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On Oubic Curves as connected with certain Triangles in Perspective.

By SAMUEL ROBERTS. Received and read May 13th, 1897.

1. Consider a triangle $A'B'O'$ in perspective with the triangle of reference ABC , the connectors AA' , BB' , CC' meeting in O , the centre of perspective (or homology). The coordinates of the vertices A' , B' , C' may be supposed proportional to

$$\begin{array}{ccc} h_1x, & y, & z, \\ x, & h_2y, & z, \\ x, & y, & h_3z, \end{array}$$

where x, y, z are the trilinear coordinates of O .

Write x', y', z' for the coordinates of A' ; then

$$\frac{x'}{x} = \frac{A'D}{OD}, \quad \frac{y'}{y} = \frac{z'}{z} = \frac{AA'}{AO},$$

D being the point in which AO meets the side of ABC opposite A .

Hence we have

$$x' : y' : z' :: \frac{AO}{OD} \cdot \frac{A'D}{AA'} x : y : z,$$

and h_1 is the anharmonic ratio of the range $AA'OD$; in like manner h_2, h_3 are the anharmonic ratios of the ranges $BB'OE, CC'OF$, if E, F are the points in which BO, CO meet the sides of ABC opposite to B, C .

2. Take other three points A'', B'', C'' whose coordinates are proportional to

$$(X_1, Y_1, Z_1),$$

$$(X_2, Y_2, Z_2),$$

$$(X_3, Y_3, Z_3),$$

respectively, and let the triangle $A''B''C''$ be in perspective with the triangle $A'B'C'$, corresponding vertices being denoted by the same letters. If α, β, γ are the coordinates of the centre of perspective, the following conditions hold:—

$$\left. \begin{aligned} xh_1(\gamma Y_1 - \beta Z_1) + y(\alpha Z_1 - \gamma X_1) + z(\beta X_1 - \alpha Y_1) &= 0, \\ x(\gamma Y_2 - \beta Z_2) + yh_2(\alpha Z_2 - \gamma X_2) + z(\beta X_2 - \alpha Y_2) &= 0, \\ x(\gamma Y_3 - \beta Z_3) + y(\alpha Z_3 - \gamma X_3) + zh_3(\beta X_3 - \alpha Y_3) &= 0. \end{aligned} \right\} \quad (1)$$

If h_1, h_2, h_3 are constant, the eliminant with regard to x, y, z , equated to zero, gives a cubic locus. The equation is

$$\begin{aligned} &\gamma^3 \alpha \{ X_1 Y_1 Y_3 (h_1 h_2 h_3 - h_2) + X_3 Y_1 Y_2 (1 - h_1) + X_1 Y_2 Y_3 (1 - h_2) \} \\ &- \beta^3 \alpha \{ X_3 Z_1 Z_2 (h_1 h_2 h_3 - h_3) + X_2 Z_1 Z_3 (1 - h_1) + X_1 Z_2 Z_3 (1 - h_2) \} \\ &+ \alpha^3 \beta \{ Y_3 Z_1 Z_2 (h_1 h_2 h_3 - h_3) + Y_1 Z_2 Z_3 (1 - h_2) + Y_2 Z_1 Z_3 (1 - h_1) \} \\ &- \gamma^3 \beta \{ X_2 X_3 Y_1 (h_1 h_2 h_3 - h_1) + X_1 X_2 Y_3 (1 - h_2) + X_1 X_3 Y_2 (1 - h_3) \} \\ &+ \beta^3 \gamma \{ X_2 X_3 Z_1 (h_1 h_2 h_3 - h_1) + X_1 X_3 Z_2 (1 - h_2) + X_1 X_2 Z_3 (1 - h_3) \} \\ &- \alpha^3 \gamma \{ Y_1 Y_3 Z_2 (h_1 h_2 h_3 - h_2) + Y_2 Y_3 Z_1 (1 - h_3) + Y_1 Y_2 Z_3 (1 - h_1) \} \\ &+ \alpha \beta \gamma \{ (X_3 Y_1 Z_2 - X_2 Y_3 Z_1) (h_1 h_2 h_3 - 1) + (X_3 Y_2 Z_1 - X_2 Y_1 Z_3) h_1 \\ &\quad + (X_2 Y_1 Z_3 - X_1 Y_3 Z_2) h_2 + (X_1 Y_3 Z_2 - X_3 Y_2 Z_1) h_3 \} = 0. \quad (2) \end{aligned}$$

The curve passes through the vertices of the triangle of reference, and also [as the form of (1) shows] through those of the triangle $A''B''C''$.

There are three constants h_1, h_2, h_3 at our disposal to determine

three other points on the curve. Any curve of the third degree can therefore be represented by an equation of the foregoing form, since we can always take six points on the curve, no three of which shall be collinear. In this case there are ten pairs of triangles, and we can take any one of the triangles for reference. Moreover the correspondence of vertices may be varied. If we include collinear triplets of points, the numbers of possible triangles of reference will be so far diminished.

