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HEAT LOSSES IN THE CONDUCTORS[1] OF ALTERNATING-CURRENT MACHINES

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ABSTRACT OF PAPER

The principal object of this paper is to show how hyperbolic functions of *complex angles* may be applied to the solution of the problem of heat losses in rectangular conductors that are embedded in open slots. A certain knowledge of the functions themselves is presupposed. Inasmuch, however, as they are handled like trigometric functions of real angles—except in regard to the plus and minus signs—it is a simple matter to acquire the requisite technical skill to use them. The hyperbolic function of a complex angle, consisting as it does of a real and an imaginary part, may represent a vector—the real part being the component of the vector along the horizontal, and the imaginary part, component along the vertical. Thus, for example, $A \sinh (x + j x)$ represents a vector just as $A e^{j \theta}$, $A \underline{/\theta}$, $A (\cos \theta + j \sin \theta)$ represent vectors.

Considerable experience has shown that the vector

1. An abstract of this paper was presented before the Research Division of the Electrical Engineering Department at the Massachusetts Institute of Technology, April 13, 1920. The author's attention was first directed to this method of analysis by Dr. A. E. Kennelly to whom great credit is due especially on account of the extensive tables he has published from which numerical computations may readily be made: "Tables of Complex Hyperbolic and Circular Functions," A. E. Kennelly.

See also:

"Eddy Currents in Large Slot-Wound Conductors," A. B. Field, TRANS., A. I. E. E., 1905, p. 761.

"Current Distribution in Armature Conductors," W. V. Lyon, *Electrical World*, July 12, 1919.

Since this present analysis was made, Mr. R. E. Gilman has presented at the Annual Convention, TRANS., A. I. E. E., 1920, p. 997 a paper entitled "Eddy Current Losses in Armature Conductors." This paper of Mr. Gilman's, although he uses real quantities exclusively, is more complete in one respect than the one here presented in that he considers the case of a laminated conductor with a finite number of strands.

method for handling a-c. problems is much superior to the original method in which simple trigonometric functions were used. With this lesson before us, it should require but little contact with the problem at hand to demonstrate the superiority of the vector method, even though it employs the possibly unfamiliar hyperbolic quantities. These hyperbolic vectors have been used for a number of years in the analysis of problems involving a-c. circuits, which have distributed inductance and capacitance, and have proved their usefulness. Here is another important problem in which they may be used to advantage, and doubtless others will appear from time to time.

The general method of solution is to obtain a vector expression for the voltage drop in the topmost element of any conductor due to its resistance, and to all slot leakage flux *below* this element, (Equation (**7a**)), together with the vector expression for the voltage drop produced in this element by the slot leakage flux within any conductor *above* it, (**8a** and **8c**). The proper combination of these two equations will give the impedance drop in any of the considered arrangements of conductors. These expressions (**7a**, **8a** and **8c**) are determined by the frequency and the resistivity and the dimensions and arrangement of the conductors. Since the impedance drop that is thus determined is in the vector form, both the resistance and reactance—due to slot leakage flux within the conductors—are determined. Only solid and finely laminated conductors are considered.

Since the present paper was written, the author has extended the method to the solution of the problem in which the conductors have a finite number of strands separated by insulation. With slight modifications expressions similar to equations (**7a**), (**8a**), and (**8c**) can be derived. The M and N functions are rather more complicated than here given, but curves similar to Figs. 3 and 4 can be plotted, by the aid of which numerical calculations can be made.

WITH the increase in the capacity of alternating-current generators, it has become more and more important to determine the exact relation between the heat developed in any conductor and the currents that it and other neighboring conductors carry. Solutions for the distribution of alternating current within conductors that are embedded in open rectangular slots have been obtained for a number of cases. The methods of attack and the results obtained have heretofore involved trigonometric and hyperbolic functions of real angles. For this reason the work has been unnecessarily complicated, and its scope considerably cramped. Had the investigators used hyperbolic functions of *complex* angles, they would have accomplished more with much less effort. These hyperbolic functions of *complex* angles are such a powerful tool in the solution of this current distribution problem that the author feels fully justified in presenting this discussion of their application, even though many of the results have been obtained before.

At the outset it should be observed that certain assumed ideal conditions are necessary in order to bring the problem within the range of our mathematical ability.[2] Briefly these conditions are (1) that an element of current in the slot produces a uniform parallel magnetic field above itself and none below it; (2) that the current density along any line parallel to the bottom of the slot is constant; (3) that the resistivity of the conductor is uniform throughout, even though more heat is developed in some portions than in others; (4) that voltages in the end connections due to leakage flux are the same for every element of the conductor.

1. The first assumption can be shown to be exactly true in the hypothetical case of an infinitely long rectangular conductor placed in an infinitely deep rectangular slot of the same width which is cut into an infinite medium of infinite permeability.

2. The second assumption is probably sufficiently accurate, except when conductors that carry different currents are placed side by side in the same slot, a condition which we will not consider.

3. The heat conductivity of copper is so high that the temperature of any one conductor is probably nearly uniform, except in the case of extremely deep conductors, and the resistivity would thus not vary appreciably from point to point. In the case of multiple layer coils, however, there may be such a difference in the amount of heat developed in successive layers that it may be desirable to use different resistivities for different layers. In order to carry out this refinement in the calculations, a considerable practical knowledge of the principles of heat radiation

2. Both A. B. Field and R. E. Gilman assume these ideal conditions in the derivation of their formulas.

in this problem would be necessary, and the final result would probably be obtained by making successive approximations.

4. The leakage flux about the end connections* is of course much less than that about the embedded portion of the coils, and is distributed in a totally different manner. Any rational consideration of the effect of this flux would lead to considerable complication in the mathematical analysis. Thus for the sake of simplicity—a feeble reason, no doubt—it is not considered. It is probable, however, that in a great many instances this coil-end leakage is of relatively minor importance.

The best test of the value of the mathematical deductions based on the foregoing assumptions lies in experimental research. The little evidence that we now have seems to confirm the theory within reasonable limits of error, and it is hoped that a more extended investigation, which is being carried on, will prove conclusive.

To the author, it seems nearer the physical reality to say that the increased heat loss in a conductor carrying alternating rather than direct current is caused by a redistribution of current over the cross-section of the conductor.[3] This redistribution is due to the electromotive forces set up by the magnetic flux *within* the conductor itself. Flux that is wholly without the conductor links all elements of it equally, produces the same voltage in each element, and thus does not affect the current distribution.

Consider any conductor lying in the midst of others, as shown in cross-section in Fig. A. Further, consider any element of this conductor of depth $d\,x$ situated x centimeters from the bottom of the conductor. With a solid conductor the length of this element should be taken only as the length of the armature core. The allowance that need be made for ventilating ducts, and more particularly for end turns, can only be

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* Within the conductor

3. Others prefer to say that the increased heating is due to the eddy currents produced by the magnetic flux within the conductor. Either description is permissible.

determined by considerable experimental research. With finely laminated conductors, the current density is constrained to be the same in, at least, the length of a half turn, and the length of the element is then that of a half turn.

The net voltage acting on the element is the sum of the resistance drop and the voltage drop due to flux linkages. The former is $l_1 \rho c$; where l_1 is the length of the core with solid conductors, or a half turn with finely laminated ones, ρ is the resistivity and c is the current density—all in c. g. s. units. The voltage drop due to flux linkages may be divided into two parts; that due to flux set up by the current in the conductor itself, and that due to flux set up by current in other conductors in the slot *below* the one we are considering. Any current above this conductor produces flux that links all elements of it *alike,* and induces the *same* voltage in each. The manner in which a *given* current is distributed in any conductor

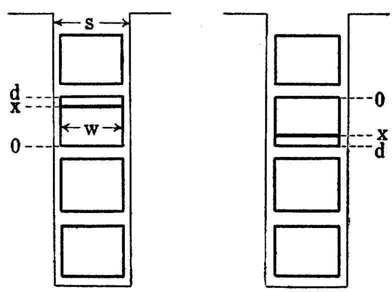

FIG. A FIG. B

below the one in question does not affect the amount of flux linking the element considered. Furthermore any flux due to these latter currents that passes across the slot above the conductor we are considering links all of its elements alike, and, since it produces the same voltage in each, need not be considered. Thus the total flux linking the element in question that need be taken into account is:

$$\varphi = 4\,\pi\, i_b\, \frac{d - x}{s}\, l_2 + \int_x^d \left\{ 4\,\pi \int_0^x w\, c\, \partial x \right\} \frac{l_2\, \partial x}{s}$$

The first term is the flux, linking the element due to a total current of i_b below the conductor of which the element is a part. The second term is the flux from x to d due to the current within the conductor itself from zero to x. l_2 is the length of the armature core, s is the width of the slot, and w that of the conductor.

The total voltage drop that need be considered is:

$$e = l_1 \rho c + \frac{\partial}{\partial t} \left\{ 4 \pi i_b \frac{d - x}{s} l_2 + \int_x^d \left\{ 4 \pi \int_0^x w c \, \partial x \right\} \frac{l_2 \, \partial x}{s} \right\} \qquad \text{(A)}$$

In a solid conductor this voltage drop is the same for every element, and $\frac{\partial e}{\partial x}$ will thus be zero. Also l_1 and l_2 are equal. Differentiating e and dividing by l_2 gives:

$$\rho \frac{\partial c}{\partial x} - \frac{4 \pi}{s} \frac{\partial i_b}{\partial t} - \frac{4 \pi}{s} \int_0^x w \frac{\partial c}{\partial t} \partial x = 0 \qquad \text{(B)}$$

With finely laminated conductors whose laminations are joined at the beginning and end of each half turn, the same result is obtained except that l_1 and l_2 are not equal. Then:

$$\frac{l_1}{l_2} \rho \frac{\partial c}{\partial x} - \frac{4 \pi}{s} \frac{\partial i_b}{\partial t} - \frac{4 \pi}{s} \int_0^x w \frac{\partial c}{\partial t} \partial x = 0$$

With finely laminated conductors whose end turns are untwisted and in which the laminations are continuous throughout a whole turn or a whole coil, the voltage in a half turn of any element is as given above. (Equation **A**). The sum of these half-turn voltages between the points at which the laminations are joined together must be the same for each lamination. This sum will consist of a number, n, of resistance drops, and an equal number of voltage drops due to flux linkages. The resultant drop is

$$e = n l_1 \rho c + \frac{\partial}{\partial t} \sum_1^n 4 \pi i_b \frac{d - x}{s} l_2$$

$$+ n \frac{\partial}{\partial t} \int_x^d \left\{ 4 \pi \int_0^x w c \, \partial x \right\} \frac{l_2 \, d x}{s} = 0$$

The current i_b, below the conductor of which the half-turn element is a part, is not the same for the different half-turn elements. Nevertheless, it is a simple matter to compute the average value of this quantity, *viz.*, $1/n \sum_1^n i_b$ for any arrangement of conductors. It is, of course, not necessary that the component i_b's should be in phase with each other. This computation is subsequently shown in detail. We will represent this average value of i_b by i_0. If the entire voltage for this element be divided by $n\, l_2$ and then differentiated with respect to x, we have:

$$\frac{l_1}{l_2}\,\rho\,\frac{\partial c}{\partial x} - \frac{4\,\pi}{s}\,\frac{\partial i_0}{\partial t} - \frac{4\,\pi}{s}\int_0^x w\,\frac{\partial c}{\partial t}\,\partial x = 0$$

