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PEAUCELLIER'S COMPOUND COMPASS AND OTHER
LINKAGES.

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The writer, in order to avoid the use of cumbersome beam compasses, having designed for the use of his students a compound compass, upon the principles first enunciated by Peaucellier, and not being aware of the existence of specific directions for the construction of a similar apparatus, believes that the following directions may be of use to the engineering profession, as being sufficiently explicit to admit of the construction of curve rulers by those able to follow the simplest mathematical work.

The theory cannot be set forth in a more interesting manner than in the subjoined translation of a paper written by the inventor, and published in the *Nouvelles Annales de Mathématiques*, Deuxieme Série, t. xii, p. 71. 1873.

NOTE UPON A QUESTION OF COMPASS GEOMETRY.

By M. PEAUCELLIER.

(This problem, dated 1864, was solved at that time, as indicated by a letter in the *Nouvelles Annales de Mathématiques*, 2e série, t. iii, p. 414.)

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We shall give the name of Compound Compass to any articulated system whatever of rigid pieces in complete connection (*a liaison complète*). We know that this name is generally applied to every assemblage of bodies in such a manner that the movement of any point whatever causes the movement of all the others.

The pantograph, the parallelogram of Watt, etc., are such assemblages, and may be considered as compound compasses.

Apparatuses of this kind possess the valuable advantage of requiring very small forces to put them in motion, and their displacements are continuous and perfectly regular, by reason of the possibility of completely suppressing all play in the articulations.

They have neither irregularities nor "lost motion," to speak technically.

These diverse properties specially recommend them to the constructor of instruments of precision, and we shall adopt them exclusively for the object that we have in view, that is to say, the mechanical tracing of plane curves.

The line that starts from any point whatever, guided by a combination of articulated pieces, is necessarily algebraical. Reciprocally one perceives that all algebraical curves can be generated by a properly arranged articulated system, because one will always have at command a sufficient number of variable elements, radii and centres of rotation to be able to satisfy the equations expressing the identity between the given curve and the described one.

Fig. 1.

In what follows, we will limit ourselves to compound compasses tracing the best known lines, the straight line, the circle of any radius, the conic sections, and lastly conchoids and cissoids.

We shall present the result of our researches in a synthetical form, the analytical processes which have guided us in the most interesting part of this work, that relative to the straight line being very complicated, and not being, in fact, a method of investigation sufficiently methodical.

Lemma.—Any point whatever, C , Fig. 1, taken upon the diagonal of a lozenge, divides it into two segments of which the product $\overline{AC} \times \overline{CB}$ is equal to the difference $\overline{AD}^2 - \overline{CD}^2$ between the squares

constructed upon the side of the lozenge and upon the distance $\overline{CD}$ of the point considered from the extremities of the other diagonal.

Prolong $\overline{CD}$ till it cuts in F and G the circle described from the point D as a centre with the radius $\overline{DA}$.

We have

$$\begin{aligned}\overline{AC} \times \overline{BC} &= \overline{CF} \times \overline{CG} = (\overline{AD} - \overline{CD})(\overline{AD} + \overline{CD}) \\ &= \overline{AD}^2 - \overline{CD}^2\end{aligned}$$

This remarkably simple demonstration has been given to us by M. Mannheim.

It follows then that if one supposes that the figure $EADBECD$ represents an assemblage of rigid rods articulated at their extremities, and that the point C remains fixed, the opposite extremities, A and B , will describe reciprocal curves. This combination will form the essential organ of the divers compound compasses that we shall have to examine.

1st. The reciprocal of a straight line being a circle passing through the pole, if the point B (Figs. 2 and 3) is forced to move in the perimeter of a circle passing through the centre of articulation C , the movement of the point A will be rectilinear. It will therefore suffice, in order to generate a straight line, to introduce in the mechanism

Fig. 2.

Fig. 3.

$EADBECD$ a new fixed centre, C' , to which one fastens the point B , the link $\overline{C'B}$ being equal in length to the distance $\overline{CC'}$ between the fixed centres.

What precedes constitutes a rigorous solution of the problem stated by Watt. It is simple enough to be advantageously employed in certain machines of long stroke.

Mr. Mannheim, in 1867, made it the subject of a communication to the Société Philomathique of Paris

2d. If the circle described by the vertex B does not pass through

the fixed centre C , the opposite vertex A describes a circle whose radius is represented by the expression

$$R = \frac{(a^2 - b^2)r}{r_2 - d^2}$$

in which one designates by a the side of the lozenge $\overline{AD}$; b the bridle rod $\overline{CD}$; r the radius $\overline{C'B}$; d the distance between the centres CC' .

