critical values in the fourth quadrant the quartic ovals become mere points or acnodes.

If $\kappa = 0$, the quartics are resolved into hyperbolæ and the line at infinity, counted twice.

If $\lambda = 0$, the quartics are bi-flecnodal, and of the sixth class.

The axes of κ, λ also constitute the second discriminating curve (§ 6).

In the first quadrant let $\kappa = 1, \lambda = 1$.

The values of x^2 in (1) are $> .6$, and the only positive root in (2) is 1.3. Hence there are four inflexions, since the curve is symmetrical. The quartic is doubly campaniform, whose two branches meet in a tacnode at infinity, thus resembling a conchoid of Nicomedes.

In the third quadrant, let $\kappa = -1, \lambda = -1$.

The values are complementary to those obtained from the first quadrant.

In (1) $x^2 > 0 < 1.62$, and in (2) x^2 has a value 1.3, the other two being negative and positive > 1.6 .

Hence the quartic is a bifolium-oval, like a double purse.

In the second quadrant, λ is negative, and two cases occur, as

$$\kappa > < -\frac{1}{4\lambda}.$$

The values of x^2 in (2) are negative, and unlimited in (1).

Hence the quartics are non-folium, resembling the pair of parabolæ, which is their limiting and critical form: the companion-quartics do or do not intersect the (λ) axis, as $\kappa > < -\frac{1}{4\lambda}$.

Fig. 1.

In the fourth quadrant, κ is negative, and $< \frac{1}{4\lambda}$.

If $\kappa = \frac{1}{4\lambda}$, $y^2 + \lambda \left(1 - \frac{x^2}{2\lambda}\right)^2 = 0$, the oval shrinks to a point.

If $\kappa < \frac{1}{4\lambda}$, the real value of x^2 in (2) is beyond the limits of the oval, as defined by (1).

The numbers in the diagrams denote the number of folium-depressions.

9. To determine all the quartics with a quadruple asymptote, which passes through the centre of its satellite ellipse.

DIVISION II.

Let a group of these central quartics be thus denoted— $y^2 = \kappa x^4 - x^2 - \lambda$.

If $-x^2$ be substituted for $+x^2$ in § 8, the analysis is applicable, and the results seen to be complementary to those there obtained.

In the first quadrant, if $\kappa = \lambda = 1$, all the values of $x^2 > 1.6$, and in the sextic (2) only one positive value exceeds this limit.

The quartic is bifolium, and doubly campaniform.

In the third quadrant, if $\kappa = \lambda = -1$, the values of x^2 in (1) are $> 0 < .62$, whereas the positive value of (2) is 1.3. The quartic is a non-folium oval.

In the second quadrant, λ is negative, and two cases should be considered.

If $\kappa = -\frac{1}{4\lambda}$, the quartic is resolved into two coaxal parabolas.

If $\kappa = 1$, $\lambda = -1$, the real value of the sextic (2) is .2, and x^2 may have all values in (1). The quartic is bifolium and doubly campaniform.

If $\kappa = 1$, $\lambda = -.1$, the values of x^2 in (1) are $< .112$ and $> .887$.

The only possible value of x^2 in (2) is 1.3. This quartic is also bifolium, but bipartite with a small oval about the origin.

In the fourth quadrant, no quartic is possible, so that the lower branch of the first discriminating hyperbola is inapplicable.

10. To determine all the quartics with a quadruple asymptote, whose satellite hyperbola is unrestricted. (Division I.)

Fig. 2.

Let a group in this general case have this equation $y^2 = \kappa x^4 + x^2 + 2x - \lambda$ (1), the satellite-conic being an equilateral hyperbola.

The following sextic gives the abscissæ of the points of inflexion in each folium:—

$$(6\kappa x^3 + 1)(\kappa x^4 + x^2 + 2x - \lambda) = (2\kappa x^3 + x + 1)^2 \dots\dots(2).$$

The first discriminating curve consists of a quintic (§ 4), and the (λ) axis,

$$\kappa(16\kappa^2\lambda^3 + 8\kappa\lambda^2 + 36\kappa\lambda + 27\kappa + \lambda + 1) = 0.$$

The second discriminating curve consists of two lines (§ 6),

$$\kappa = 0, \quad \lambda + 1 = 0.$$

If κ is positive, real branches touch the line at infinity in a tacnode; if κ is negative, the tacnode is imaginary, and the quartics consist of one or two closed ovals.

Critical values occur in all quadrants but the first.