In the case of finely laminated conductors that are twisted in the end connections so that the top lamination of one half turn becomes the bottom lamination of the next half turn, a similar equation may be derived. When the end turn is twisted in passing from one coil side to the next, the flux within the conductor linking the half-turn element of one side has already been given. (Equation **A**). The flux linking the next half turn of this element is, (Fig. B):

$$\rho = 4\,\pi\,i_b\,\frac{x\,l_2}{s} + \int_0^x \left\{ 4\,\pi \int_x^d w\,c\,\partial x \right\} \frac{l_2\,\partial x}{s}$$

The first term is the flux from zero to x due to a current of i_b below this conductor. The second term is the flux from zero to x due to current from x to d within the conductor itself. The second term may be rewritten in this way:

$$\int_0^x 4\,\pi \left\{ \int_0^d w\,c\,\partial x - \int_0^x w\,c\,\partial x \right\} \frac{l_2\,\partial x}{s}$$

The voltage drop in this half-turn element becomes:

$$e = l_1\,\rho\,c + \frac{\partial}{\partial t}\left\{ 4\,\pi\,i_b\,\frac{x\,l_2}{s} \right.$$

$$+ \int_0^x 4\,\pi \int_0^d w\,c\,\partial x\,.\,\frac{l_2\,\partial x}{s}$$

$$\left. - \int_0^x 4\,\pi \int_0^x w\,c\,\partial x\,.\,\frac{l_2\,\partial x}{s} \right.$$

$\int_0^d w\,c\,\partial x$ is the entire current in the conductor of which the element is a part. Represent this current by i. Differentiate e with respect to x and divide by l_2. We have:

$$\frac{l_1}{l_2}\rho c + \frac{4\pi}{s}\frac{\partial i_b}{\partial t} + \frac{4\pi}{s}\frac{\partial i}{\partial t} - \frac{4\pi}{s}\int_0^x w\frac{\partial c}{\partial t}\partial x = 0 \qquad \text{(C)}$$

Thus in every case that we shall discuss, the following equation may be written:

$$\rho\frac{\partial c}{\partial x} - \frac{\partial}{\partial t}\frac{4\pi i_0}{s} - \frac{\partial}{\partial t}\frac{4\pi}{s}\int_0^x w\,c\,\partial x = 0$$ [4]

ρ is the resistivity in the case of solid conductors, and is the resistivity multiplied by the ratio of the length of a half turn to the length of the core in the case of infinitely laminated conductors. c is the instantaneous current density in amperes per square centimeter along a line x centimeters from the bottom of the conductor. s is the width of the slot and w, that of the conductor. The first term is the differential of the resistance drop per centimeter in any element. The second term is the differential of the voltage per centimeter due to the flux produced by other currents in the same slot. As we shall see, i_0 is determined by the arrangement of the conductors and the currents they carry. The third term is the differential of the voltage per centimeter due to the flux produced by current within the conductor itself below the element considered. If the currents vary sinusoidally with

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4. An equation of this sort applies only to the conductor in which the current density is a continuous function with respect to x, such as is the case with solid or finely laminated conductors. When the laminations have appreciable depth the current density changes abruptly as we pass from one lamination to the next. The effect of this is that the vector constant i_0 is different in successive laminations. A comparison of equations (**B**) and (**C**) shows that twisting the end connections reverses the effect of the current, i_b, below the conductor we are considering.

the time, the differential operator $\frac{\partial}{\partial t}$ may be symbolically represented by $j\,\omega$; $j = \sqrt{-1}$, $\omega = 2\,\pi\,f$ where f = frequency.

The equation may now be written in the complex or vector form:

$$\frac{\partial \underset{\cdot}{c}}{\partial x} - j\,\frac{4\,\pi\,\omega}{\rho\,s}\,I_0 - j\,\frac{4\,\pi\,\omega}{\rho\,s}\int_0^x w\,\underset{\cdot}{c}\,\partial x = 0 \qquad (1)$$

Differentiating a second time gives:

$$\frac{\partial^2 \underset{\cdot}{c}}{\partial x^2} = j\,\frac{4\,\pi\,\omega}{\rho\,s}\,w\,\underset{\cdot}{c}$$

or:

$$\frac{\partial^2 \underset{\cdot}{c}}{\partial x^2} = \alpha^2\,\underset{\cdot}{c} \qquad (2)$$

where

$$\left.\begin{aligned} \alpha^2 &= j\,\frac{8\,\pi^2\,w\,f}{\rho\,s} \\ \text{or} \quad \alpha &= \sqrt{\frac{8\,\pi^2\,w\,f}{\rho\,s}}\ \underline{/45^\circ} \end{aligned}\right\}$$

The solution of equation (**2**) may be written in the form

$$\underset{\cdot}{c} = \underset{\cdot}{A}\cosh\alpha\,x + \underset{\cdot}{B}\sinh\alpha\,x \qquad (3)$$

This is a vector equation for the root-mean-square current density at a point x centimeters from the bottom of the conductor. The constants of integration $\underset{\cdot}{A}$ and $\underset{\cdot}{B}$ are vector quantities and are usually determined by the current in the conductor considered and by the vector I_0 in equation (**1**). Substitution of equation (**3**) in equation (**1**) shows that:

$$\underset{\cdot}{B} = \alpha/w\,I_0$$

If the depth of the conductor is d centimeters, the current in it is

$$I_1 = \int_0^d w\,\underset{\cdot}{c}\,\partial x$$

from which it follows that:

$$\underset{\cdot}{A} = \frac{\alpha}{w}\left\{\frac{I_1}{\sinh\alpha\,d} - I_0\tanh\frac{\alpha\,d}{2}\right\}$$

Therefore, the general solution for the vector current density may be written:

$$\underset{\cdot}{c} = \frac{1}{w\,d} \left\{ \frac{I_1\,\alpha\,d \cosh \alpha\,x}{\sinh \alpha\,d} \right.$$

$$\left. - I_0\,\alpha\,d \tanh \frac{\alpha\,d}{2} \cosh \alpha\,x + I_0\,\alpha\,d \sinh \alpha\,x \right\} \qquad (4)$$

A numerical calculation is helpful. Consider the case of two solid rectangular conductors situated as shown in Fig. 1. The 60-cycle currents are each 1000 amperes but the lower current leads the upper

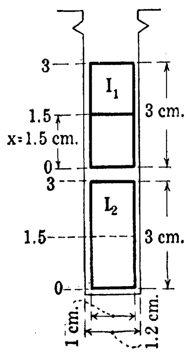

FIG. 1

by 60 degrees as would be the case in some of the slots of a three-phase fractional pitch winding. As we shall presently see $I_0 = I_2$ in this case.

$$\begin{aligned} \rho &= 2100 \text{ c. g. s. units of resistance} \\ w &= 1 \text{ cm.} \\ s &= 1.2 \text{ cm.} \\ d &= 3 \text{ cm.} \\ f &= 60 \text{ cycles} \end{aligned}$$

$$\alpha\,d = 3 \sqrt{\frac{8\,\pi^2 \times 1 \times 60}{2100 \times 1.2}} \;\underline{/45^\circ}$$

$$\begin{aligned} &= 4.11 \;\underline{/45^\circ} \\ \alpha\,x &= 2.06 \;\underline{/45^\circ} \\ \sinh \alpha\,d &= 9.12 \;\underline{/166.^\circ 4} \\ \tanh \frac{\alpha\,d}{2} &= 1.11 \;\underline{/1^\circ.41} \end{aligned}$$

$$\sinh \alpha x = 2.24 \underline{/84.^\circ 1}$$
$$\cosh \alpha x = 2.03 \underline{/82.^\circ 7}$$

If we choose I_1 along the horizontal, *i. e.*, with zero phase:

$$I_1 = 1000 \underline{/0^\circ}, \; I_0 = 1000 \underline{/60^\circ}$$

$$c_{1.5} = \frac{1}{1 \times 3} \left\{ (1000 \underline{/0^\circ} \times 4.11 \underline{/45^\circ} \times 2.03 \underline{/82.^\circ 4} \div 9.12 \underline{/166.^\circ 4}) - (1000 \underline{/60^\circ} \times 4.11 \underline{/45^\circ} \times 1.11 \underline{/1.^\circ 41} \times 2.03 \underline{/82.^\circ 7}) + (1000 \underline{/60^\circ} \times 4.11 \underline{/45^\circ} \times 2.24 \underline{/84.^\circ 1}) \right\} = \frac{1}{1 \times 3} \left\{ 915 \underline{/-39.^\circ 0} - 923 \underline{/189.^\circ 1} + 923 \underline{/189.^\circ 1} \right\}$$

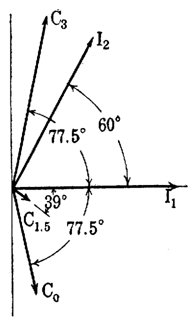

FIG. 2—CURRENT DENSITIES IN UPPER CONDUCTOR

Referred to I_1 the current densities are:

$$c_0 = 1620 \underline{/-77.^\circ 5} \; \frac{\text{amperes}}{\text{sq. cm.}} \; \underline{/\quad}$$

$$c_{1.5} = 305 \underline{/-39.^\circ} \; \frac{\text{amperes}}{\text{sq. cm.}} \; \underline{/\quad}$$

$$c_3 = 2490 \underline{/77.^\circ 5} \; \frac{\text{amperes}}{\text{sq. cm.}} \; \underline{/\quad}$$

$$c_{1.5} = 305 \underline{/-39.^\circ 0}$$
$$c_0 = 1620 \underline{/-77.^\circ 5}$$
$$c_3 = 2490 \underline{/77.^\circ 5}$$

Fig. 2 is the vector diagram showing the current densities at the bottom, middle and top of the conductor, together with the total current in it, I_1, and the current below it, I_2.

The current density at the center, where αx equals $\frac{\alpha d}{2}$, is:

$$\underset{\cdot}{c} = \frac{I_1}{w d} \frac{\frac{\alpha d}{2}}{\sinh \frac{\alpha d}{2}}$$

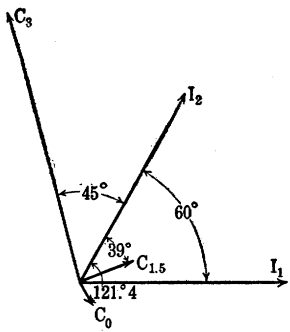

FIG. 2A—CURRENT DENSITIES IN LOWER CONDUCTOR

Referred to I_2 the current densities are:

$$\underset{\cdot}{C}_0 = 150 \angle{-121.^\circ 4} \ \frac{\text{amperes}}{\text{sq. cm.}}$$

$$\underset{\cdot}{C}_{1.5} = 305 \angle{-39^\circ} \ \frac{\text{amperes}}{\text{sq. cm.}}$$

$$\underset{\cdot}{C}_3 = 1380 \angle{45^\circ} \ \frac{\text{amperes}}{\text{sq. cm.}}$$

It depends only upon the average current density and the angular depth, αd, and is in no way affected by the position of the conductor in the slot. It is the same for solid and finely laminated conductors.