If $r = d$ we have $R = \infty$ which corresponds to the case of a straight line.

This combination of articulated pieces will then permit, if we vary one of the elements, for example the distance CC' , the tracing of arcs of circles of every curvature in a continuous manner.

Designers know that this operation, apparently so simple, is not less complicated than the tracing of any geometric curve whatever, when it happens to be a curve surpassing the extreme limits of the beam compass.

3d. Let us now consider the conic sections.

We know there exists an infinity of theorems leading to their determination by means of the straight line and the circle, if successive points of these lines are given. Of the articulated systems coming in question, permitting so to say, the materializing of every combination of straight lines and of circles, without recourse to other organs of transmission, have articulated rods, we foresee that there ought to exist an infinity of compound compasses adapted to the

Fig. 4.

tracing of lines of the second order.

Thus a finite straight line, AB , of the same length as OA , Fig. 4, being moved, for example, with its extremities in the circumference of a circle, and upon one of its diameters, $\overline{OB}$, each point of the movable line will describe an ellipse. This theorem furnishes a very simple means of constructing an elliptic compass, since we are able to guide the straight line $\overline{AB}$, by its extremities as the data of the problem require.

But there is a more general solution, capable, by means of the same combination of lines, of describing all the conic sections without distinction.

The polar equation of these curves is

$$\rho = \frac{p}{1 + \epsilon \cos. \omega}$$

and represents ellipses, parabolas or hyperbolas, according as $\epsilon < 1$; $\epsilon = 1$ or $\epsilon > 1$.

If we consider the reciprocal curve of the preceding, we will have for the equation

$$\rho^1 = \frac{k^2 (1 + \epsilon \cos. \omega)}{p}$$

k being a constant. It is a limaçon de Pascal, of which we shall describe the mechanical generation.

For this, let us return to the compass relating to the straight line, to which we will add the bridle rod, $\overline{C_1 I}$, Fig. 5, where the straight line described by A cuts the line of centres CC' . If we fix the point C_1 as well as the vertex A , and make $C_1 I = C_1 A$, and let all the other points of the figure remain free, it is apparent that any point M of the perpendicular IM to CC' will describe a limaçon de Pascal. Of which the equation will be

$$\rho^1 = 2 \overline{C_1 I} \cos. \omega + IM$$

Its reciprocal relatively to A is a conic.

We shall obtain this latter curve by connecting the point M to the free vertex of an articulated system whose fixed centre is A .

This point will become the focus of the conic of which $\overline{AC'}$ will be the direction of the principal axis, and which will be

$$\left. \begin{array}{l} \text{An ellipse,} \\ \text{A parabola,} \\ \text{A hyperbola,} \end{array} \right\} \text{if } \frac{IM}{2C_1 I} \left\{ \begin{array}{l} < 1 \\ = 1 \\ > 1 \end{array} \right.$$

It is besides evident that by giving to IM and to $C_1 I$ convenient values it will be possible to trace with the same compound compass all the curves of the second degree.

Conchoids of the Circle and of the Straight Line, Cissoid, etc.

We have just seen how one is able to arrange for tracing the conchoid of the circle, or limaçon de Pascal; by modifying the combination relative to this curve in a similar manner to that pursued for the conics we shall generate the conchoid of the straight line. We find in

Fig. 5.

the same manner the combinations suitable for the lemniscata, but they are for the most part very complicated. We shall content ourselves, in closing this short summary, by indicating the compass for the cissoid, which is extremely simple. The centre of articulation of the links $\overline{CD}$ $\overline{CE}$ (Fig. 6) is forced to move in circle passing through the fixed vertex A of the lozenge, whose diameter is equal to $\sqrt{b^2 - a^2}$.

As before, b designating the length CD and a the length AD .

The opposite vertex B under these conditions will trace the cissoid, as is easy to see by forming the equation of the described curve.—*End of translation.*

This translation is made with the desire that the fortunate inventor of this remarkable linkage be awarded by English readers the credit justly his due; not only as an inventor but as a mathematician capable of understanding and discussing his invention in an able manner.

Referring to Fig. 1 and, for the sake of brevity, designating the radii vectores AC by ρ^1 , BC by ρ and the constant $\overline{AD}^2 - \overline{CD}^2 = a^2 - b^2$ by m^2 , called the modulus of the linkage, we have

$$\rho \rho^1 = m^2 \quad (1)$$

Fig. 6.