The description given in § 8 is here applicable, except for the looped portion in the third and fourth quadrants.

For points on the loop, the quartic is acnodal, and for values of (κ, λ) inside the loop, the quartic is bipartite, one of the ovals being bifolium, if it lie to the left of the discriminating line $(\lambda + 1 = 0)$,* and in other cases non-folium.

11. To determine all the quartics with a quadruple asymptote, whose satellite-ellipse is unrestricted. (Division II.)

Let such a group be thus denoted—

$$y^2 = \kappa x^4 - x^2 - 2x - \lambda.$$

The satellite-conic is selected, without sacrificing generality, to be the circle of curvature at the points of double contact.

Now, the calculations necessary to draw the quartics corresponding to the several positions of (κ, λ) in the hyperbolic group of § 10, will suffice here also, if $-\kappa, -\lambda$ be written for κ, λ .

The first discriminating curve is identical with the former, when turned upside down.

The second discriminating line is the (λ) axis alone, since $y^2 + (x+1)^2$ denotes a pair of imaginary lines.

Fig. 3.

Fig. 4.

* The true position of the digit (2) in the wood-cut can hardly be discerned.

The description given in § 10 can be adopted, except for the loop in the first and second quadrants.

For values of (κ, λ) on the loop, the quartics are crunodal, and values within the loop, bipartite; the inner portion is a closed oval, and the outer doubly campaniform, whose branches unite and touch the line at infinity at the tacnode.

If (κ, λ) is on the first discriminating curve in the fourth quadrant, the quartic is reduced to an acnode.

The preceding quartics are of the eighth class, unless they are critical, when they are of the sixth class.

12. In the special case, when one satellite-asymptote coincides with the line at infinity, the tacnode at infinity becomes a stapete-point, and the quartics are of the fourth class. (Division I.) (Dr. Salmon's Genus IX.)

If the equation to the group be taken to be

$$\kappa x^4 = 2ex + 2dy + \lambda,$$

its tangential equivalent is (§ 3), after rejecting η^6 , since two cusps and a node are merged in a stapete-point,

$$16\kappa\eta(\lambda\eta + 2d)^3 + 27(e\eta - d\xi)^4 = 0.$$

There is no critical quartic, and all have points of undulation (§ 6).

13. In the special case, when one satellite-asymptote is also a quadruple asymptote, the tacnode becomes either a tacnode-cusp or a triple point, and the quartics are of the fifth class. (Dr. Salmon's Genus VIII.) (Division I.)

Let the equation to the group be

$$\kappa x^4 = 2bxy + 2dy + \lambda.$$

Then, by § 3, the tangential equivalent is found from the invariants

$$S = \frac{1}{3}b^3\xi^3\eta^3 - \eta\kappa^3(\lambda\eta + 2d),$$

$$T = \frac{1}{3}\eta b^3\xi^3\eta^3 + \frac{1}{3}\kappa b\xi\eta^4(\lambda\eta - d) + \frac{1}{4}\kappa\eta^4(d\xi + b)^3.$$

In simplifying the equivalent ($S^3 = 27T^2$), η^7 is rejected, which represents a cusp at infinity on the line $(bx + d = 0)$, and the tacnode at infinity.

The line $(b^4\lambda = d^4\kappa)$ is the locus of points (κ, λ) , which correspond to quartics, which are resolved into cubics and lines

$$\kappa(b^4x^4 - d^4) = 2b^4(bx + d)y.$$

The axes $(\kappa = 0, \lambda = 0)$ distinguish unifolium from non-folium quartics. The unifolium quartics have two concurrent cusps at the extremity of the (y) axis, whose dual form exhibits two points of inflexion on a common stationary tangent, the (x) axis. The non-folium quartics

have a triple point, in the same position at infinity, at which an ordinary tangent passes through a cusp. The former curves may be viewed as quartic tridents, with the folium-depression in each for a handle, the latter as Beta-shaped.

14. To determine all the quartics with quadruple asymptotes, whose satellite-parabola is unrestricted. (Division III.)

Such a group may be written

$$\kappa x^4 = cy^3 + 2ex + \lambda,$$

whose first discriminating curve is the quartic of fourth class and order

$$16\kappa\lambda^3 + 27e^4 = 0.$$

The (λ) axis discriminates the bifolium from the non-folium quartics; points (κ, λ) above the (λ) axis give the former curves.