The above result may be stated thus :—If through the vertices of a triangle ABO a variable triplet of right lines concurrent at O be drawn meeting the opposite sides in D, E, F , and the vertices of variable triangles $A'B'O'$ be determined so that the anharmonic ratios of the ranges $AA'OD, BB'OE, CC'OF$ are constant, the locus of the centres of homology of such of these triangles as are in perspective with a fixed triangle $A''B''C''$ is a cubic curve through the vertices of the two fixed triangles.

We may take the second triangle for reference and represent the same curve by an equation of the same form.

$$3. \text{ Let } \left. \begin{aligned} A_1\alpha + B_1\beta + C_1\gamma &= 0, \\ A_2\alpha + B_2\beta + C_2\gamma &= 0, \\ A_3\alpha + B_3\beta + C_3\gamma &= 0, \end{aligned} \right\} \quad (3)$$

be the equations of the sides of the second triangle opposite respectively to the vertices $A''B''C''$ which correspond to A, B, C .

The coordinates of the intersections of three straight lines through the point (α, β, γ) with the sides (3) respectively are proportional to

$$\begin{array}{ccc} -\frac{B_1\beta + C_1\gamma}{A_1}, & \beta, & \gamma, \\ \alpha, & -\frac{A_2\alpha + C_2\gamma}{B_2}, & \gamma, \\ \alpha, & \beta, & -\frac{A_3\alpha + B_3\beta}{C_3}. \end{array}$$

Now, take straight lines through these points respectively, and through the corresponding opposite vertices of the second fixed triangle $A''B''C''$. The equations are of the form

$$x(Y_1\gamma - Z_1\beta) + y\left(-\frac{B_1\beta + C_1\gamma}{A_1} - X_1\gamma\right) + z\left(\frac{B_1\beta + C_1\gamma}{A_1} + X_1\beta\right) = 0.$$

&c. &c.

For brevity, denote $Y_1\gamma - Z_1\beta$ by $(Y_1\gamma)'$, and so in like cases. The condition that the above lines shall meet in a point is

$$\begin{vmatrix} A_1(Y_1\gamma)', & B_1(Y_1\gamma)' - \Delta\gamma, & C_1(Y_1\gamma)' + \Delta\beta \\ A_2(Z_2\alpha)' + \Delta\gamma, & B_2(Z_2\alpha)', & C_2(Z_2\alpha)' - \Delta\alpha \\ A_3(X_3\beta)' - \Delta\beta, & B_3(X_3\beta)' + \Delta\alpha, & C_3(X_3\beta)' \end{vmatrix} = 0, \quad (4)$$

where Δ means the determinant

$$\begin{vmatrix} X_1Y_1Z_1 \\ X_2Y_2Z_2 \\ X_3Y_3Z_3 \end{vmatrix},$$

A_1, B_1, C_1 are $(Y_2Z_3)'$, $(Z_2X_3)'$, $(X_2Y_1)'$, respectively, and the sets A_2, B_2, C_2 , and A_3, B_3, C_3 have analogous values.

The determinant (4) is separable into

$$\begin{aligned} & \begin{vmatrix} A_1(Y_1\gamma)', & B_1(Y_1\gamma)', & C_1(Y_1\gamma)' \\ A_2(Z_2\alpha)', & B_2(Z_2\alpha)', & C_2(Z_2\alpha)' \\ A_3(X_3\beta)', & B_3(X_3\beta)', & C_3(X_3\beta)' \end{vmatrix} \quad (a) \\ + & \begin{vmatrix} A_1(Y_1\gamma)', & B_1(Y_1\gamma)', & \Delta\beta \\ A_2(Z_2\alpha)', & B_2(Z_2\alpha)', & -\Delta\alpha \\ A_3(X_3\beta)', & B_3(X_3\beta)', & 0 \end{vmatrix} \quad (b) \\ + & \begin{vmatrix} 0, & B_1(Y_1\gamma)', & C_1(Y_1\gamma)' \\ \Delta\gamma, & B_2(Z_2\alpha)', & C_2(Z_2\alpha)' \\ -\Delta\beta, & B_3(X_3\beta)', & C_3(X_3\beta)' \end{vmatrix} \quad (c) \\ + & \begin{vmatrix} A_1(Y_1\gamma)', & -\Delta\gamma, & C_1(Y_1\gamma)' \\ A_2(Z_2\alpha)', & 0, & C_2(Z_2\alpha)' \\ A_3(X_3\beta)', & \Delta\alpha, & C_3(X_3\beta)' \end{vmatrix} \quad (d) \\ + & \begin{vmatrix} 0, & B_1(Y_1\gamma)', & \Delta\beta \\ \Delta\gamma, & B_2(Z_2\alpha)', & -\Delta\alpha \\ -\Delta\beta, & B_3(X_3\beta)', & 0 \end{vmatrix} \quad (e) + \begin{vmatrix} 0, & -\Delta\gamma, & C_1(Y_1\gamma)' \\ \Delta\gamma, & 0, & C_2(Z_2\alpha)' \\ -\Delta\beta, & \Delta\alpha, & C_3(X_3\beta)' \end{vmatrix} \quad (f) \\ + & \begin{vmatrix} A_1(Y_1\gamma)', & -\Delta\gamma, & \Delta\beta \\ A_2(Z_2\alpha)', & 0, & -\Delta\alpha \\ A_3(X_3\beta)', & \Delta\alpha, & 0 \end{vmatrix} \quad (g) + \begin{vmatrix} 0, & -\Delta\gamma, & \Delta\beta \\ \Delta\gamma, & 0, & -\Delta\alpha \\ -\Delta\beta, & \Delta\alpha, & 0 \end{vmatrix} \quad (h). \end{aligned}$$