The current density is nowhere greater than at the top of the conductor, where it is:

$$\underset{\cdot}{c}_d = \frac{1}{w d} \left\{ I_1 \alpha d \coth \alpha d + \frac{I_0}{2} \alpha d \, 2 \tanh \frac{\alpha d}{2} \right\} \tag{5}$$

The ratio of this maximum current density to the average is:

$$\frac{\varrho_d}{\varrho_{av}} = \alpha\, d \coth \alpha\, d + \frac{I_0}{2\, I_1}\, \alpha\, d\, 2 \tanh \frac{\alpha\, d}{2} \qquad (6)$$

The complex quantities $\alpha\, d \coth \alpha\, d$ and $\alpha\, d\, 2 \tanh \frac{\alpha\, d}{2}$ occur in all of the expressions which we shall

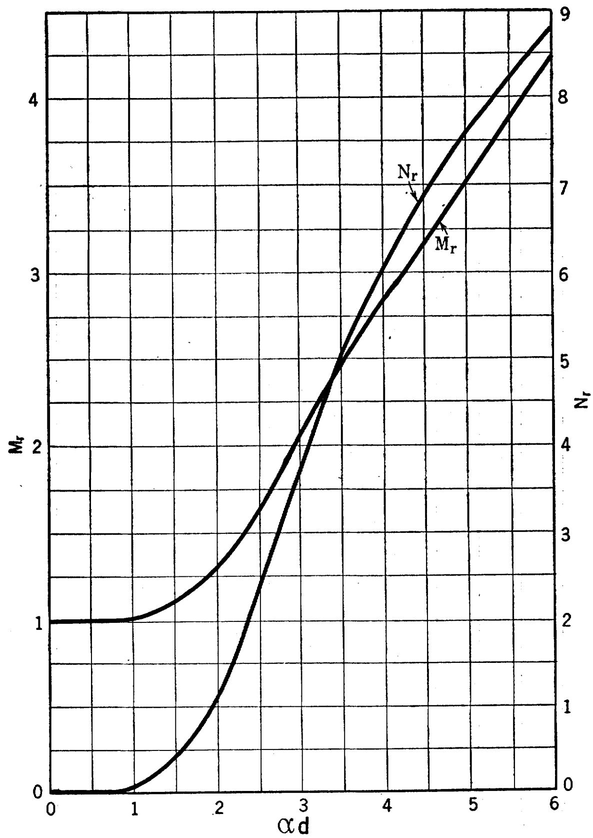

FIG. 3

develop, and thus it will be simpler to represent them by single letters. Hereafter

$$\alpha\, d \coth \alpha\, d = M = M_r + j\, M_x$$

$$\alpha\, d\, 2 \tanh \frac{\alpha\, d}{2} = N = N_r + j\, N_x$$

In Figs. 3 and 4 the abscissas are the numerical values of αd and the ordinates are M_r, M_x, N_r, N_x. The real portions of M and N, *viz.*, M_r and N_r, appear in the expressions for resistance, and the imaginary portions, M_x and N_x, in the expressions for reactance.

The voltage drop per centimeter in the conductor considered due to its own resistance and to all of the leakage flux below its topmost layer is:

$$\rho\, \dot{c}_d = \frac{\rho}{w\, d} \left\{ I_1 M + \frac{I_0}{2} N \right\} \qquad (7)$$

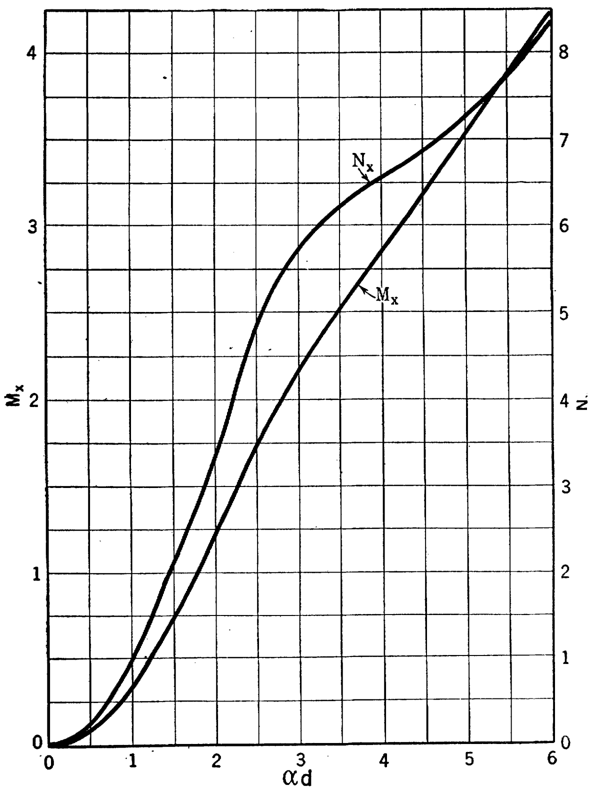

Fig. 4

The flux within the conductor due to its own current and all of the current, I_b, below it is:

$$\varphi = \int_0^d \partial x \frac{4\pi}{s} \int_0^x w\, \dot{c}\, \partial x + \frac{4\pi d}{s} I_b$$

On integration the expression for the flux may be written:

$$\varphi = \frac{1}{j\,\omega}\,\frac{\rho}{w\,d}\left\{\left(\frac{I_1}{2} + I_0\right)N + (I_b - I_0)\,\alpha^2\,d^2\right\}$$

The voltage drop per centimeter produced by this flux in each conductor below the one considered is $j\,\omega\,\varphi$, which may now be written:

$$E = \frac{\rho}{w\,d}\left\{\left(\frac{I_1}{2} + I_0\right)N + (I_b - I_0)\,\alpha^2\,d^2\right\} \qquad (8)$$

The relations expressed in equations (7) and (8) are very important since their proper combination will give the leakage impedance drop due to resistance and leakage flux within the conductors themselves for any arrangement of solid or infinitely laminated conductors. The reactance due to slot-leakage flux which does not pass through the conductors can be calculated by well-known methods and need not be considered here.

If the equations (7) and (8) are each multiplied by the length of the core, l, they will apply to the embedded portion of a solid conductor, the true resistance of which is R, and to the half turn of an infinitely laminated one, the true resistance of which is also R. Thus (7) and (8) may be written:

$$l\,\rho\,\varsigma_d = R\left\{I_1 M + \frac{I_0}{2}N\right\} \qquad (7a)$$

$$l\,E = R\left\{\left(\frac{I_1}{2} + I_0\right)N + (I_b - I_0)\,\alpha^2\,d^2\right\} \qquad (8a)$$

The *additional* voltage produced in all conductors below the one in question by flux within it due to its own current is

$$l\,E' = R\left\{\left(\frac{I_1}{2} + I_0\right)N - I_0\,\alpha^2\,d^2\right\} \qquad \textbf{(8b)}$$

Having found the vector impedance drop in the conductors of one phase by properly applying equations (**7a**) and (**8a**), the effective resistance and reactance of that phase are determined by dividing this drop by the vector current. The real portion of the result is the effective resistance and the imaginary portion, the effective reactance. This method, however, gives no indication of the distribution of the copper loss among the several conductors of the phase. This may be of considerable importance, especially in the case of solid conductors when the heat developed in the topmost conductor in a slot may be several times that developed in the bottom conductor.

The current through a conductor is non-uniformly distributed on account of flux *within* the conductor. This flux is due only to the current in the conductor itself and to current below it in the slot. Any current above the conductor in question has no effect on the current distribution within it. The heat generated within a conductor depends only upon the manner in which the current is distributed. The current density is completely determined by equation (**1**) and the total current in the conductor itself. For any particular value of I_0 (equation **1**) the heat developed by a given current in the conductor does not depend upon whether I_0 is some particular current or some combination of currents. Thus if it is possible to find the heat loss when I_0 is some particular current, we will have obtained a general expression for the loss in terms of the current in the conductor, I_1, and the constant, I_0, in equation (**1**). The particular case which we will consider is that of a solid conductor carrying a current of I_1 with a total current of I_b below it. The special form of equation (**1**) is then,

$$\frac{\partial\,\dot{c}}{\partial\,x} - j\,\frac{4\,\pi\,\omega}{\rho\,s}\,I_b - j\,\frac{4\,\pi\,\omega}{\rho\,s}\int_0^x w\,\dot{c}\,\partial\,x = 0 \qquad \textbf{(1a)}$$

Thus for this arrangement, $I_0 = I_b$.

The heat loss in the conductor is equal to the total power supplied both to this conductor and to all of those below it less the power supplied to those below it when the conductor is removed from the slot. The addition of the conductor in question increases the power supplied to the lower conductors without increasing the heat loss in them on account of its mutual inductive effect upon them *i. e.*, as some would say, on account of the eddy currents which are produced in it by the total current I_b, below it. The power supplied to the upper conductor is, symbolically, since $I_0 = I_b$

$$I_1 R \left\{ I_1 M + \frac{I_b}{2} N \right\} \qquad \textbf{(See 7a)}$$

This indicates the product of the numerical values of the current, I_1, the voltage applied to the conductor and the cosine of the phase angle between them. Flux above the conductor produces a quadrature voltage and thus does not affect the power. The *additional* power supplied to the conductors below this one is symbolically

$$I_b R \left\{ \left(\frac{I_1}{2} + I_b \right) N - I_b \alpha^2 d^2 \right\}$$

(See 8b)

The actual heat loss in the conductor is thus symbolically:

$$I_1 R \left\{ I_1 M + \frac{I_b}{2} N \right\} + I_b R \left\{ \left(\frac{I_1}{2} + I_b \right) N - I_b \alpha^2 d^2 \right\}$$

This reduces to:

$$R \left\{ I_1^2 M_r + (I_b^2 + I_1 I_b \cos \delta) N_r \right\} \text{[5]} \qquad \textbf{(9a)}$$

where I_1 and I_b are the numerical values of the currents and δ is the phase angle between them. Therefore the general expression for the heat loss in any conductor, solid or infinitely laminated, is:

$$R \left\{ I_1 M_r + (I_0^2 + I_1 I_0 \cos \delta) N_r \right\} \qquad \textbf{(9)}$$

where I_1 is the numerical value of the current in the

---

5. M_r and N_r are calculated from the data pertaining to the conductor in question and bear no relation to other conductors.

conductor, I_0 is the numerical value of the vector constant in the differential equation (**1**), and δ is the phase angle between I_1 and I_0. The ratio of alternating-current to direct-current resistance is thus:

$$K = M_r + \left(\left| \frac{I_0}{I_1} \right|^2 + \left| \frac{I_0}{I_1} \right| \cos\delta \right) N_r \qquad (\mathbf{10})$$

The vertical lines | | indicate that the division is one of numerical values and not of vector values. The first term M_r accounts for the natural non-uniformity of current distribution due to the action of the current upon itself. The second term

$$\left(\left| \frac{I_0}{I_1} \right|^2 + \left| \frac{I_0}{I_1} \right| \cos\delta \right) N_r,$$

accounts for the additional heating produced by the "eddy currents" due to the action of I_0.