By writing the equation to the critical curves in the form

$$16\lambda^3 y^3 + (2\lambda + 3ex)^2 \{ (ex - 2\lambda)^2 + 2e^2 x^2 \} = 0,$$

it is seen that acnodes correspond to positive values of λ , and crunodes to negative values.

Non-singular quartics in this division are of the eighth class.

If $e=0$, the satellite is resolved into two parallel lines, or asymptotes whose points of contact are points of undulation ($cy^3 + \lambda = 0$).

The class of these quartics is also the eighth.

15. To determine all the quartics with quadruple asymptotes, which are also the diameters of their satellite parabola. (Division III.) (Dr. Salmon's Genus IX.)

Let the equation to this group be

$$\kappa x^4 = ax^3 + 2dy + \lambda.$$

Each quartic has a stapede-point at infinity, which reduces the class to the fourth.

The tangential equivalent is

$$S^3 = 27T^2,$$

where

$$\frac{S}{\eta^3} = \frac{1}{18}\eta - \kappa(\lambda\eta + d),$$

$$\frac{T}{\eta^4} = \frac{1}{18}\eta^2 - \frac{1}{3}\kappa\eta(\lambda\eta - d) + \frac{1}{4}\kappa d^2 \xi^2 - \frac{1}{4}\kappa d\eta.$$

It is seen that η^3 is rejected as a factor, so that the class is the fourth.

There are no critical quartics in this group, but the hyperbola

$$4\kappa\lambda + a^2 = 0$$

determines the (x) axis to be a bitangent.

The (λ) axis discriminates between unifolium and non-folium quartics.

16. To show that there is a focus in the tacnode.

Let the group be denoted by the trilinear equation

$$\kappa a^4 a^4 = bc\beta\gamma(2\Delta)^3 + \lambda(2\Delta)^4.$$

If the line ($aap + b\beta q + c\gamma r = 0$) touches the quartic, $S^3 = 27T^3$, where

$$S = \frac{1}{1^3} (r-p)^2 (p-q)^2 - \kappa\lambda (r-\phi)^4 + \kappa q r (r-q)^2,$$

$$T = -\frac{1}{2 \cdot 1^3} (r-p)^3 (p-q)^3 - \frac{1}{3} \kappa\lambda (r-q)^4 (p-q)(r-p) \\ + \kappa (r-q)^2 \left\{ \frac{1}{1^3} p^3 (r-q)^2 - \frac{1}{1^3} (r-p)(p-q)qr \right\}.$$

The class is the eighth, as has been shown, $(r-q)^4$ being rejected for the tacnode. When this octavic ($S^3 = 27T^3$), thus reduced, is expanded in a series of powers of $(r-q)^2$, the first term is

$$\frac{1}{1^3} \kappa\lambda (r-p)^4 (p-q)^4 - \frac{1}{1^3} \kappa^2 q^2 r^2 (r-p)^2 (p-q)^2.$$

If we substitute for $(r-p)(p-q)$ its value derived from the fundamental relation

$$4\Delta^2 = a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q),$$

the octavic takes the form, which proves the property in question

$$(r-q)u_7 = 4\Delta^2 v_6,$$

since $(r-q=0)$ denotes the extremity of BC , the side of reference. There is no trace of an analogue to the theorem in Spherics, that if a quartic have a quadruple cyclic arc, the quadrantal pole of that arc is a quadruple focus.

In the case where one satellite-asymptote of these quartics coincides with the line at infinity, the focus at the extremity of the (y) axis is thus shown.

Let the equation of § 12, ($\kappa x^4 = 2dy + \lambda$),

be transformed to

$$\kappa x^4 = 2fy.$$

Its tangential equivalent is $27f\xi^4 + 32\kappa\eta = 0$.

Let the constants be so chosen that, for convenience,

$$2\xi^4 + \eta = 0.$$

By Plücker's definition of a focus (x', y') , the quartic must be touched by the imaginary asymptote drawn through it,

$$x - x' + i(y - y') = 0.$$

Hence the tangential equivalent must be satisfied by the coordinates

$$\xi : \eta : 1 :: 1 : i : (x' + iy')^{-1}.$$

Therefore

$$2 + i(x + iy)^3 = 0,$$

with a factor $(x + iy)^{-1}$, if the accents are dropped from x', y' . So also

$$2 - i(x - iy)^3 = 0.$$

The foci are found from the simultaneous equations

$$\left. \begin{aligned} 2-y(3x^2-y^2) &= 0 \\ x(x^2-3y^2) &= 0 \end{aligned} \right\}.$$

Three real foci have the coordinates

$$x = 0, \quad y + \sqrt[3]{2} = 0; \quad x^2 = 3y^2, \quad y = \frac{1}{3}\sqrt[3]{2}.$$

The situation of the fourth as of the first real focus is indicated by the form of the tangential equation

$$-2(\xi^2 + \eta^2)^2 + 4\eta^2(\xi^2 + \eta^2) = \eta(2\eta^3 + 1).$$

In like manner, the foci may be found in all the other cases; and one real focus, single or double, lies at the extremity of the (y) axis.