But (e) + (f) + (g) gives

$$\begin{aligned} & \alpha^2 \gamma (Y_1 Y_2 Z_3 + Y_2 Y_3 Z_1) - \alpha^2 \beta (Y_1 Z_2 Z_3 + Y_2 Z_1 Z_3) \\ & \quad + \beta^2 \alpha (X_2 Z_1 Z_3 + X_1 Z_2 Z_3) - \&c. \\ & + \alpha \beta \gamma (X_3 Y_1 Z_2 - X_2 Y_3 Z_1) + 2 (Y_1 \gamma)' (Z_2 \alpha)' (X_3 \beta)'. \end{aligned}$$

Moreover (a) + (b) + (c) + (d) gives

$$-2 (Y_1 \gamma)' (Z_2 \alpha)' (X_3 \beta)',$$

and (h) is zero. Therefore the net result is

$$\alpha^2 \gamma (Y_1 Y_2 Z_3 + Y_2 Y_3 Z_1) - \&c. + \alpha \beta \gamma (X_3 Y_1 Z_2 - X_2 Y_3 Z_1) = 0.$$

This equation is the same, except the reversal of signs, as that obtained from (2) by making

$$h_1 = h_2 = h_3 = 0,$$

and the new variable triangle is inscribed in the second triangle, while the locus is the same.

4. The foregoing result depends on the following geometrical theorem, which is probably known, though I have not met with it: Given two triangles ABO , $A'B'O'$, and three right lines through the vertices of ABO concurrent in O_1 , and meeting the opposite sides in D , E , F ; if DEF , $A'B'O'$ are in perspective, and P is the centre of perspective, right lines through A , B , O concurrent in P will meet corresponding sides of $A'B'O'$ in points D' , E' , F' such that right lines joining these points to opposite vertices will be concurrent in O_2 .

Thus, in Fig. 1, we have

$$\begin{aligned} \frac{AF}{FB} &= \frac{\sin APF}{\sin FPB} \frac{AP}{BP}, & \frac{BD}{DC} &= \frac{\sin BPD}{\sin DPC} \frac{BP}{CP}, & \frac{CE}{EA} &= \frac{\sin CPE}{\sin EPA} \frac{CP}{AP}, \\ \frac{A'F'}{F'B'} &= \frac{\sin DPC}{\sin CPE} \frac{A'P}{B'P}, & \frac{B'D'}{D'C'} &= \frac{\sin EPA}{\sin APF} \frac{B'P}{C'P}, & \frac{C'E'}{E'A'} &= \frac{\sin FPB}{\sin BPD} \frac{C'P}{A'P}, \end{aligned}$$

and since, by the theorem of Ceva,

$$\frac{AF}{FB} \frac{BD}{DC} \frac{CE}{EA} = 1,$$

we have also

$$\frac{A'F'}{F'B'} \frac{B'D'}{D'C'} \frac{C'E'}{E'A'} = 1.$$

FIG. 1.

5. It appears that (2) gives the same equation when, instead of being zero, h_1, h_2, h_3 have the harmonic value -1 . In fact, the variable triangle and the centre of perspective relative to the triangle of reference then constitute a quadrangle whose harmonic triangle is that of reference. Thus in (Fig. 2) a, b, c, d being the quadrangle, the right lines through any point P and through a, b, c concurrent in P meet the corresponding sides of the harmonic triangle ABC in D, E, F , and right lines through these points and corresponding vertices meet in O .

6. When h_1, h_2, h_3 are made equal to h in (2), the equation becomes, after division by $1-h$,

$$(h^2 + h) \{ (Y_1\gamma)' (Z_2\alpha)' (X_3\beta)' \} \\ + \{ (Z_1\alpha)' (X_2\beta)' (Y_3\gamma)' + (X_1\beta)' (Y_2\gamma)' (Z_3\alpha)' \} = 0.$$

The function multiplied by $h^2 + h$, equated to zero, represents the right lines joining corresponding vertices of the two fixed triangles;

the remaining function, equated to zero, gives the locus for $h = 0$ or -1 . Thus we have for different values of the anharmonic ratio h a sheaf of cubics passing through the same nine points on the three right lines joining corresponding vertices of the two triangles.

FIG. 2.