Referring to Figs. 2 and 3, Section 1, we see that the fixed centre C' , midway between B and C and the radius bar $C'B$, forces the vertex B to move in the perimeter of a circle passing through the pole C (fixed point), the prime radius of which is its diameter. This gives

$$\rho = 2r \cos. \alpha$$

in which $r = CC' = C'B$ and α = the angle through which the point B is moved with respect to the fixed line ACB . Substituting this value in Equation (1), we have

$$\rho^1 = \frac{m^2}{2r \cos. \alpha} = \frac{m^2}{2r} \sec. \alpha \quad (2)$$

Which is the polar equation of a straight line at right angles to the prime radius ABC , and we see that the point A will describe a straight line.

The form of cell outlined in Fig. 2 is commonly called a negative cell, and that in Fig. 3 a positive cell.

Referring to Section 2, and using the notation there given, we have for any position of C' between B and C

$$\begin{aligned}\rho^2 - 2 d \rho \cos. a &= r^2 - d^2 \\ \rho &= d \cos. a \pm \sqrt{r^2 - d^2 \sin.^2 a}\end{aligned}\quad (3)$$

Substituting for ρ its value $\frac{m^2}{\rho^1}$ we have

$$\begin{aligned}\frac{m^4}{\rho^{12}} - \frac{2 d m^2}{\rho^1} \cos. a &= r^2 - d^2 \text{ and reducing} \\ \rho^1 &= - \frac{m^2 d}{r^2 - d^2} \cos. a \pm \sqrt{\left(\frac{m^2 r}{r^2 - d^2}\right)^2 - \left(\frac{m^2 d}{r^2 - d^2}\right)^2 \sin. a}\end{aligned}\quad (4)$$

If in this equation we place

$$D = \frac{m^2 d}{r^2 - d^2} \text{ and } R = \frac{m^2 r}{r^2 - d^2}$$

Equation (4) becomes

$$\rho^1 = D \cos. a \pm \sqrt{R^2 - D \sin.^2 a} \quad (5)$$

and we observe that Equations (3) and (5) are similar; therefore while the vertex B describes an arc of a circle as stated with the radius r and centres at a distance, d , from the pole C , the vertex A will describe an arc of a circle whose centre is on the prime radius at a distance, D , $= \frac{m^2 d}{r^2 - d^2}$ from the pole C with a radius

$$R = \frac{m^2 r}{r^2 - d^2} = \frac{(a^2 - b^2)r}{r^2 - d^2} \quad (6)$$

If we make the distance between B and C (Figs. 2 and 3) constant and $= 2 e$, and denote the distance of the point C^1 from a point bisecting the line BC by x , we can simplify this equation.

Let $d = e \pm x$

“ $r = e \mp x$

We have $R = \frac{m^2 r}{r^2 - d^2} = - \frac{m^2(e-x)}{4 e x}$ or $= \frac{m^2(e+x)}{4 e x}$

Still further reducing, we have for the value of x

$$+ x = \frac{m^2}{\frac{m^2}{e} + 4 R} \quad (7)$$

$$\text{or } - x = \frac{m^2}{\frac{m^2}{e} - 4 R} \quad (8)$$

since in the latter case R is of the contrary sign.

We have now a ready means of computing the distance x to the right or left of the central point of the line BC , in order to describe an arc of any required radius, R .

When x is +, *i. e.*, moved to the right, the arc is convex to the pole C , and Equation (7) should be used; when $x = 0$ we have an arc of infinite radius, a straight line; when x is —, *i. e.*, moved to the left, the arc is concave to the pole C , and Equation (8) should be used in computing the radius.

The compound compass shown in Fig. 7 is adapted to drawing arcs of a little over 3 feet in length.

The construction will, in the light of previous explanations, be readily understood; in it we have taken $a = 12$ inches

$$b = 8 \quad "$$

$$2c = 12 \quad "$$

$$\text{giving } m^2 = 144 - 64 = 80$$

This will be found a most convenient size for the ordinary purposes of the draughtsman.

At C^1 is a split joint, such as is used for proportional dividers, and the vertex B is steadied by means of a hole and pin while the joint is being adjusted at any required distance from C^1 , this joint sliding in the slots in the radius bar C^1B and the ruler CB . A scale reading to hundredths of an inch is laid off from the central point C^1 three inches both ways.