This equation (**10**) enables us to compute the ratio of alternating-current to direct-current resistance for any conductor carrying a specified current and for which a differential equation of the form given equation (**1**) can be written. In the case of solid conductors, the ratio only applies to the embedded portion. The following are the resistance ratios for some of the simpler arrangements of conductors.

1. The heat loss in an open-circuited bar with I_b amperes below it is:

$$\text{heat loss} = R\, I_b^2\, N_r \qquad (\text{equation } \mathbf{9a},\ I_1 = 0)$$

2. The resistance ratio for the pth conductor of a one-coil-side-per-slot bar winding is:

$$K = M_r + [\,(p-1)^2 + (p-1)]\, N_r$$
$$= M_r + (p^2 - p)\, N_r$$

3. The resistance ratio for a one-coil-side-per-slot winding having n layers is:

$$K = 1/n \sum_1^n \,[M_r + (p^2 - p)\, N_r]$$
$$= M_r + \frac{n^2 - 1}{3}\, N_r$$

This is also the ratio for the lower coil side of any bar winding having n layers. The upper coil side has no effect on the resistance of the lower coil side.

4. The resistance ratio for the upper coil side of a

two-coil-side-per-slot fractional pitch winding having n layers per coil side reduces to:

$$K = M_r + \left(\frac{4\,n^2 - 1}{3} + n^2 \cos\theta \right) N_r$$

θ is the phase angle between the currents in the upper and lower coil sides.

By combining this with the ratio just preceding, we obtain the resistance ratio for a coil, one side of which is above a coil side carrying a current which differs in phase by θ.

The hottest conductor is the one at the top of the coil side which has beneath it current of the same phase. The fact that, with solid bar windings, the heat developed is not uniformly distributed throughout the winding may be no inconsiderable argument against their use.

5. Our method of attack enables us to obtain a simple solution for the relation between the currents in a double squirrel-cage winding. Neglect the effect of the end rings. In this case the constant of integration, A, in equation (3) is determined by the fact that the resistance drop in the lowest element of the upper bar is the same as the impedance drop in the lower bar due to its resistance and to all of the leakage flux which does not link any portion of the upper bar. The vector equation for the current density in the upper bar may be written:

$$\dot{c} = \frac{I_2 \dot{Z}_2}{\rho} \cosh \alpha x + \frac{I_2 R_1}{\rho} \alpha d \sinh \alpha x$$

$I_2 \dot{Z}_2$ is the vector impedance drop in the lower bar per centimeter;[6] ρ and R_1 are respectively the resistivity and the true resistance per centimeter of the upper bar whose depth is d centimeters. α is calculated for the upper bar.

The vector current in the upper bar is:

$$I_1 = \frac{I_2 \dot{Z}_2}{R_1} \frac{\sinh \alpha d}{\alpha d} + I_2 \cosh \alpha d - I_2$$

6. $I_2 Z_2$ is the voltage drop due to resistance and flux that does not link the upper bar.

The process of calculating the heat loss in the upper bar by substitution in equation (**9a**) is much simplified if we let

$$P = \frac{m\, Z_2}{R_1} \cos(\theta_2 + \beta) + n \cos \delta$$

and

$$Q = \frac{m\, Z_2}{R_1} \sin(\theta_2 + \beta) + n \sin \delta$$

where:

$$Z_2 = Z_2 \,/\underline{\theta_2}\,; \quad \frac{\sinh \alpha\, d}{\alpha\, d}\text{[7]} = m \,/\underline{\beta}\,;$$

$$\cosh \alpha\, d = n\, /\delta$$

The expression for the loss in the upper bar is:

$$I_2^2 R_1 \{ [(P-1)^2 + Q^2]\, M_r + P\, N_r \}$$

This method of solution for the relation between the currents in a double squirrel cage should prove of considerable value in any analysis of the design of such windings.

Finely laminated windings may be of three types: Those in which the laminations are joined at the ends of each half turn; those in which they are joined at the ends of each turn; and those in which the laminations are continuous throughout a single coil. The first is like a solid bar winding except that, as noted previously, the real resistivity of the copper should be multiplied by the ratio of the length of the half turn to the length of the core. The resistance ratio then applies to the whole winding and not to the embedded portion solely. The resistance ratios for the second type depend upon whether the end turn between the coil sides is untwisted or twisted. The resistance ratio for the third type depends upon whether the end turns are untwisted, twisted on one side only, or twisted on both sides. One considerable advantage of continuous laminations is that the heat developed is the same in all of the conductors.

6. Type two: End turn untwisted. The arrangement of the coil sides is shown in Fig. 5. The heavy

7. See Table IV, Tables of Complex Hyperbolic and Circular Functions.

line across the conductors indicates the same lamination. The current, I_2, below the upper coil side may be in phase with the current above it or differ from it by 60 or 90 degrees. The differential equation of the form (1) applying to this case for the pth layer from the bottom is:

$$2\rho \frac{\partial \dot{c}}{\partial x} - j\frac{4\pi\omega}{s}\left\{ 2(p-1)I_1 + I_2 \right\}$$

$$- j\frac{4\pi\omega}{s} 2\int_0^x w\, \dot{c}\, \partial x = 0$$

Comparing this equation (1) shows that the vector constant I_0 equals $(p-1) I_1 + \frac{I_2}{2}$; $I_2 = n I_1 /\underline{\theta}$,

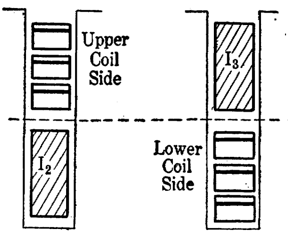

FIG. 5

n is the number of layers in the coil side and θ the phase angle between the currents in the upper and lower coil sides.

Making this substitution, equation (9) reduces to:

$$\text{Heat loss} = R I_1^2 \{ M_r + [p^2 - p + n(p - 1/2)\cos\theta + n^2/4] N_r \}$$

The resistance ratio for the entire coil is one-nth of the summation of this expression from $p = 1$ to $p = n$. It reduces to:

$$K = M_r + \left(\frac{7n^2 - 4}{12} + \frac{n^2}{2}\cos\theta \right) N_r$$

7. Type three: End turns untwisted. In this case the heat developed is the same in each conductor. The differential equation now becomes:

$$2n\rho\frac{\partial \dot{c}}{\partial x} - j\frac{4\pi\omega}{s}\sum_1^n \{2(p-1)I_1 + I_2\}$$

$$- j \frac{4 \pi \omega}{s} \; 2 n \int_0^x w \, c \, \partial x = 0$$

In this case $I_0 = \frac{n-1}{2} I_1 + \frac{I_2}{2}$. Making this substitution in equation (**10**) gives the resistance ratio for the whole coil.

$$K = M_r + \left(\frac{2 n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N_r$$

8. Type two: End turns twisted. When the end turns are twisted on one side only the top laminations in one coil side become the bottom laminations of the other coil side in corresponding layers. See Fig. 6. The lines across the layers trace the positions of one continuous lamination—type three. In type two, however, the laminations are joined at the beginning and end of each turn. The differential equation which applies to the pth layer is:

$$2 \rho \frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{s} I_2 + \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} \{ (p-1) I_1 (d - x) + (p-1) I_1 x \} - j \frac{4 \pi \omega}{s} \left\{ \int_0^x w \, c \, \partial x + \int_d^x w \, c \, \partial x \right\} = 0 \qquad (\mathbf{11})$$

This readily reduces to:

$$\frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left(\frac{I_2}{2} - \frac{I_1}{2} \right) - j \frac{4 \pi \omega}{s \rho} \int_0^x w \, c \, \partial x = 0$$

Notice that the mutual effect of the layers upon each other is eliminated, by twisting the end connection (third term (11)). The resistance ratio is thus the same for each turn. It is independent of the current in the lower coil side, but it does depend upon the number of layers in the coil side.

$$K = M_r + \frac{n^2 - 1}{4} N_r$$

9. Type three: End connections twisted on one side only. The differential equation of the form (1) is:

$$2 n \rho \frac{\partial \underset{\cdot}{c}}{\partial x}$$

$$+ \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_2 \{ (d - x) + x + (d - x) + \cdots\cdots$$

$$+ \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_1 \left\{ \begin{array}{l} (d - x) + 2x + 3(d - x) + \cdots\cdots \\ + x + 2(d - x) + 3x + \cdots\cdots \end{array} \right\}$$

$$- j \frac{4 \pi \omega}{s} \{ n \int_0^x w \underset{\cdot}{c} \, \partial x + n \int_d^x w \underset{\cdot}{c} \, \partial x \}$$

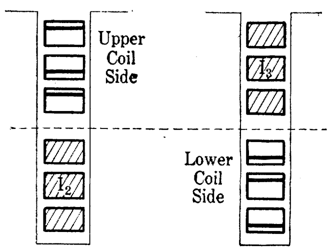

FIG. 6

The third term of this equation is always zero, but the second term may or may not be. It is thus apparent that there are two cases to be considered, *viz.*, when the number of layers is even and when it is odd. If n is even the second term is zero and the equation reduces to:

$$\frac{\partial \underset{\cdot}{c}}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left(- \frac{I_1}{2} \right) - j \frac{4 \pi \omega}{s \rho} \int_0^x w \underset{\cdot}{c} \, \partial x = 0$$

Notice that this equation is independent of the number of layers and the current in the lower coil side. This arrangement of an even number of continuously laminated layers whose end turns are twisted on one side only gives the smallest resistance ratio of any of the cases considered.

$$K = M_r - 1/4\, N_r$$

From the forms of M_r and N_r this is readily shown to be the M_r for a conductor one-half as deep. This same condition of current distribution is obtained by having an even number of laminated conductors side by side in the slots if the end connections are twisted on one side only. The resistance ratio is then independent of the number of layers in the coil. If n is odd, the differential equation reduces to:

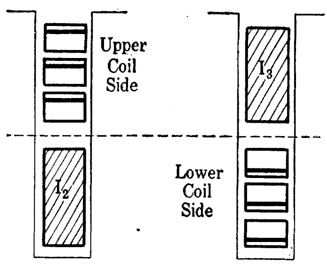

FIG. 7

$$\frac{\partial c}{\partial x} - j\,\frac{4\,\pi\,\omega}{s\,\rho}\left\{\frac{I_2}{2\,n} - \frac{I_1}{2}\right\} - j\,\frac{4\,\pi\,\omega}{s\,\rho}\int_0^x w\,c\,\partial x = 0$$

The resistance ratio is now:

$$K = M_r$$

This ratio is the same as in the case of a *single* solid conductor of the *same* depth, whereas if there are an even number of layers the ratio is the same as for a single solid conductor of one-half the depth of the laminated one.