7. If, from the conditions (1), α, β, γ are eliminated, instead of x, y, z , we get the equation of another cubic locus.

In fact, the conditions may be written

$$\alpha (yZ_1 - zY_1) + \beta (zX_1 - xZ_1h_1) + \gamma (xY_1h_1 - yX_1) = 0,$$

$$\alpha (yZ_2h_2 - zY_2) + \beta (zX_2 - xZ_2) + \gamma (xY_2 - yX_2h_2) = 0,$$

$$\alpha (yZ_3 - zY_3h_3) + \beta (zX_3h_3 - xZ_3) + \gamma (xY_3 - yX_3) = 0.$$

Now, if we divide the left-hand member of the first of these equations by h_1 , it becomes

$$\alpha \frac{1}{h_1} (yZ_1 - zY_1) + \beta \left(z \frac{X_1}{h_1} - xZ_1 \right) + \gamma \left(xY_1 - y \frac{X_1}{h_1} \right) = 0,$$

practically the same form as the first of the set (1), (α, β, γ) and (x, y, z) being interchanged, and $\frac{1}{h_1}, \frac{X_1}{h_1}$ being written for h_1 and X_1 .

Applying the same remark, *mutatis mutandis*, to the remaining equations, we can derive the locus when α, β, γ are eliminated from that obtained when x, y, z are eliminated.

The equation thus obtained is

$$\begin{aligned} & z^2x \{ X_1Y_1Y_3(1-h_1h_3) + X_3Y_1Y_2(h_1h_2-h_3) + X_1Y_2Y_3(h_3-1) \} \\ & - y^2x \{ X_3Z_1Z_2(1-h_1h_3) + X_2Z_1Z_3(h_1h_3-h_2) + X_1Z_2Z_3(h_2-1) \} \\ & + \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ & + xyz \{ (X_3Y_1Z_2 - X_2Y_3Z_1)(1-h_1h_2h_3) + (X_2Y_1Z_3 - X_3Y_2Z_1)h_1 \\ & \quad + (X_1Y_3Z_2 - X_2Y_1X_3)h_2 + (X_3Y_2Z_1 - X_1Y_3Z_2)h_3 \} = 0. \quad (5) \end{aligned}$$

The coefficient xyz here is the same as that of $\alpha\beta\gamma$ in (2) with changed sign, showing that this term disappears from the two equations simultaneously, if at all.

The curve passes through the points given by

$$\begin{aligned} & \frac{X_1}{h_1}, \quad Y_1, \quad Z_1, \\ & X_1, \quad \frac{Y_2}{h_2}, \quad Z_2, \\ & X_3, \quad Y_3, \quad \frac{Z_3}{h_3}, \end{aligned}$$

and through the vertices of the triangle of reference. Of course, it is the locus of the centres of homology of the triangle of reference and the variable triangle.

8. It appears from (2) and (5) that, if $h_1 = h_2 = h_3$, and the coefficients of $\alpha\beta\gamma$ or xyz vanish, the triangle $A''B''C''$ must be in perspective with the triangle ABC . By way of introducing this condition, write

$$\begin{aligned} & k_1X, \quad Y, \quad Z, \quad \text{for} \quad X_1, \quad Y_1, \quad Z_1, \\ & X, \quad k_2Y, \quad Z, \quad ,, \quad X_2, \quad Y_2, \quad Z_2, \\ & X, \quad Y, \quad k_3Z, \quad ,, \quad X_3, \quad Y_3, \quad Z_3. \end{aligned}$$

The first equation becomes

$$\begin{aligned} & \gamma^2\alpha XY^2 \{ h_1h_2h_3 - k_2h_1 - h_2 - k_1k_2h_3 + k_1k_2 + k_3 \} \\ & - \beta^2\alpha XZ^2 \{ h_1h_2h_3 - k_3h_1 - k_1k_3h_2 - h_3 + k_1k_3 + k_3 \} \\ & + \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ & + \alpha\beta\gamma XYZ \{ (k_2 - k_3)h_1 + (k_3 - k_1)h_2 + (k_1 - k_2)h_3 \} = 0; \quad (6) \end{aligned}$$

the curve now passes through the point (X, Y, Z) , a seventh point determined.

The second equation becomes

$$\begin{aligned} & y^2 x Z^3 X (1 + h_1 h_3 k_3 + k_1 k_3 h_3 - k_3 h_1 - h_1 h_3 - k_1 k_3) \\ & - x^2 x Y^3 X (1 + h_1 h_3 k_3 + k_1 k_3 h_3 - k_3 h_1 - h_1 h_3 - k_1 k_3) \\ & + \dots \dots \dots \dots \dots \dots \dots \dots \\ & + xyz XYZ \{ (k_3 - k_1) h_1 + (k_3 - k_1) h_2 + (k_1 - k_3) h_3 \} = 0, \end{aligned} \quad (7)$$

and may be derived from the previous form by writing $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$ for h_1, h_2, h_3 , $\frac{k_1}{h_1}, \frac{k_2}{h_2}, \frac{k_3}{h_3}$ for k_1, k_2, k_3 , and x, y, z , for α, β, γ , respectively, and *vice versa*.