The better method of locating the mid point C^1 is to adjust it tentatively until the vertex A describes a perfectly straight line. The following table of values of x for circles of radii between 10 and 120 inches has been computed by means of Formulæ (7) and (8), which reduce to

Radius	Convex arcs	Concave arcs
	$+x =$	$-x =$
10 in.	1.500	3.000
20 "	0.857	1.200
30 "	0.600	0.750
40 "	0.462	0.545
50 "	0.375	0.428
60 "	0.316	0.353
70 "	0.273	0.300
80 "	0.240	0.261
90 "	0.214	0.231
100 "	0.194	0.207
110 "	0.177	0.188
120 "	0.162	0.171

$$\pm x = \frac{60}{10 \pm 3R} \quad (9)$$

and can be used for computing the value of x for circles of any radius from $3\frac{1}{2}$ inches to infinity or a straight line, with an instrument of the described proportions.

In order to use the instrument, place the ruler CB on any normal to the circle, adjust the joint at the required distance x and cause the pencil in the vertex A to coincide with the starting point of the circle sliding the ruler upon the normal until the proper position is reached. The circle can then be described.

Fig. 7.

Referring to the special elliptic linkage, Section 3 of Peaucellier's paper Fig. 4, if $OA = AB$, we have Fig. 8, $r = OA = AB$ and $d = AP$ and

$$y^2 = \left(\frac{r-d}{r+d} \right)^2 (2(r+d)x - x^2)$$

the equation of an ellipse referred to its vertex of which the transverse and conjugate semi-axes are $(r+d)$ and $(r-d)$.

The case is different when AB is greater than OA and of interest, as covering the ordinary instance of a point on a connecting rod which describes an egg-shaped oval closely approximating to an ellipse.

Fig. 8.

The equation of the curve described by a point P at a distance d from A on the line AB , Fig. 8, when referred to rectangular co-ordinates with the origin in the perimeter at a distance $(r+d)$ from O , and with OB for the axis of abscissas is, representing AB by l ; OA by r .

$$x = r + d - \frac{\sqrt{r^2(l-d)^2 - y^2 l^2} + \sqrt{d^2(l-d)^2 - d^2 y^2}}{l-d}. \quad (10)$$

For $d = 0$ we have $y^2 = 2rx - x^2$ the equation of a circle with the radius r .

For $d = l$ we have $x = \infty$ which is the equation of a straight line coinciding with the axis of abscissas.

If we cause equation (10) to take the form

$$x = r \left(1 - \sqrt{1 - \frac{y^2 l^2}{(l-d)^2 r^2}} \right) + d \left(1 - \sqrt{1 - \frac{y^2}{(l-d)^2}} \right) \quad (11)$$

and neglect the last term as inconsiderable when d is relatively small or l very great, we have

$$x = r - \sqrt{r^2 - \frac{y^2 l^2}{(l-d)^2}}$$

and reducing

$$y^2 = \frac{(l-d)^2}{l^2} (2rx - x^2) \quad (12)$$

which is the equation of an ellipse referred to its vertex and of which the major and minor semi-axes are r and $\frac{l-d}{l} r$.

The special linkage shown in Fig. 9 was devised by Mr. Freeland, University of Pennsylvania, and will describe a hyperbola, it is a positive cell having the longer arms split and turned through a fixed angle, being fastened together at F , the arm AFK forming one piece,

Fig. 9.

and the arm BFG another. The point L being fixed, the point F is forced to move along the straight line LM , the point P will describe a hyperbola. The equation $\rho\rho^1 = m^2$ being that of a hyperbola referred to its asymptotes. In this case $m^2 = \frac{A^2 + B^2}{4}$, and A and B are

Fig. 10.

the major and minor semi-axes of the hyperbola.

The limaçon of Pascal is a curve traced by a point on a line which projects a constant distance beyond the circumference and passes through a fixed point on the circumference of a circle.

In the combination of linkages shown in Fig. 5, the function of the cell ACC^1B is by means of the bar C^1CI to cause the prolongation of IM to pass through the point A in all its positions, as IM is rigidly fastened to ICC^1 and at right angles to it. In this linkage, the only

points fastened to the surface, upon which these two linkages move, are A and C .

This assemblage, although theoretically fulfilling the required conditions, is too complicated for ordinary purposes.

A good mechanical substitute is shown in Fig. 10, which is lettered as in Fig. 5. The place of the cell $ACCB$, Fig. 5, is supplied by the slotted ruler $MIAP$, and the radius bar C_1I . The same cell may be used as that shown in Fig. 7, the alteration required being in the ruler used as a basis. The pin C_1 is fixed and the radius bar can be made adjustable at I in the slot IA , and to any length. The point A is fixed, the slotted ruler sliding upon it. The point M is fastened to the ruler. The point P will describe conic sections as in Fig. 5.