10. Type three: End connections twisted on both sides (See Fig. 7). The differential equation of the form (1) is:

$$2\,n\,\rho\,\frac{\partial c}{\partial x}$$

$$+ \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_2 (d - x) n$$

$$+ \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_1 \left\{ \begin{array}{l} (d - x) + 2 x + 3 (d - x) \\ \qquad\qquad\qquad + \ldots\ldots \\ + x + 2 (d - x) + 3 x \\ \qquad\qquad\qquad + \ldots\ldots \end{array} \right\}$$

$$- j \frac{4 \pi \omega}{s} \left\{ n \int_0^x w \underset{\cdot}{c} \, \partial x + n \int_d^x w \underset{\cdot}{c} \, \partial x \right\} = 0$$

This reduces to:

$$\frac{\partial \underset{\cdot}{c}}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left(\frac{I_2}{2} - \frac{I_1}{2} \right)$$

$$- j \frac{4 \pi \omega}{s \rho} \int_0^x w \underset{\cdot}{c} \, \partial x = 0$$

The resistance ratio is:

$$K = M_r + \frac{n^2 - 1}{4} N_r$$

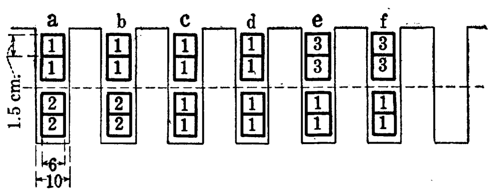

Fig. 8

It is independent of the current in the lower coil side. Notice that this is the same ratio as was obtained for case 8.

Enough illustrations of the method and the simplicity of its application have been given. There follows a numerical calculation of the resistance ratios for a given arrangement of conductors of various types. The winding data are: Three-phase with four slots per pole per phase; coil pitch of 10 slots; two turns per coil; conductors 1.5 cm. deep; length of embedded portion and of end turns the same; frequency 60 cycles, the ratio of width of copper to width of slot, 0.6; resistivity, 2100 c. g. s. units.

For solid conductors:

$$\alpha d = 1.5 \sqrt{\frac{8 \pi^2 \times 60 \times 0.6}{2100}} \underline{/45°}$$

$$= 1.74 \underline{/45°}$$

From curve

$$M_r = 1.20$$
$$N_r = 0.73$$

For laminated conductor:

$$\alpha d = 1.5 \sqrt{\frac{8 \pi^2 \times 60 \times 0.6}{2 \times 2100}} \underline{/45°}$$

$$= 1.23 \underline{/45°}$$

From curve

$$M_r = 1.05$$
$$N_r = 0.20$$

The arrangement of the conductors of one phase before one pole is shown in Fig. 8. The resistance ratios for phase one are given in the following table for the various cases considered.

TABLE I

	Ratio
Solid Conductors	
Lower coil sides (slots c, d, e & f)	1.93*
Upper " " (" a and b)	6.31*
" " " (" c and d)	7.77*
Entire winding including end turns	2.75†
Infinitely Laminated Conductors, Type 1.	
Lower coil sides (slots c, d, e & f)	1.25
Upper " " (" a and b)	2.45
" " " (" c and d)	2.85
Entire winding	1.95
Infinitely Laminated Conductors, Type 2.	
Untwisted (slots c and d)	1.85
" (" a, b, e and f)	1.65
" (entire winding)	1.75
Twisted (entire winding)	1.20
Infinitely Laminated Conductors, Type 3.	
Untwisted (slots c and d)	1.80
" (" a, b, e and f)	1.60
" (entire winding)	1.70
Twisted in one end connection (entire winding)	1.005
Twisted in both end connections (entire winding)	1.20

*Embedded portion.

†The ratio of the heat developed in the top conductor of slots, c or d to that developed in the bottom conductor is as $\frac{9.96}{1.20}$ or as 8.3 to 1.

This may be a very important consideration, even more than the resistance ratio for the entire winding.

Leakage Reactance Volts. As previously stated, the entire leakage impedance due to resistance and slot leakage flux lying wholly within the conductors themselves may be computed by the proper combination of equations (**7a**) and (**8a**). This method offers little or no advantage when calculating the resistance. It has been shown that in the case of solid bar winding I_0 equals I_b, (equation **8a**). Thus it is probable that the expressions for reactance are similar to those for resistance except that M_x and N_x would replace M_r and N_r. Such proves to be the case. In the case of laminated conductors, however, there is an added term in the expressions for reactance. Consider the general case of a three-phase fractional pitch winding, a typical arrangement of which is shown in Fig. 8. There will usually be slots in which both coil sides are in the same phase. There will also be slots in which the top coil side is in phase one, for example, and the lower coil side in phase two, together with an *equal* number of slots in which the lower coil side is in phase *one* and the upper coil side in phase three. In the general polyphase case of p phases there will be slots occurring in pairs in which the currents differ in phase by plus and minus π/p radians. Let us designate these symmetrical pairs as fractional pitch slots and those in which the upper and lower coil sides are in the same phase as full pitch slots. Accordingly there are two fractional pitch and two full pitch slots per pole per phase in the winding illustrated in Fig. 8. Let n be the number of layers per coil side and R the true resistance of the conductors considered.

1. Solid conductors: Full pitch slots. Apply equations (**7a**) and (**8a**), but divide by the phase current, I_1, in order to obtain the impedance directly. With solid conductors the coefficient of $\alpha^2 d^2$ in equation (**8a**) is zero. Let p be the number of any conductor measured from the bottom of the slot. There are $2n$ conductors in both layers in the slot.

The vector expression for the impedance is:

$$Z = \frac{R}{2n} \left\{ \sum_{1}^{2n} \left(M + \frac{p-1}{2} N \right) \right.$$

$$+ \sum_{1}^{2n} (p-1)(1/2 + p - 1) N \Big\}$$

The first term is the summation of the resistance drops (equation **7a**) divided by the current; the second term is the summation of the reactive drops per ampere produced in each of the $(p - 1)$ conductors below the pth conductor (equation **8a**) by the flux within the latter.
This reduces to:

$$Z = R \left\{ M + \frac{4n^2 - 1}{3} N \right\}$$

The resistance is the real portion of this expression and the reactance the imaginary portion. Thus:

$$r = R \left\{ M_r + \frac{4n^2 - 1}{3} N_r \right\}$$

$$x = R \left\{ M_x + \frac{4n^2 - 1}{3} N_x \right\}$$

2. Solid conductors: Fractional pitch slots (taken in pairs as described). The expression for the impedance reduces to

$$Z = \frac{R}{2n} \Big\{ \sum_{1}^{n} \left[M + \left(\frac{p-1}{2} + n \,\underline{/\theta} \right) N \right]$$

$$+ \sum_{1}^{n} \left[M + \frac{p-1}{2} N \right]$$

$$+ \sum_{1}^{n} (p-1) \left[\frac{1}{2} + (p-1) + n \,/\, \theta \right] N$$

$$+ n \sum_{1}^{n} \left[\frac{/-\theta}{2} + (p-1) \,/-\theta + n \right] N$$

$$+ \sum_{1}^{n} (p-1) \left[\frac{1}{2} + (p-1) \right] N \Big\}$$

The first two terms are respectively the summations of the resistance drops per ampere (equation **7a**) in the upper and lower coil sides; the third term is the summation of the reactive drops per ampere produced in each of the $(p-1)$ conductors below the pth conductor of the upper coil side (equation **8a**) by the flux within the latter due to its own current and all of that below it in the slot; the fourth term is the reactive drop (equation **8a**) in the lower coil side due to the flux within the coil side above it which carries a current having a relative phase angle of $\underline{/-\theta}$; the last term is the summation of the reactive drops per ampere produced in each of the $(p-1)$ conductors below the pth conductor of the lower coil side (equation **8a**) by the flux within the latter.
This reduces to:

$$Z = R\left\{ M + \left(\frac{5\,n^2 - 2}{6} + \frac{n^2}{2}\cos\theta \right) N \right\}$$

The resistance is:

$$r = R\left\{ M_r + \left(\frac{5\,n^2 - 2}{6} + \frac{n^2}{2}\cos\theta \right) N_r \right\}$$

The reactance is:

$$x = R\left\{ M_x + \left(\frac{5\,n^2 - 2}{6} + \frac{n^2}{2}\cos\theta \right) N_x \right\}$$

This expression is general for both fractional and full pitch slots. For the latter θ equals zero.

3. Finely and continuously laminated conductors (type three) with untwisted end connections. Consider the general case of symmetrical pairs of fractional pitch slots in which the currents in the coil sides lying in the same slot differ in phase by plus and minus θ. R is the true resistance of one coil. In this case

$$I_0 = \frac{n-1}{2} I_1 + \frac{I_2}{2} \; ;$$

where I_2 is the current in the lower coil side.

$$Z = \frac{R}{2\,n}$$

$$\Big\{ 2 \sum_1^n \Big[M + \Big(\frac{n-1}{2 \times 2} + \frac{n \angle \theta}{2 \times 2} \Big) N \Big]$$

$$+ \sum_1^n (p-1) \Big[\Big(\frac{1}{2} + \frac{n-1}{2} + \frac{n \angle \theta}{2} \Big) N$$

$$+ \Big(p - 1 + n \angle \theta - \frac{n-1}{2} - \frac{n \angle \theta}{2} \Big) \alpha^2 d^2 \Big]$$

$$+ n \sum_1^n \Big[\Big(\frac{\angle - \theta}{2} + \frac{n-1}{2} \angle - \theta + \frac{n}{2} \Big) N$$

$$+ \Big((p-1) \angle - \theta + n - \frac{n-1}{2} \angle - \theta$$

$$- \frac{n}{2} \Big) \alpha^2 d^2 \Big]$$

$$+ \sum_1^n (p-1) \Big[\Big(\frac{1}{2} + \frac{n-1}{2} + \frac{n \angle \theta}{2} \Big) N$$

$$+ \Big(p - 1 - \frac{n-1}{2} - \frac{n \angle \theta}{2} \Big) \alpha^2 d^2 \Big] \Big\}$$

The first term is the summation of the resistance drops per ampere (equation **7a**) in the upper and lower coil s˙des. Due to the fact that laminations are continuous the current distribution is the same in each conductor of the coil. Thus the resistance drops are also the same for each conductor. The second, third and fourth terms respectively correspond to the third, fourth and fifth terms in the preceding case.
This reduces to:

$$Z = R \Big\{ M + \Big(\frac{2n^2 - 1}{4} + \frac{n^2}{2} \cos\theta \Big) N$$

$$+ \frac{4n^2 - 1}{12} \alpha^2 d^2 \Big\}$$

The resistance and reactance are respectively the real and imaginary portions of this expression. Thus:

$$r = R \Big\{ M_r + \Big(\frac{2n^2 - 1}{4} + \frac{n^2}{2} \cos\theta \Big) N_r \Big\}$$

$$x = R \left\{ M_x + \left(\frac{2 n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N_x + \frac{4 n^2 - 1}{12} \left| \alpha \right|^2 d^2 \right\}$$

$| \alpha |^2$ is the square of the numerical value of α, *viz.*, $\dfrac{8 \pi^2 w f}{\rho s}$.