The coefficients of $\alpha\beta\gamma, xyz$ vanish for

$$h_1 = h_2 = h_3 \quad \text{or} \quad k_1 = k_2 = k_3.$$

$$\text{If} \quad h_1 = h_2 = h_3 = 0 \quad \text{or} \quad -1,$$

$$\text{or if} \quad k_1 = k_2 = k_3 = 0 \quad \text{or} \quad -1,$$

the tangents to either of the curves (6) and (7) at the vertices of the triangle of reference meet in one point, and the equations are reducible directly to the form

$$A \frac{\alpha}{m} \left(\frac{\beta^2}{n^2} - \frac{\gamma^2}{p^2} \right) + B \frac{\beta}{n} \left(\frac{\gamma^2}{p^2} - \frac{\alpha^2}{m^2} \right) + C \frac{\gamma}{p} \left(\frac{\alpha^2}{m^2} - \frac{\beta^2}{n^2} \right) = 0,$$

which I formerly treated of (*Quarterly Journal of Mathematics*, Vol. v., pp. 54 *et seq.*). The equation has also been the subject of papers by Michael Roberts (*Q. J.*, Vol. v., p. 363), by Herr Eckardt ("Ueber eine gewisse Classe von Curven dritter Ordnung," *Schlämilch Z.*, XIII., pp. 263-6), and geometrically the same curves have been noticed by Emil Weyr ("Zur Erzeugung der Curven dritter Ordnung," *Wien, Sitzungsberichte der Kaiserl. Akad.*, LVIII., pp. 633-44).

It would not be convenient to restate here the results arrived at by the examination of the equation. Maclaurin gave the theorem that, if four points of contact of tangents to a cubic curve from a point thereon are given, the harmonic or diagonal triangle of the quadrangle has its vertices on the curve. The equation in question shows that the converse is true. The form is possible for a general curve having no double point, but it is not possible when there is one

double point. Mr. Michael Roberts gave the discriminant in the form

$$\sqrt{-R} = \frac{3\omega(1-\omega)}{m^3 n^3 p^3} (A^3 - B^3)(A^3 - C^3)(B^3 - C^3),$$

where ω is a cube root of unity.

The equality of two of the squares A^3, B^3, C^3 indicates the breaking up of the curve into a conic and right line; the other factor $\frac{1}{m^3 n^3 p^3}$, equated to zero, also breaks up the curve.

Thus, if, in (6), we make

$$k_1 = k_2 = k_3 = -1,$$

and

$$h_1 h_2 h_3 + h_1 - h_2 - h_3 = h_1 h_2 h_3 - h_1 + h_2 - h_3,$$

or

$$h_1 = h_2,$$

we get

$$\begin{aligned} (h_1^2 h_3 - h_3) \left\{ \frac{\alpha}{X} \left(\frac{\beta^3}{Y^3} - \frac{\gamma^3}{Z^3} \right) + \frac{\beta}{Y} \left(\frac{\gamma^3}{Z^3} - \frac{\alpha^3}{X^3} \right) \right\} \\ - (h_1^2 h_3 - 2h_1 + h_3) \frac{\gamma}{Z} \left(\frac{\beta^3}{Y^3} - \frac{\alpha^3}{X^3} \right) = 0, \end{aligned}$$

and the curve consists of the line

$$\frac{\beta}{Y} - \frac{\alpha}{X} = 0$$

and the conic

$$(h_1^2 h_3 - h_3) \left(\frac{\gamma^3}{Z^3} + \frac{\alpha\beta}{XY} \right) - (h_1^2 h_3 - 2h_1 + h_3) \frac{\gamma}{Z} \left(\frac{\beta}{Y} + \frac{\alpha}{X} \right) = 0,$$

the line passing through two points of the quadrangle and one vertex of the harmonic triangle, the conic passing through the remaining two points of the quadrangle and the remaining vertex of the harmonic triangle.

9. A case will now be briefly treated in which the variable triangle degenerates into a triplet of collinear points. We shall suppose a linear transversal to meet the sides of the triangle of reference in three points (8).

The coordinates of these points may be taken to be $(0, \beta_1, \gamma_1)$, $(\alpha_2, 0, \beta_2)$, $(\alpha_3, \beta_3, 0)$, and they will be collinear if

$$\alpha_2 \beta_3 \gamma_1 + \alpha_3 \beta_1 \gamma_2 = 0. \quad (9)$$

As previously, let

$$\begin{aligned}(X_1, Y_1, Z_1), \\ (X_2, Y_2, Z_2), \\ (X_3, Y_3, Z_3)\end{aligned}$$

be proportional to the coordinates of the vertices of a second fixed triangle. The linear segments determined by the points (8) will be in perspective with this triangle if the following conditions hold :—

$$\begin{aligned}\alpha(\beta_1 Z_1 - \gamma_1 Y_1) + \beta(\gamma_1 X_1 - \gamma \beta_1 X_1) &= 0, \\ -\alpha \gamma_2 Z_2 + \beta(\gamma_2 X_2 - \alpha_2 Z_2) + \gamma \alpha_2 Y_2 &= 0, \\ \alpha \beta_3 Z_3 - \beta \alpha_3 Z_3 + \gamma(\alpha_3 Y_3 - \beta_3 X_3) &= 0.\end{aligned}$$

By means of the condition (9) we can eliminate the coordinates of the points (8), and the result is

$$(\alpha Z_1)'(\beta X_2)'(\gamma Y_3)' - (\beta X_1)'(\gamma Y_2)'(\alpha Z_3)' = 0.$$

