A METHOD OF ESTABLISHING THE 27-LINE CONFIGURATION ON A CUBIC SURFACE

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1. The present communication gives a method whereby, when a certain homographic correspondence has been established, the configuration and chief fundamental properties of the 27 lines on a cubic surface become almost intuitive. The method was suggested by, and is the extension of, a method published in the *Quarterly Journal*, No. 167, 1911, in which I gave an elementary proof of Salmon's theorem in connexion with the cross-ratio of the four tangents to a cubic curve drawn from a point on itself. Thereafter I used the method to prove the chief focal and bitangent properties of circular cubics and bicircular quartics in two papers published in the *Proceedings of the Edinburgh Mathematical Society*, Session 1910–1911. The present paper exhibits many of the properties of the 27 lines on the cubic surface as extensions of corresponding properties in the plane cubic.

2. If we assume that the existence of one line on the cubic surface has been established, it is easy to see that any plane containing this line cuts the surface besides in a conic and that, for *five* positions of this plane, the conic degenerates into two straight lines. We shall hereafter refer to these five planes as the five tangent planes drawn from this line to the surface.

Let AB be a line on the surface, BC and CA two other lines on the surface meeting AB . Let DAB be one (other than CAB) of the five planes through AB containing degenerate conics of section, and let D be the intersection of the two lines on the surface lying in this latter plane. The cubic surface then assumes the form

$$ax^2t + by^2t + cz^2t + 2dt^2z + 2fyzt + 2gzxt + 2hxyt + 2kxyz = 0 \quad (1)$$

if we take $ABCD$ as the tetrahedron of reference.

Let $t = \lambda x$ be a plane passing through the line BC and cutting the surface again in the conic S . Then the equation to the cone DS is

$$ax^2 + by^2 + cz^2 + 2\left(f + \frac{k}{\lambda}\right)yz + 2(g + d\lambda)zx + 2hxy = 0. \quad (2)$$

Similarly, if the plane $t = \mu y$ through CA cut the surface again in the conic S' , the equation to the cone DS' is

$$ax^2 + by^2 + cz^2 + 2(f + d\mu)yz + 2\left(g + \frac{k}{\mu}\right)zx + 2hxy = 0. \quad (3)$$

Thus the two cones DS and DS' will be identical if

$$\lambda\mu = \frac{k}{d}. \quad (4)$$

Thus, if a plane be drawn through BC and if its conic of section S be joined to D , the second degree cone DS cuts the cubic surface in a curve of the sixth degree consisting of S , the two lines on the surface through D and another conic S' whose plane passes through CA .

A one-to-one algebraic correspondence exists between the above two planes containing S and S' respectively in virtue of (4).

Any one ray of the cone DS (other than the lines on the surface through D) is sufficient to identify the plane containing S' and thus to find corresponding members of the homography.

3. If S breaks up into two lines, it is plain that S' will also consist of two straight lines, and thus the four planes through BC containing degenerate conics correspond to the four planes through CA containing degenerate conics. Furthermore, the line joining the point of intersection of the two lines composing S to D plainly passes through the meeting point of the two lines composing S' , and a similar property holds whatever point D we choose of the four points of contact of the planes containing degenerate conics which can be drawn through AB . We thus have the following theorems:—

The three groups of four tangent planes (other than the plane containing the three given lines on the surface) that can be drawn from three coplanar lines on the surface have equal cross-ratios.

The points of contact constitute the vertices of three "desmic tetrahedra."

If a and b be two coplanar lines on the surface, the cross-ratio of the four planes joining a to the points of contact of the four tangent planes from b is equal to the cross-ratio of the four planes joining b to the points of contact of the four tangent planes from a .

For, let α, β be the points of contact of representative tangent planes from a, b respectively. Then the line $\alpha\beta$ passes through one or other of the points of contact of tangent planes from c , where c is the line on the surface coplanar with a and b . Hence, by Art. 2, $\alpha\beta$ and ba are corresponding planes according to one of the four homographic systems defined by the points of contact of the four tangent planes from c . The theorem is then evident.

4. The geometrical configuration of the 27 lines, and in particular the intersection diagram of Mr. G. T. Bennett, given in the *Proc. London Math. Soc.*, Series 2, Vol. 9, Part 5, can now be quite easily verified. For example, in terms of the usual notation let a_1, b_2, c_{12} be three coplanar lines of the system, and let a_2, b_1 be two other coplanar lines of the system meeting c_{12} . Consider a_3, a_4, a_5, a_6 . The planes $b_1a_3, b_1a_4, b_1a_5, b_1a_6$ meet the surface again in $c_{13}, c_{14}, c_{15}, c_{16}$ which are thus determined. The lines meeting a_1 , complementary to $c_{13}, c_{14}, c_{15}, c_{16}$, are respectively b_3, b_4, b_5, b_6 , which joined to a_2 give $c_{23}, c_{24}, c_{25}, c_{26}$ respectively. Thus all the lines meeting a_1 and b_2 are arranged as far as a_2 and b_1 are concerned. Similarly, we can arrange all the lines meeting a_1 and b_2 with reference to another pair of coplanar lines meeting c_{12} .