$IM = \frac{m^2}{p}$, m being the modulus of the cell and p = the semi-latus rectum or half the principal parameter.

$C_1I = \frac{m^2 \varepsilon}{2p}$ in which ε = the eccentricity of the conic section.

We have now a ready mechanical means of describing any required straight line, circle, ellipse, parabola or hyperbola as the case may require.

Letting A = the transverse and B = the conjugate semi-axes, we have for an ellipse

$$IM = \frac{m^2 A}{B^2} \text{ and } C_1 I = \frac{m^2 \sqrt{(A^2 - B^2)}}{2 B^2}$$

For a hyperbola

$$IM = \frac{m^2 A}{B^2} \text{ and } C_1 I = \frac{m^2 \sqrt{(A^2 + B^2)}}{2 B^2}$$

For a parabola

$$IM = \frac{m^2}{p} \text{ and } C_1 I = \frac{m^2}{2p}$$

Referring to the linkage for the cissoid, Fig. 6, we have

$$AB = CB - CA = \frac{m^2}{CA} - CA; \text{ or,}$$

$$\rho = \frac{m^2}{\rho^1} - \rho^1 = \frac{m}{\cos. a} - m \cos. a = m \frac{\sin.^2 a}{\cos. a} = m \sin. a \tan. a$$

which is the polar equation of the cissoid under the condition placed

C^1A , being the polar axis and m the diameter of the circle passing through A of a radius $= C^1A = \frac{m}{2}$.

Fig. 11.

A traces a circle of the radius C^1A , and the point B traces an ellipse OB . The point opposite to E in the large rhomb and the two unlabeled points in the other, also describe ellipses.

Fig. 12.

We will close this brief essay by an explanation of Hart's Perfect Parallel Motion, which is described by Mr. Kempe in "How to Draw a Straight Line."

This linkage is remarkable, because only four rods are required where six are used in Peaucellier's cell. It will be perceived that this linkage CD_1DC_1C is a parallelogram in a reversed position, *i. e.* the rods CD_1 and C_1D instead of being parallel are crossed (Fig. 12).

Let a straight line AP_1PA_1 be drawn parallel to C_1D_1 , cutting all the links in AP_1PA_1 .

Let $AP = \rho$

" $AP_1 = \rho^1$

" $AC_1 = a_1$

" $AC = a_2$

" $CD = b$

" $C_1D_1 = b_1$

" $CC_1 = a_3$

" $C_1E = D_1F = d$

" $CE = DF = h$

$C_1D = b_3$

The similarity of the triangles CC_1D_1 and CAP and also C_1CD and C_1AP_1 which is true for all positions gives the two proportions

$$\frac{\rho}{b_1} = \frac{a_2}{a_3} \text{ and } \frac{\rho^1}{b} = \frac{a_1}{a_3}$$

$$\text{Therefore, } \rho\rho^1 = \frac{a^1a^2}{a_3^2} bb_1$$

$$\text{But } bb^1 = b(b + 2d) = b^2 + 2bd.$$

From the right angled triangles C_1FD and C_1EC we have $(b + d)^2 + h^2 = b_3^2$ and also

$d^2 + h^2 = a_3^2$, subtracting the latter from the first we have

$$b_3^2 - a_3^2 = b^2 - 2bd = bb_1$$

$$\text{Therefore, } \rho\rho^1 = \frac{a_1a_2}{a_3^2} (b_3^2 - a_3^2) = m^2.$$

All the terms of the second member of this equation being constants, we have a linkage with the same properties as the Peaucellier Compound Compass. The point A being fixed and either of the points P or P_1 being forced to move in a circle with the circumference, passing through A the other will describe a straight line. These points are interchangeable.

There are, of course, many other forms of linkages besides the ones described, and it is to be hoped that some mentioned in this paper can be greatly simplified.

The degree of exactitude which is required in the mechanical workshops of the United States is daily becoming greater, and at present tries our machinery to the utmost.

With the possibility of a mathematically straight line before us, with a means of describing an arc of a circle of any radius, in a convenient shape, and with the conic sections almost as easily describable as a circle; there is no reason why our mechanics should not produce machinery of an exactitude, and in forms hitherto deemed unattainable.

University of Pennsylvania, February 18th, 1879.

Wealth of France and England.—The national wealth of England is estimated at \$39,200,000,000, that of France at \$40,300,000,000.—*Fortsch. der Zeit.*