4. Finely and continuously laminated conductors (type three) with end connections twisted on both sides. Consider the general case of symmetrical pairs of fractional pitch slots in which the currents in the coil sides lying in the same slot differ in phase by θ. Due to the twist in the end connections the current density is the same at points equally distant from the bottom of half of the conductors, and from the top of the other half. The expression for the flux within the conductor has already been given for the first condition. When current density is measured from the top of the conductor, the expression for the flux within it is:

$$\varphi = \frac{4 \pi}{s} \int_d^0 \partial x \int_d^x w \, \underset{\cdot}{c} \, \partial x$$

This readily reduces to:

$$\varphi = \frac{1}{j \omega} \frac{\rho}{w d} \left\{ (I_1 + I_0) \alpha^2 d^2 - \left(\frac{I_1}{2} + I_0 \right) N \right\}$$

The total flux within the conductor including that produced by the current, I_b, below it is:

$$\varphi_0 = \frac{1}{j \omega} \frac{\rho}{w d} \left\{ (I_1 + I_0 + I_b) \alpha^2 d^2 - \left(\frac{I_1}{2} + I_0 \right) N \right\}$$

The voltage produced in every conductor below the one in question by this flux is:

$$E = R \left\{ (I_1 + I_0 + I_b) \alpha^2 d^2 - \left(\frac{I_1}{2} + I_0 \right) N \right\}$$

where R is the true resistance of a half turn.

For conductors in which the current density is given for values of x measured from the top, rather than from the bottom, the resistance drop $\rho\, c$ is that in the bottom element. The flux within this conductor then produces an *additional* voltage in it. In this case

$$I_0 = \frac{I_2}{2} - \frac{I_1}{2}$$

$$Z = \frac{R}{2\,n}$$

$$\Bigg\{ 2 \sum_1^n \left[M + \left(\frac{n \angle \theta}{2 \times 2} - \frac{1}{2 \times 2} \right) N \right]$$

$$+ \sum_1^n (p-1) \left[\left(\frac{1}{2} + \frac{n \angle \theta}{2} - \frac{1}{2} \right) N \right.$$

$$\left. + \left((p-1) + n \angle \theta - \frac{n}{2} \angle \theta + \frac{1}{2} \right) \alpha^2 d^2 \right]$$

$$+ n \sum_1^n \left[\left(\frac{\angle -\theta}{2} + \frac{n}{2} - \frac{\angle -\theta}{2} \right) N \right.$$

$$\left. + \left((p-1) \angle -\theta + n - \frac{n}{2} + \frac{\angle -\theta}{2} \right) \alpha^2 d^2 \right]$$

$$+ \sum_1^n p \left[\left(1 + \frac{n}{2} \angle \theta - \frac{1}{2} + (p-1) \right) \alpha^2 d^2 \right.$$

$$\left. - \left(\frac{1}{2} + \frac{n}{2} \angle \theta - \frac{1}{2} \right) N \right] \Bigg\}$$

In this expression R is the true resistance of the coil. The terms are written in the same order as in the previous case.

This reduces to:

$$Z = R \left\{ M + \frac{n^2 - 1}{4} N + \left(\frac{7\,n^2 - 1}{12} \right. \right.$$

$$\left. \left. + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2 \right\}$$

The method of calculating the impedance should now be sufficiently clear. The final equations for the impedance of finely and continuously laminated conductors whose end turns are twisted on one side only are given without showing their detailed construction.

There are two cases to consider,—one with an even number of layers per coil side, and the other with an odd number of layers.

For n, even

$$Z = R\left\{ M - \frac{N}{4} + \left(\frac{10\,n^2 - 1}{12} + \frac{n^2}{2}\cos\theta \right) \alpha^2 d^2 \right\}$$

For n, odd

$$Z = R\left\{ M + \left(\frac{10\,n^2 - 1}{12} + \frac{n^2}{2}\cos\theta \right) \alpha^2 d^2 \right\}$$

The formulas are given in such detail that it must be evident how the effects of unbalanced currents may be calculated. If there are marked harmonics in the currents the heating loss for each harmonic may be calculated as if the others were absent. The resulting loss is the sum of the component losses. The resistance ratios increase with the frequency so that higher harmonics of any considerable magnitude may prove troublesome. For example, if the currents should contain 20 per cent fifth and seventh harmonics, the resistance ratio for the entire winding—solid conductors—would increase from 2.75 as given in Table I to 2.95. This neglects any skin effect in the end turns which would probably be considerable for these harmonics. It also neglects the fact that with higher harmonics there would be a marked magnetic skin effect in the laminations surrounding the conductor which might raise the saturation to such a point that the fundamental assumptions would no longer hold.

The increase in the ratio for the embedded portion only is much more marked. The ratio for the embedded portion of the entire winding as calculated from Table I is

$$\frac{4 \times 1.93 + 2 \times 6.31 + 2 \times 7.77}{8} = 4.49$$

The ratio for the entire embedded portion with harmonics present becomes 6.73. The ratio of the heats developed in top and bottom conductors of slots *c* or *d* becomes $\frac{15.8}{1.39}$ or 11.4 when these harmonics are present instead of the value of 8.3 as given in the table.

By making the proper assumptions, this method of analysis allows us to account for the hysteresis and eddy current losses in the armature teeth and core due to the leakage flux, the effect of which we are discussing. Assume that, due to these iron losses, each tube of flux lags behind the net current that is producing it by the same angle, η. If this be the case the reactive drop will lead the resistance drop by $(\pi/2 - \eta)$ radians instead of by $\pi/2$ radians as we have assumed.

Thus $$\alpha^2 = \frac{8\,\pi^2 f\,w}{\rho\,s} \;\angle \frac{\pi}{2} - \eta$$

and $$\alpha = \sqrt{\frac{8\,\pi^2 f\,w}{\rho\,s}} \;\angle \frac{\pi}{4} - \frac{\eta}{2}$$

New values of the complex quantities M and N may be calculated for this value of α and substituted in the expressions for effective resistance and reactance already obtained. Whether or not this method will produce accurate results can only be determined by much experimental research.

SUMMARY OF FORMULAS

SOLID CONDUCTORS

Ratio $\dfrac{\text{Alternating-current resistance}}{\text{Direct-current resistance}}$

*p*th conductor from bottom one-coil-side-per-slot bar winding.

$$M_r + p\,(p - 1)\,N_r$$

One-coil-side-per-slot with *n* layers, or lower coil side with *n* layers.

$$M_r + \frac{n^2 - 1}{3}\,N_r$$

Upper coil side, n layers, two-coil-side-per slot, fractional pitch.

$$M_r + \left(\frac{4\,n^2 - 1}{3} + n^2 \cos\theta \right) N_r$$

Finely Laminated Conductors (laminations soldered at beginning and end of each turn) fractional pitch[1]

$$\text{Ratio } \frac{\text{A-C. resistance.}}{\text{D-C. resistance}}$$

End turn untwisted, pth conductor from bottom of upper coil side.

$$M_r + [p^2 - p + n\,(p - 1/2)\cos\theta + n^2/4]\,N_r$$

End turn untwisted, each coil side.

$$M_r + \left(\frac{7\,n^2 - 4}{12} + \frac{n^2}{2} \cos\theta \right) N_r$$

End turn twisted, each coil side.

$$M_r + \frac{n^2 - 1}{4}\; N_r$$

Finely Laminated Conductors, soldered at the beginning and end of each coil. Ratio of impedance to direct-current resistance is given for a pair of coil sides below one of which is current lagging by θ and abovethe other current leading by θ. n layers per coil side.[2]

End turns untwisted.

$$M + \left(\frac{2\,n^2 - 1}{4} + \frac{n^2}{2} \cos\theta \right) N + \frac{4\,n^2 - 1}{12}\;\alpha^2\,d^2$$

End turns twisted both sides.

$$M + \frac{n^2 - 1}{4}\,N + \left(\frac{7\,n^2 - 1}{12} + \frac{n^2}{2} \cos\theta \right) \alpha^2\,d^2$$

End turns twisted one side, n even.

$$M - \frac{N}{4} + \left(\frac{10\,n^2 - 1}{12} + \frac{n^2}{2} \cos\theta \right) \alpha^2\,d^2$$

End turns twisted one side, n odd.

$$M + \left(\frac{10\,n^2 - 1}{12} + \frac{n^2}{2} \cos\theta \right) \alpha^2\,d^2$$

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1. Two coil sides per slot, n layers per coil side.
2. In calculating the impedance only leakage flux that lies within the conductors is considered. There are well known methods for calculating the reactance due to other leakage flux.

Discussion on "Synchronous Motors for Ship Propulsion" (Henningsen), "Magnetic Properties of Compressed Powdered Iron" (Speed and Elmen), and "Heat Losses in Conductors in A-C. Machines" (Lyon), Salt Lake City, Utah, June 24, 1921.

Wm. J. Foster: There has always been more or less uncertainty as to proper field of application of the induction and synchronous motor. I think Mr. Henningsen's paper describes the application of the synchronous motor in this particular case in a manner to justify it. I will not review that, but I recall a somewhat similar case many years ago where it had been generally decided that the proper motor for motor-generator sets, for transforming alternating current transmitted a distance to direct current for use on the Edison three-wire system was the induction motor, and one of our large companies installed a number of what were then very large motor-generator sets, with the idea that the induction motor was simpler in operation and in that particular application the synchronous motor was not to be considered. Well, it was not very long before the engineers of that company had come to the conclusion that they would try the synchronous motor, when increasing the size of the substation and it was discovered—probably they anticipated quite well what would happen—that the synchronous motor was superior to the induction motor in certain respects, such as its ability to stay in step. An argument frequently used for the induction motor is that you can get it back into step much easier. In that particular case it was found that if the potential was considerably reduced on the line, as happens occasionally the synchronous motor had the ability to carry the load and remain in step, which we all now know is characteristic of the synchronous motor with a given excitation. Another surprising difference was the convenience in manipulation, in starting up, as it was their practise after one set was in operation to start the other sets from the d-c. end, and it would have been expected in advance that the switch might be thrown on the induction motor without serious disturbance. It was found that the only way of deciding when to throw the switch was to go by the sound, and that operators were throwing in the switch on the induction motor in such a manner that it gave bad jolts to the system, whereas the synchronous motor could be put on the line without jolt. Regarding ship propulsion I wish to say that when the synchronous motor was first suggested it was looked upon as rather absurd, but if I am not mistaken there are one or two manipulations of the ship where the synchronous motor can do a little bit better than an induction motor. I hope to see not fewer applications of the induction motor but more of synchronous motors or synchronous condensers, as I think it will help out the situation of generator design very materially especially the steam turbine generators of the largest size and highest speed. It would be much better for the life of the machine and from consideration of the holding together of different parts of systems, if the requirement of low power factor on such generators could be dispensed with, and in the case of the largest sizes that they be built for unity power factor. I was pleased in listening to Mr. Baum's paper the other day to find him so strongly recommending the installation of synchronous condensers in connection with the high-voltage transmission that he is now working on. By such installation of synchronous condensers I see no reason why the generator should any longer be handicapped by the requirement of operation at 80, 85 or 90 per cent power factor. If that were removed then any given generator could be made suitable for operating on higher transmission circuits than at present, which is a very desirable feature, *i. e.*, it could be built so as to better hold down the potential for a given charging current

F. G. Baum: We have here given an illustration of the remarkable application of the synchronous motor, and my purpose now is to try to impress on the power men the great advantage of adding synchronous machinery to their system. Make your customer put them in wherever you can. You are adding to your generating capacity in doing so. The work of design will be very much reduced. Everybody today knows and says they want a generator designed for 80 per cent power factor, another 85 per cent and 50 per cent overload. If you will get rid of that power factor feature in designing generators, you will make the generator cheaper, simpler, and much less liable to overvoltage due to speed fluctuations of the system. Every synchronous motor added to the system is an asset, adding to generating capacity.