17. The general form of quartics, which have a tacnode at infinity, and quadruple asymptotes, is not here considered,

$$(x+c_1)(x+c_2)(x+c_3)(x+c_4) = u_1.$$

By quadruple asymptote is meant any line, which meets the curve in two coincident points of the tacnode, in whose direction the directions of the four possible asymptotes to a quartic are merged.

It appears from § 4, Cor. 5, that a change in the coordinates of the quartic group alters the degree of the discriminating curve.

Thursday, March 9th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. A. Buchheim, B.A., Scholar of New College, Oxford, was elected a member.

The following communications were made:—

“In how many independent ways can a Polygon of $2n$ sides be divided into Quadrilaterals by means of non-intersecting diagonals? with an extension to the general case of division into p -gons:” Mr. H. M. Taylor and Prof. Rowe (by whom the paper was communicated).

“Systems of Formulæ for the m , cn , and dn of $u+v+w$:” Prof. W. Woolsey Johnson (communicated by Mr. Glaisher).

“Remarks on the preceding paper,” and “Some Elliptic Function Formulæ:” Rev. M. M. U. Wilkinson.

Two Notes—(1) “Proof that

$$\int_0^\infty \frac{X^{m-1}dx}{X^{2p+1}+1} = \frac{\pi}{(2p+1) \sin \frac{m\pi}{2p+1}},$$

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if $m < 2p + 1$, and both m and p positive integers;" (2.) "On the surface

$$\frac{a^2x^2}{r^2(a^2-r^2)} + \frac{b^2y^2}{r^2(b^2-r^2)} + \frac{c^2z^2}{r^2(c^2-r^2)} = 0,$$

its Form, Conical Points, and singular Tangent Planes:" Mr. C. E. Bickmore.

"Note of Proofs of the Addition-Theorem for the Second Elliptic Integral, and of Fagnani's Theorem by Confocal Conics:" Mr. J. J. Walker.

The following presents were received:—

"Educational Times," March, 1882.

"Reprint of Mathematics from the Educational Times," Vol. xxxvi.: from the Publishers.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück 2, No. 2.

"Bulletin de la Société Mathématique de Paris," Tome ix., No. 5.

"Monatsbericht," December, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 6°; Roma.

"The Mathematical Magazine, a Journal of Elementary Mathematics," edited and published by Artemas Martin, M.A., Erie, Pa., Pt. 1; 1882.

"Sitzungsberichte der Physikalisch-medizinischen Societät zu Erlangen," 13 Heft; November 1880, bis August 1881.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. Fr. Gomes Teixeira, Vol. iii., Nos. 4, 5, 6; Coimbra.

"Boletin de la Institucion libre de Ensenanza," Año vi., Dec. 31, 1881.

Systems of Formulæ for the sn, cn, and dn of $u+v+w$. By WILLIAM WOOLSEY JOHNSON, Professor of Mathematics in the Naval Academy, Annapolis, Maryland, U.S.

[Read March 9th, 1882.]

1. In the *Messenger of Mathematics*, for July, 1881, Mr. Glaisher has given formulæ for the sn, cn, and dn of $u+v+w$, in which the common denominator is a rational function of $\text{sn}^2 u$, $\text{sn}^2 v$, and $\text{sn}^2 w$; and he has also recorded formulæ first given by Professor Cayley, in which the common denominator is not rational, but which reduce when $w=0$ to the usual forms of the sn, cn, and dn of $u+v$, in which the common denominator is rational in $\text{sn}^2 u$ and $\text{sn}^2 v$. This communication contains the two systems of formulæ for $u+v+w$ of which those just mentioned are members.

2. If for sn, cn and dn we substitute $\frac{s}{n}$, $\frac{c}{n}$ and $\frac{d}{n}$, and use the suffixes 1 and 2 to refer to the arguments u and v , and the large letters VOL XIII.—NO. 186.

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