The expanded form is

$$\begin{aligned}\alpha^3 \{ (Y_1 Z_3)' Y_2 \gamma + (Z_1 Y_2)' Z_3 \beta \} + \beta^3 \{ (Z_2 X_1)' Z_3 \alpha + (Z_3 X_2)' X_1 \gamma \} \\ + \gamma^3 \{ (X_3 Y_2)' X_1 \beta + (X_1 Y_2)' Y_2 X \} + \alpha \beta \gamma \{ Z_1 X_2 Y_3 + X_3 Z_2 Y_1 - 2 X_1 Y_2 Z_3 \} \\ = 0.\end{aligned}\tag{10}$$

If we place the two fixed triangles in perspective, by writing for X_1, Y_1, Z_1 , &c.,

$$\begin{aligned}k_1 X, \quad Y, \quad Z, \\ X, \quad k_2 Y, \quad Z, \\ X, \quad Y, \quad k_3 Z,\end{aligned}$$

the equation takes the form

$$\begin{aligned}\alpha^3 \{ Y^2 Z (k_3 - 1) k_2 \gamma + Z^2 Y (k_2 - 1) k_3 \beta \} \\ + \beta^3 \{ Z^2 X (k_1 - 1) k_3 \alpha + X^2 Z (k_3 - 1) k_1 \gamma \} \\ + \gamma^3 \{ X^2 Y (k_2 - 1) k_1 \beta + Y^2 X (k_1 - 1) k_2 \alpha \} \\ + 2 \alpha \beta \gamma X Y Z \{ 1 - k_1 k_2 k_3 \} = 0;\end{aligned}$$

when $k_1 = k_2 = k_3 = k$, this is

$$(\alpha Y + \beta X)(\beta Z + \gamma Y)(\gamma X + \alpha Z) - 2 \alpha \beta \gamma X Y Z \frac{(k+1)^3}{k} = 0,$$

and the curve is transformed into itself by writing

$$\frac{X}{\alpha}, \frac{Y}{\beta}, \frac{Z}{\gamma} \quad \text{for} \quad \frac{\alpha}{X}, \frac{\beta}{Y}, \frac{\gamma}{Z}.$$

10. We have seen that right lines through the point α, β, γ , and the respective vertices of the triangle of reference meet the corresponding sides of the second fixed triangle in the points whose coordinates are proportional to

$$\left(-\frac{B_1\beta+C_1}{A_1}, \beta, \gamma\right),$$

$$\left(\alpha, -\frac{A_2\alpha+C_2\gamma}{B_2}, \gamma\right),$$

$$\left(\alpha, \beta, -\frac{A_3\alpha+B_3\beta}{C_3}\right),$$

and, if these are collinear, we get the locus

$$\alpha^2 \{ (A_2B_1)' A_3\beta + (A_3C_1)' A_2\gamma \} + \beta^2 \{ (B_1A_2)' B_3\alpha + (B_3C_2)' B_1\gamma \} \\ + \gamma^2 \{ (C_1A_3)' C_3\alpha + (B_3C_2)' C_1\beta \} + \alpha\beta\gamma \{ B_1A_3C_2 + C_1A_2B_3 - 2A_1B_2C_3 \} = 0.$$

The coefficient of $\alpha\beta\gamma$ is

$$A_3(B_1C_2)' + B_3(C_1A_3)' + A_2(C_1B_3)' + C_3(B_1A_2)'.$$

Now, if we replace A_1 by $(Y_3Z_3)'$, B_1 by $(Z_3X_3)'$, C_1 by $(X_3Y_3)'$, A_2 by $(Y_3Z_1)'$, B_2 by $(Z_3X_1)'$, C_2 by $(X_3Y_1)'$, A_3 by $(Y_1Z_3)'$, B_3 by $(Z_1X_3)'$, C_3 by $(X_1Y_3)'$, the above equation reduces to the same form as (10).

This result depends on the following theorem:—If three concurrent lines pass severally through a different vertex of a fixed triangle, and also through the points where a transversal meets the sides of a second fixed triangle, a triplet of right lines through the same point of concurrence passing severally through a different vertex of the second triangle will determine three collinear points on the corresponding sides of the first triangle. The geometrical proof of this is easy. For, following the notation of Fig. 3, we have

$$\frac{AY}{YO} = \frac{\sin APY}{\sin YPO} \frac{PA}{PO}, \quad \frac{OX}{XB} = \frac{\sin OPX}{\sin XPB} \frac{PC}{PB},$$

$$\frac{BZ}{ZA} = \frac{\sin BPZ}{\sin ZPA} \frac{PB}{PA},$$

and, according to the theorem of Menelaus, we assume

$$\frac{AY}{YO} \cdot \frac{OX}{XB} \cdot \frac{BZ}{ZA} = -1.$$

FIG. 3.

$$\text{Again, } \frac{A'Y'}{Y'O'} = \frac{\sin XPB}{\sin BPZ} \frac{PA'}{PO'}, \quad \frac{O'X'}{X'B'} = \frac{\sin ZPA}{\sin APY} \frac{PC'}{PB'},$$

$$\frac{B'Z'}{Z'A'} = \frac{\sin YPO}{\sin OPX} \frac{PB'}{PA'};$$

and therefore

$$\frac{A'Y'}{Y'O'} \cdot \frac{O'X'}{X'B'} \cdot \frac{B'Z'}{Z'A'} = -1.$$

11. Another case is noteworthy, in which a variable triplet of collinear points occurs.

The equations

$$\beta y - \gamma z = \gamma x - \alpha z = \alpha y - \beta x = 0, \quad (11)$$

represent three concurrent lines passing respectively through the vertices of the triangle of reference ABC .