It is found, also, that the correspondences of the special planes which give line-pairs occur with double exchanges of these line-pairs. We proceed to tabulate these exchanges. Let the tangent planes from a_1 be $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and let their corresponding planes with respect to one of the points of contact of the four tangent planes from b_2 be $\beta_1, \beta_2, \beta_3, \beta_4$ as explained in Art. 2. Denoting the four homographies thus obtained by I, II, III, IV respectively, we obtain the following sets of corresponding planes:—

$$[\alpha_1 \alpha_2 \alpha_3 \alpha_4] \equiv [\beta_1 \beta_2 \beta_3 \beta_4], \quad (\text{I})$$

$$[\alpha_1 \alpha_2 \alpha_3 \alpha_4] \equiv [\beta_2 \beta_1 \beta_4 \beta_3], \quad (\text{II})$$

$$[\alpha_1 \alpha_3 \alpha_2 \alpha_4] \equiv [\beta_3 \beta_4 \beta_1 \beta_2], \quad (\text{III})$$

$$[\alpha_1 \alpha_2 \alpha_3 \alpha_4] \equiv [\beta_4 \beta_3 \beta_2 \beta_1]. \quad (\text{IV})$$

Let	α_1	contain the lines	$b_3, c_{18},$
	α_2	„ „	$b_4, c_{14},$
	α_3	„ „	$b_5, c_{15},$
	α_4	„ „	$b_6, c_{16},$
	β_1	„ „	$a_3, c_{23},$
	β_2	„ „	$a_4, c_{24},$
	β_3	„ „	$a_5, c_{25},$
	β_4	„ „	$a_6, c_{26}.$

Next, let ab denote the plane containing the lines a, b , and let us rewrite the above correspondences (I), (II), (III), (IV) in terms of the lines lying in the various planes, with the proviso that lines in the same vertical column are coplanar. We thus obtain the following:—

I.

$b_3 c_{13}(\alpha_1)$	$b_4 c_{14}(\alpha_2)$	$b_5 c_{15}(\alpha_3)$	$b_6 c_{16}(\alpha_4)$
$c_{23} a_3(\beta_1)$	$c_{24} a_4(\beta_2)$	$c_{25} a_5(\beta_3)$	$c_{26} a_6(\beta_4)$
$a_2 b_1$	$a_2 b_1$	$a_2 b_1$	$a_2 b_1.$

II.

$b_3 c_{13}(\alpha_1)$	$b_4 c_{14}(\alpha_2)$	$c_{15} b_5(\alpha_3)$	$c_{16} b_6(\alpha_4)$
$a_4 c_{24}(\beta_2)$	$a_3 c_{23}(\beta_1)$	$c_{26} a_6(\beta_4)$	$c_{25} a_5(\beta_3)$
$c_{34} c_{56}$	$c_{34} c_{56}$	$c_{34} c_{56}$	$c_{34} c_{56}.$

III.

$b_3 c_{13}(\alpha_1)$	$c_{14} b_4(\alpha_2)$	$b_5 c_{15}(\alpha_3)$	$c_{16} b_6(\alpha_4)$
$a_5 c_{25}(\beta_3)$	$c_{26} a_6(\beta_4)$	$a_3 c_{23}(\beta_1)$	$c_{24} a_4(\beta_2)$
$c_{35} c_{46}$	$c_{35} c_{46}$	$c_{35} c_{46}$	$c_{35} c_{46}.$

IV.

$b_3 c_{13}(\alpha_1)$	$c_{14} b_4(\alpha_2)$	$c_{15} b_5(\alpha_3)$	$b_6 c_{16}(\alpha_4)$
$a_6 c_{26}(\beta_4)$	$c_{25} a_5(\beta_3)$	$c_{24} a_4(\beta_2)$	$a_3 c_{23}(\beta_1)$
$c_{36} c_{45}$	$c_{36} c_{45}$	$c_{36} c_{45}$	$c_{36} c_{45}.$

To explain the above table, we note that the order of the lines b_3, c_{13} in the plane α_1 is kept unaltered throughout. Then, when we make interchanges of the “ β ” planes (*e.g.*, in passing from I to II, β_1 and β_2 , β_3 and β_4), we interchange the order of the lines in each of α_3 and α_4 , and also in each of β_1 and β_2 . Thus we obtain II from I. Again, in passing from II to III, we interchange β_1 and β_4 , β_2 and β_3 . We therefore interchange the lines in a_2 and a_3 , β_1 and β_4 , and so on.

5. Some of the other theorems described in Mr. Bennett's paper also follow at once. Thus with the relation $\lambda\mu = \text{const.}$, zero and infinity are a pair of corresponding values, and hence the plane BCA through BC corresponds to CAD through CA , and BCD through BC corresponds to CAB through CA . Interpreting this result in the notation of the double-six, we obtain

$$a_1[b_2c_{12}, b_3c_{13}, b_4c_{14}, b_5c_{15}, b_6c_{16}] \equiv b_2[a_2b_1, c_{23}a_3, c_{24}a_4, c_{25}a_5, c_{26}a_6],$$

where b_2c_{12} denotes the intersection of the lines b_2 and c_{12} , etc. In this result is contained Dixon's theorem, viz.,

$$a_1[b_3b_4b_5b_6] = b_2[a_3a_4a_5a_6],$$

where a_1b_3 denotes the plane containing a_1 and b_3 , etc. The various perspectives are also evident, *e.g.*, $a_1[b_3b_4b_5b_6]$ joined respectively to $b_2[c_{23}c_{24}c_{25}c_{26}]$ meeting in a_2c_{12} , where a_1b_3 denotes the intersection of a_1 and b_3 , etc.