With respect to Mr. Lyon's paper. It is a peculiar characteristic of almost all electrical apparatus that it works best when kept in service all the time. The reason for that is that deterioration is largely due to the maximum temperature ranges that occur and the number of those cycles that occur in a given time. Putting a generator into service, loading it up so that it gets extremely hot and taking it out of service every day, is as though you take a pipe line out every day and fill it up and then empty it. You destroy that pipeline in a very short time and we found that out fifteen years ago in our generating stations. When the load is light we take a machine out of service at night and start it up again, a great deal of trouble results eventually from the extreme temperature changes. We should keep those machines going all of the time.

Wm. Fondiller: I wish to bring out several points in connection with the development of the powdered iron core which I think will be of interest.

The authors have referred briefly to the experiments leading up to the present form of powdered iron core. It became evident after the work of testing different iron mixtures was started, that special testing means must be devised for determining the characteristics in order to bring the development work to a conclusion in a reasonable time and at a reasonable cost. When the number of possible combinations is considered of varying pressure, insulating materials, composition of iron, fineness, etc., it will be appreciated that a formidable problem was presented from the laboratory standpoint.

To meet the situation, special apparatus was designed enabling tests for permeability and iron losses to be made without the necessity of applying a winding to the core rings. These two devices comprising a permeammeter and a core loss tester, enabled fairly accurate measurements to be made at telephone frequencies with great dispatch. It is hoped to make these new testing instruments the subject of a future paper.

Messrs. Speed and Elmen have described the major operations in the production of the powdered iron cores. It should be understood that in the actual carrying out of the processes, great care is needed in order to secure the uniformity demanded by the close limits imposed on telephone apparatus. This makes necessary careful checks on the successive operations by means of tests while the material is in process of manufacture. In this way it has been possible to keep the initial permeability of core rings made from powdered iron within a few per cent of the nominal value; for example, the permeability of the grade "A" core material is regularly maintained within limits of 58 ± 5. So far as I know this is not possible with any other ferrous material commerically available.

One of the most important properties of the powdered iron core is its self-demagnetizing characteristics. This is shown clearly in Fig. 17 on the return curve after superposing comparatively large values of direct current. It will be noted that for the grade "B" material the a-c. permeability is unaffected when the superposed direct current is reduced to zero and in the case of grade "A" material it is reduced by only 4 per cent. This must be compared with an alteration in the a-c. permeability in the wire core of approximately 40 per cent for the same magnetic experience. Even for values of direct current producing the saturation value of residual magnetism the effect on the a-c. permeability is quite small.

The authors have referred to the extensive use of the cores which they have described in the Bell plant in this country. I would add that the high efficiency of loading coils using these cores together with their satisfactory characteristics in connection with telephone repeater operation have caused their adoption as the standard core for loading coils in toll cables, not only in this country, but in Europe. The powdered core loading coil has been introduced abroad by the Western Electric Co. with great success. This improved loading coil has been adopted by the British Post Office for its toll cables and is being installed at the present time by the Western Electric Company in Sweden in the Stockholm-Goteborg cable.

H. L. Hibbard: I presume Mr. Henningsen's paper is simply to explain the successful application of synchronous motors, and I agree with Mr. Henningsen that probably a great many applications of this kind will be made in the future, since I think the day of the electrically operated ship is here; but this is the question I want to ask—Are we to understand from this paper that the author recommends the synchronous motor as a proper application for all merchant ships? I simply raise the question because it seems to me a large field is opening up along the line of oil engine propulsion, where d-c. apparatus would be found better suited.

G. Semenza: I only want to support what Mr. Foster has said about the synchronous motors with an instance coming from the other side of the ocean. In 1895 after the electrification of the tramways in Milan, a rather large system, synchronous motors had been used for motor-generators. After two years, enlarging the station somebody suggested to put in an induction motor because they thought it would be better under certain points of view. Well, after three years that motor was taken out and a synchronous motor substituted. That shows that the conclusions on the two sides of the ocean are just the same.

C. A. Copley: It might be of interest to have somebody explain why it was that synchronous motors were not used in the new pumping plant which is about 7 or 8 miles from here. I believe it contains about 2400 h. p. in induction motors and lifts the water 300 feet.

W. E. Thau (read by W. E. Skinner): The author has outlined two of the special operating requirements of ship propulsion machinery, namely, the torque requirement during rough weather and the torque requirement during reversing, and has shown how these conditions can be surmounted by the synchronous motor. There is a third special requirement, characteristic, particularly of multiple screw ships, namely, the excess torque required by the inboard screw, or screws, when making a turn with hard-over rudder (by inboard screws is meant the screws toward the center of the circle described while turning). In this case, if the propeller speed is maintained constant and the delivered power unlimited, the torque required particularly by the inboard screws is considerably in excess of the normal zero rudder requirement, and will easily reach 150 per cent or more of the normal. To overcome this condition, as far as practicable, the steam flow is limited to an amount corresponding to from 5 per cent to 10 per cent above normal. With the steam so limited, the additional torque required by the inboard screws causes a decrease in the speed of the propeller and prime mover when the ships is making the turn with hard-over rudder. With separate prime movers for the port and starboard sides, this effect is most noticeable, and in such cases the speed will drop from 20 to 25 per cent of normal, and as the steam flow, and consequently the kw. input to the motors is limited as stated, the excess torque will amount to 30 to 35 per cent.

Usually the speed of a ship is reduced when running in a heavy sea for reasons other than the load conditions on the propeller. Reducing the speed in this manner will decrease the excess torque requirements to values slightly in excess of the normal capacity of the machine.

These special torque requirements not only influence the motor characteristics, but also have an important bearing on the generator design as the generator must at all times be capable of maintaining voltage above the breakdown point for the excess load conditions. These conditions, particularly the rough sea and turning requirements, therefore, are the determining factors in the amount of generator field current to be carried for uninterrupted service. In the case of the synchronous motor, where special efforts are necessary to obtain sufficient torque for quick reversals, the generator field requirement under such conditions is of considerably more importance than in the case of a wound secondary induction motor drive. As the propulsive equipment for a ship is a self-contained unit, the requirements outlined necessitated special generator design, and the ordinary maximum rated machine such as is used in central stations, is not applicable.

The principal attraction of a synchronous motor drive as compared with the ordinary induction motor drive is the unity power factor with its consequent decrease in the cost and weight, as the author states. The third gain mentioned, namely, better efficiency, is of little importance as the net gain in the plant efficiency is inconsequential for the reason that the excitation for the synchronous motor must be supplied through an auxiliary d-c. turbine set, the unit steam consumption of which is at least twice that of the main turbine through which the excitation for the induction motor is supplied. From a ship viewpoint, therefore, the economy has little advantage, if any, over the ordinary induction motor.

However, the unity power factor advantage of the synchronous motor is equally obtainable by means of an induction motor system using a phase advancer for power factor correction to unity. The phase advancer is a small, simple, commutating machine, which is connected to the motor secondary (wound secondary). The function of the phase advancer is to supply the excitation for the induction motor just as d-c. excitation is supplied for the synchronous motor.

Thus, the induction motor drive with the phase advancer not only provides the advantages resulting from unity power factor, but in addition provides the superior torque characteristics of the wound secondary induction motor for maneuvering, and also obviates the necessity for synchronous operation with the generator.

The generator requirements are substantially the same for this type of ship drive as for that using the synchronous motor, and the net results of weight and cost are approximately the same.

As a matter of interest, the induction motor system using the phase advancer has the advantages of superior torque characteristics and of dissipating the energy of reversal in resistance external to the driving machinery. Furthermore, the control is simple, being no more complicated than that for the ordinary wound secondary induction motor.

In connection with the discussion of special features incorporated in the design of a-c. turbine electric systems, to meet the load requirements of ship propellers, it is important to note that no special features whatsoever must be considered in the case of the Diesel-electric d-c. system of propulsion. This system employs a number (three to six, depending upon the power required) of moderately high speed, small, reliable, Diesel engines driving direct-connected d-c. generators which supply power to direct-connected motors, the speed of which is regulated in the simplest and most economical manner by voltage control. With this system of drive, the prime movers, generators and motors need be designed for only the continuous normal requirements and no special precautions or additional material in the machines are necessary to keep the motors and generators in step when subjected to the overload torque conditions. The inherent characteristics of the machines are such that the latter automatically adjust themselves to the abnormal load conditions. The overloads are of short duration, and no cognizance need be taken of the consequent heating.

Paul P. Ashworth: The question was asked, why 2800 h. p. in induction motors have been installed in the pumping plant 7 or 8 miles north of Salt Lake. I had nothing to do with the installation and yet I can see fairly good reasons why motors of that type were chosen. One reason, of course—the most obvious reason—is the matter of cost, that, however, is not the controlling reason. Induction motors in pumping plants of that size always operate fully loaded, under which condition they have a power factor of somewhere around 85 to 93 per cent,—not a bad power factor at all. A further point is the character of the service—three or four months during the year and this during the time of minimum load on the power system; therefore, there is at that time a large excess in generating capacity—in current carrying capacity. Following out the suggestion of Mr. Baum that there is no particular objection, in fact there is an advantage, in loading the equipment to somewhere near capacity, so from that standpoint the pumping plant having induction motors is no disadvantage.

There is one point that might be mentioned in connection with our experience here with synchronous motors. There is a certain danger which should not be overlooked. We found for example at a large mining installation where a hoist operated by a synchronous motor was installed— a large 300 or 400 h. p. hoist operating 24 hours a day the year around,—that when the brushes on the exciter would begin to spark due to improper setting or inadequate maintenance, the operator would decrease the exciting current to the motor, to reduce sparking, so that instead of carrying unity power factor the motor was carrying a power factor of 50 per cent or less,—much less than any induction motor would carry under the same conditions. We have found it advisable to ask customers using synchronous motors to adjust the excitation to give 100 per cent power factor under approximately full-load conditions, and if possible to throw the rheostat handle down the mine or put it where the operator was not likely to use it to prevent sparking brushes. I merely want to indicate this danger, that with the use of the synchronous motor there is a possibility of getting very low power factor and thus to defeat one of the main reasons for installing synchronous equipment. The entrance of power factor into rate schedules will tend to force the installation and proper operation of equipment which will maintain good power factor. The nature of the service in any particular case will determine the type of motor equipment to be used.

S. P. Grace: A few weeks ago I had occasion to present a statement to the Public Service Commission of the City of New York showing the number of loading coils in use in New York City together with economies which resulted from their use. In the City of New York there are in use 100,000 loading coils in approximately 198,000 miles of telephone circuit. If the same transmitting efficiency had been obtained by the use of larger size copper wires there would have been an additional investment in the City of New York of $25,000,000, representing an annual charge of something like $4,000,000. You therefore can appreciate the very great savings which have been brought about through the loading coils.

A. M. Maccutcheon: I would like to ask Mr. Henningsen how they handle the synchronous motor when maneuvering a ship. It is very clear how they get the direct reversal, but to anyone who has been on a ship coming into a harbor they sometimes have to use two or three speeds coming up the channel,—forward speed, part speed, full speed, reversal; constant changes in the speed of the propellers.