Let $A\alpha + B\beta + C\gamma = 0$

be the equation of any straight line. Its intersections with the lines (11) have coordinates proportional to

$$-\frac{By+Cz}{A}, \quad y, \quad z, \quad (1)$$

$$x, \quad -\frac{Ax+Cz}{B}, \quad z, \quad (2)$$

$$x, \quad y, \quad -\frac{Ax+By}{C}; \quad (3)$$

take, as before, three points whose coordinates are

$$(X_1, Y_1, Z_1), \quad (4)$$

$$(X_2, Y_2, Z_2), \quad (5)$$

$$(X_3, Y_3, Z_3). \quad (6)$$

Right lines through (1) and (4), (2) and (5), (3) and (6), respectively, are given by

$$\left. \begin{aligned} \alpha \left(Y_1 z - Z_1 y \right) - \beta \left(Z_1 \frac{By+Cz}{A} + X_1 z \right) + \gamma \left(Y_1 \frac{By+Cz}{A} + X_1 y \right) &= 0, \\ \alpha \left(Z_2 \frac{Ax+Cz}{B} + Y_2 z \right) + \beta \left(Z_2 x - X_2 z \right) - \gamma \left(X_2 \frac{Ax+Cz}{B} + Y_2 x \right) &= 0, \\ -\alpha \left(Y_3 \frac{Ax+By}{C} + Z_3 y \right) + \beta \left(X_3 \frac{Ax+By}{C} + Z_3 x \right) + \gamma \left(X_3 y - Y_3 x \right) &= 0. \end{aligned} \right\} \quad (12)$$

Write P, Q, R for $AX_1 + BY_1 + CZ_1, AX_2 + BY_2 + CZ_2, AX_3 + BY_3 + CZ_3$, and the above system becomes

$$\alpha A + \beta \left(B - \frac{Px}{(Y_1 z)} \right) + \gamma \left(C + \frac{Py}{(Y_1 z)} \right) = 0,$$

$$\alpha \left(A + \frac{Qz}{(Z_2 x)} \right) + \beta B + \gamma \left(C - \frac{Qx}{(Z_2 x)} \right) = 0,$$

$$\alpha \left(A - \frac{Ry}{(X_3 y)} \right) + \beta \left(B - \frac{Rx}{(X_3 y)} \right) + \gamma C = 0.$$

Eliminating α, β, γ , and equating to zero, we get

$$\begin{vmatrix} A, & B - \frac{Pz}{(Y_1z)',} & C + \frac{Py}{(Y_1z)',} \\ \frac{Qz}{(Z_2x)',} & \frac{Pz}{(Y_1z)',} & -\frac{Qx}{(Z_2x)',} - \frac{Py}{(Y_1z)',} \\ \frac{Qz}{(Z_2x)',} + \frac{Ry}{(X_3y)',} & -\frac{Rx}{(X_3y)',} & -\frac{Qx}{(Z_2x)',} \end{vmatrix} = 0.$$

Expanding and reducing this result, we find without difficulty

$$(Ax + By + Cz) \{ PQz (X_3y)' + QRx (Y_1z)' + PRy (Z_2x)' \} = 0,$$

so that the locus consists of the given line and a conic.

12. On the other hand, we may eliminate x, y, z instead of α, β, γ .

Write L for $A\alpha + B\beta + C\gamma$, P, Q, R retaining their meaning. The first equation of the system (12) is

$$\begin{aligned} z \left(Y_1\alpha - \beta Z_1 \frac{C}{A} - \beta X_1 + \gamma Y_1 \frac{C}{A} \right) \\ + y \left(\gamma X_1 + \gamma Y_1 \frac{B}{A} - \beta Z_1 \frac{B}{A} - Z_1x \right) = 0, \end{aligned}$$

$$\text{or} \quad z (Y_1L - P\beta) + y (P\gamma - Z_1L) = 0,$$

and in like manner the other two equations are

$$x (Z_2L - Q\gamma) + z (Q\alpha - X_2L) = 0,$$

$$y (X_3L - R\alpha) + x (R\beta - Y_3L) = 0.$$

Hence, finally,

$$(Y_1L - \beta P)(Z_2L - \gamma Q)(X_3L + R\beta)$$

$$- (Z_1L - P\gamma)(X_2L - Q\alpha)(Y_3L - R\beta) = 0,$$

$$\text{or} \quad L^3 (X_3Z_2Y_1 - X_2Y_3Z_1)$$

$$\begin{aligned} + L^3 \{ R(Z_1X_2\beta - Y_1Z_2\alpha) + Q(Z_1Y_3\alpha - Y_1X_3\gamma) \} + P(X_2Y_3\gamma - Z_2X_3\beta) \\ + L \{ QR\alpha (Y_1\gamma)' - PR(Z_2\alpha)' \beta + PQ(X_3\beta)' \gamma \} = L \cdot \psi = 0, \end{aligned}$$

when $\psi = 0$ gives a conic having the same chord as the former one cut off from the line $L = 0$.

13. The foregoing results give the following construction. Take two conics (I.) and (II.) having common intersections P and Q . From A, B, C , three points on (I.), draw three straight lines $AO, BO,$

OO , meeting in O , a point on the same conic, and cutting the chord PQ (produced if necessary) in a, b, c . Through a, b, c draw three straight lines $A'O', B'O', C'O'$ meeting the conic (II.) in O' , and again in A', B', C' respectively. Then, if through A, B, C we draw straight lines AO'', BO'', CO'' meeting the conic (I.) in O'' , and the line PQ in a', b', c' , the straight lines $a'A', b'B', c'C'$ will intersect in O''' , a point on (II.).

But, since two lines determine their own intersection, it is not necessary to pay attention to more than two lines of the variable triplets through A, B, C . Therefore, if we take any conic (I.), and two points P, Q upon it, and are given besides three points A', B', C' , which determine with P, Q a second conic (II.), this conic can be constructed by the process indicated.

For suppose we draw straight lines $A'O', B'O'$, meeting the straight line through P and Q in a, b , and through a point O on (I.) we draw AO, BO , meeting the same conic again in A, B ; we can then proceed to determine points on (II.) by drawing AO', BO' through O' , a point on (I.), and cutting PQ in $a'b'$. For straight lines $a'A', b'B'$ will meet in a point O'' on (II.).

Now it will be remarked that we do not make explicit use of the points P, Q , and this is important, for we have a method of describing a conic through three given real points and two imaginary points given as the intersections of a circle with a given real external straight line.

We may take a circle for conic (I.) and for the line PQ a real line external to the circle. The construction is easily verified by taking two circles and their radical axis. From a geometrical point of view, the straight lines through A, B , and a variable point O of the conic (I.) determine two corresponding homographic ranges on the line PQ . The double points of these are points on the curve. Having therefore determined the points A, B by means of A', B', C' on the conic (II.), this can be constructed through the points A', B', C', P, Q .

Also a pair of real straight lines may be taken for the conic (I.). The five points through which the constructed conic passes are then manifestly real.

14. The form of the equation

$$(Z_1\alpha')(X_2\beta')(Y_3\gamma') - (Y_1\alpha - X_1\beta)(X_3\gamma - Z_3\alpha)(Z_2\beta - Y_2\gamma) = 0$$

shows that the product of three perpendiculars from a point P on the curve to the lines BA', CB', AC' is in a constant ratio to the product

of three perpendiculars to the lines CA' , BC , AB' , and this is easily proved geometrically.

Write Pa_2 for the perpendicular from P on BC' , and so forth. Then (Fig. 4)

$$\frac{Pa_2}{Pb_1} = \frac{\sin BC'Z}{\sin ZO'A}, \quad \frac{Pc_1}{Pa_3} = \frac{\sin AB'Y}{\sin YB'O}, \quad \frac{Pb_3}{Pc_2} = \frac{\sin CA'X}{\sin XA'B},$$

$$\frac{AY}{YO} = \frac{\sin AB'Y}{\sin YB'O} \frac{AB'}{OB'}, \quad \frac{OX}{XB} = \frac{\sin CA'X}{\sin XA'O} \frac{CA'}{BA'}, \quad \frac{BZ}{ZA} = \frac{\sin BC'Z}{\sin ZO'A} \frac{BC'}{AO'}.$$

FIG. 4.

and, since AX , BY , OZ are concurrent,

$$\frac{Pa_2 \cdot Pc_1 \cdot Pb_3}{Pb_1 \cdot Pa_3 \cdot Pc_2} = \frac{AO' \cdot BA' \cdot OB'}{AB' \cdot CA' \cdot BO'}.$$

In like manner, when the triangle of reference is cut by a linear transversal, the equation

$$(Z_1\alpha)'(X_2\beta)'(Y_3\gamma)' + (\gamma_1\alpha)'(X_3\gamma)'(Z_2\beta)' = 0$$

leads us, following an analogous notation, to the form

$$\frac{Pa_2 \cdot Pc_1 \cdot Pb_3}{Pb_1 \cdot Pa_3 \cdot Pc_2} = - \frac{AO' \cdot BA' \cdot OB'}{AB' \cdot CA' \cdot BO'}.$$

The results may be included in one form by postulating

$$\frac{BX \cdot OY \cdot AZ}{XC \cdot YA \cdot ZB} = \mu,$$

and the condition connecting the coordinates of X , Y , Z becomes

$$\gamma_1\alpha_2\beta_3 - \mu\beta_1\gamma_2\alpha_3 = 0.$$