H. W. Taylor (by letter): In the most general investigation of the problems dealt with in Mr. Lyon's paper there is no doubt that the use of hyperbolic functions of the complex variable afford the simplest form of solution. To those, however, who absorbed in practical work, or for other reasons prefer that the solutions of their problems should be presented to them if possible in still simpler forms, it is interesting to know that all the problems of eddy currents in stator conductors which occur in practical work can be dealt with by simple algebra.

In a paper by the present writer published in the *Journal of the Institution of Electrical Engineers,* in April 1920 (Vol. 58, page 279) the most general case involving the use of the complex hyperbolic functions was discussed, but the main part of the paper was devoted to the development of algebraic formulas.

The present occasion affords an opportunity of republishing the principal formulas in a form which continued practise with them has shown to be most useful.

The formulas for eddy current losses in stator conductors consist essentially of two factors, the first involving the mechanical dimensions of the conductor and the frequency, and the second involving the way in which the conductors are arranged in the winding.

The factors involved in the first factor of the formula are given in the expression

$$D = \frac{h^4 f^2 a^2}{80}$$

where h is the height of a lamination or section of the conductor in inches
f is the frequency in cycles per second
and a is the ratio in a slot of the copper width to the total slot width.

The numerical co-efficient assumes the copper of the conductor is operating at approximately 100 deg. cent.

If the height of the lamination is given in centimeters the numeric 80 becomes 2.

The expressions for the second factor of the complete expression may be divided under four headings as follows:

1. When the conductors are solid.
2. When the conductor is sub-divided with solid connections at the end of each half turn.
3. When the sub-divisions of a conductor are continued throughout a coil.
4. When the sub-divisions of a conductor are continued throughout a winding and are successively transferred in position.

Solid Conductors

Where each lamination may be considered as a separate conductor, the *extra* loss factor for the complete slot is given by the formula

$$D\left(\frac{4}{45} + \frac{p^2 - 1}{9}\right) \tag{1}$$

or

$$D\left(\frac{p^2}{9} - \frac{1}{45}\right) \tag{1a}$$

where p is the number of conductors under consideration; in the present instance, the total number of separate conductors in the slot.

If the slot conductors are divided into an upper and a similar lower layer and it is required to know the separate *extra* loss factors for each layer, then for the bottom layer we have the average loss factor

$$D\left(\frac{4}{45} + \frac{p^2 - 1}{9}\right) \tag{1}$$

as before, p being the number of conductors under consideration, in this case the number of separate conductors in each layer, and for the upper layer the average extra loss factor is

$$D\left(\frac{4}{45} + \frac{7\,p^2 - 1}{9}\right) \tag{2}$$

When the upper layer carries current of a different phase to the bottom layer the last formula is modified as follows:

$$D\left(\frac{4}{45} + \frac{7\,p^2 - 1 + 6\,p^2 \sin^2 \phi/2}{9}\right) \tag{3}$$

or the equivalent expression

$$D\left(\frac{4}{45} + \frac{4\,p^2 - 1 + 3\,p^2 \cos \phi}{9}\right) \tag{3a}$$

The former of these two expressions is preferable in that it shows directly what reduction in loss is produced by a different phase of current in the two layers.

Sub-Division of Conductors with Solid Connections at End of each Half Turn

The formula for calculating this case consists of two parts, the first portion dealing with the eddy currents still localized in the section of each lamination and in the core portion of the length of it, and the second portion dealing with the eddy currents which circulate around the various sections and flow over the whole length of the conductor.

Two new co-efficients are introduced in the second portion of the formulas, *viz*:

b = ratio of core length to half the length of turn and c = ratio of the total height of the assembled laminations of the conductor to net copper height in the conductor.

The first co-efficient allows for the extra resistance experienced by the eddy currents in circulating around the more extended path, and the second co-efficient takes account of the flux which passes through the insulation between the sections.

If now n is the number of sections in each conductor, and a given conductor is situated at position p in the slot, counting upwards from the bottom, 1, 2, 3, etc. the extra loss factor in p^{th} conductor is given by the algebraic expression

$$D\left[\left\{\frac{n^2(p^2-p)}{3}+\frac{n^2-1}{9}+\frac{4}{45}\right\}+b^2c^2(n^2-1)\left\{\frac{4n^2-1}{45}+\frac{n^2(p^2-p)}{3}\right\}\right] \quad (4)$$

If there are in all, p such conductors under consideration, the average loss factor for such a group is

$$D\left[\left\{\frac{n^2(p^2-1)}{9}+\frac{n^2-1}{9}+\frac{4}{45}\right\}+b^2c^2(n^2-1)\left\{\frac{4n^2-1}{45}+\frac{n^2(p^2-1)}{9}\right\}\right] \quad (5)$$

If there is a further layer of p such conductors, (p^2-1) in both portions of the above expression is changed to $(7p^2-1)$ and when the phase of the current is different it is replaced by $(7p^2-1-6p^2\sin^2\phi/2)$ just as in the expressions (**2**) and (**3**) previously given for solid conductors.

SUBDIVISION CONTINUED THROUGHOUT A COIL

The formulas are similar to (**4**) and (**5**) just given but in all cases the second part of the expression is the same for each conductor or for each layer, and a value of p is determined for use in this part of the expression from a knowledge of the arrangement of the order of the laminations in the successive conductors.

The usual arrangement hitherto employed has been to have the order of the laminations in the top layer reverse of the order of the laminations in the bottom layer. In this case the value of p to be used in the second part of the expression is $\left(\frac{p+1}{2}\right)$ if p is the number of conductors in each layer.

In order to avoid confusion this equivalent value for the complete coil will be written as p in the formula about to be given.

Various improved arrangements were discussed in the writer's paper and equivalent values of P were derived for all cases arising in practise, including that shown as Fig. 6 in the present paper. It was shown, moreover, how the imperfection still existing in the coil with an odd number of turns can be remedied.

The formula for the extra loss factor in the bottom layer of a coil consisting of p turns, the conductor consisting of n separately insulated laminations is

$$D\left[\left\{\frac{n^2(p^2-1)}{9}+\frac{n^2-1}{9}+\frac{4}{45}\right\}\right.$$
$$\left.+\,b^2c^2(n^2-1)\left\{\frac{4n^2-1}{45}+\frac{n^2(P^2-P)}{3}\right\}\right] \quad (\mathbf{6})$$

For the top layer, when the currents in the slot are all in the same phase the (p^2-1) in the first portion only of the formula gives place to $(7p^2-1)$ and when the current in the bottom portion of the coil is in a different phase to $(7p^2-1-6p^2\sin^2\phi/2)$ as in the previous formulas (**2**) and (**3**). The second part of the expression is the same as for the bottom layer.

Subdivision of Conductors Throughout a Winding with Successive Transference of Lamination

This method was described by Gilman in his paper before the Institute last year, and the author's firm has used the principle in practical work since 1913. It consists essentially of choosing the number of laminations in the conductor the same as the number of coils in the winding, by continuously insulating the laminations of the conductor throughout the winding and by successively changing the position of the laminations in the conductor from coil to coil so that each lamination occupies each position in the conductor once by the time the winding is completed.

It will be obvious that when the phase relations of the currents in the upper and bottom layers are the same in each slot throughout the winding, there are no losses except those localized in each lamination in the core portion of the winding, in which case formulas (**1**), (**2**), (**3**) are directly applicable, taking p as the number of laminations in each layer or in the whole slot as required.

Where, however, the upper layers of conductors are not all influenced by the same phase of currents in the lower layers then some extra loss will occur as a result of currents circulating between the laminations.

The greatest effect will be produced when half the upper layers are influenced by one phase of current in the bottom layer and when the other half are influenced by another phase of current as in a three-phase winding with 5/6 pitch. The extra loss factor, in addition to those calculated from formulas (**1**), (**2**), (**3**) will be

$$\frac{D\, b^2\, c^2\, n^2\, (n^2 + 8\, Q^2)}{192\, Q^2} \qquad (7)$$

where the conductor has n laminations and the winding consists of n coils which are distributed over Q poles.

Other cases lying between the perfect and the most imperfect one may be readily estimated by interpolation.

Finally, all the above formulas are accurate to within a few per cent as long as $(n^4\, b^2\, D)$ is less than 2. Formula (**7**) is accurate for a much higher value.

These limits will be found to cover all cases arising in practise and the accuracy of the formulas have been confirmed by running tests on a number of large machines in which temperature detectors have been embedded.

Further at the conclusion of a paper read on the same subject by R. E. Gilman at the summer convention of last year, the results of a number of experiments on eddy current heating were given and it will be found that the present writer's formulas check up with the experiment results in all cases, except where the conductor consists of six sections solidly connected together at the end and giving an *extra* loss factor of 3.225, an arrangement obviously outside the range allowable in practise.

E. S. Henningsen: In answer to Mr. Hibbard's question as to whether it is considered that the synchronous motor is the proper application for all types of merchant ships. The answer is, of course, that it is not; any more than the synchronous motor or the induction motor can be always said to be particularly applicable to any and all installations. There are many installations, particularly of small ships where I think the d-c. equipment and the Diesel engine drive will be found more practical and more efficient than either the induction motor or the synchronous motor. The d-c. equipment, cannot, of course, compete in the large fields.

In answer to the question brought up in the letter read by Mr. Skinner. I was very much interested in the scheme that was outlined and I certainly hope it has a chance to be tried out. This is a brand new field; there is very little information available, and the more methods that are tried out the better the final solution will be. This country needs a merchant marine and certainly anything that will aid to build up this merchant marine and arrive at the best solution of the question is to be desired.

The matter of torque during turning has been alluded to in the paper. The requirement here is no different with the synchronous than with the induction motor, both have to supply torque enough to take care of this condition. This has been covered in the paper. In the matter of efficiency, even counting the much higher rating of the exciters required for the synchronous motor proposition, there results a saving of something like 2 per cent in the total steam consumption for a 3000-horse power equipment. Not a very tremendous saving, but something nevertheless.

In answering Mr. MacCutcheon's question as to the maneuvering: When a ship is coming into or going out of a port, it is practically always operated at either one quarter or half of normal speed. At low speed, as I pointed out, the torque required by the propellers is very materially less than that required at high speed, hence the control can be arranged with the control handle something similar to a street car controller, if you like, and the operation or reversal at these lower speeds is accomplished in a very short interval of time. On trial trips on one of our ships we answered 75 bells in 2½ minutes—quarter speed ahead to quarter speed reversed, throwing the lever from ahead position straight through to the astern position and bringing it back almost as fast as a man could operate the lever. Even under the full speed condition where we require almost 100 per cent of normal torque to reverse, the time that is required for full-speed reversal is rather remarkable. Using the 250-volt, *i. e.*, total excitation across the motor field the propeller is stopped from full speed in 3½ seconds and when you consider the fastest time that can be made with a steam engine is about a minute and a half, and that is expert manipulation, 3½ seconds is quite remarkable. On a trial using only the induction motor characteristic without making use of the braking characteristic, the full-speed stop was accomplished in 13 seconds, so that it will be seen that with the lower speed of the ship the reversal of the propeller can be accomplished very rapidly.
