Remarks on the preceding Paper, by Mr. J. CLERK-MAXWELL, F.R.S.

It appears from this paper that it has been stated, that when a hole is made in one side of a vessel containing water, so as to allow the water to escape, the pressure on the opposite side of the vessel is thereby increased. This statement Mr. Hanlon shows to be erroneous, and asserts that the immediate effect of opening the hole, and allowing motion to take place, is to diminish the pressure in every part of the vessel.

He then shows, by a calculation of the momentum communicated in a unit of time to the fluid which is projected from an orifice in the side of a vessel, that the force acting on it is measured by $\iint \rho v^2 dS$, where ρ is the density, v the velocity at the Vena Contracta where the pressure vanishes, and dS an element of the section of the stream.

Now at the Vena Contracta $v^2 = 2gz$ approximately, where z is the depth, so that the force of propulsion is $F = 2g\rho \iint z dS$; or if $\bar{z}$ is the depth of the centre of gravity of the section S , $F = 2g\rho \bar{z} S$. But at a distance from the hole, where the velocity is small, the pressure is the hydrostatic pressure $p = g\rho z$.

Let us assume (for an instant) that this is the value of the pressure on every part of the surface of the vessel, in which there is a hole whose section is S' , and whose centre of gravity is at a depth $\bar{z}'$. The resultant of the pressures on the sides will be $F' = g\rho \bar{z}' S'$, and this must be equal to F , because action and reaction are equal and opposite. This gives

$$2\bar{z}S = \bar{z}'S'.$$

There is very little difference between $\bar{z}$ and $\bar{z}'$, the depths of the centre of gravity at the hole and at the Vena Contracta. Hence we ought to have $S = \frac{1}{2}S'$, or section of Vena Contracta = .5 of the hole.

I am not aware how far the author is original in this method of treating the subject; but it is certainly very clear and intelligible, without the aid of symbols, which is very important in practical matters. Rankine, however (*Steam Engine*, p. 96), gives $\cdot 6zS'$ as the section of the Vena Contracta for round holes, and I have not seen any experimental result as low as $\cdot 5S'$.

This arises partly from our having made an erroneous assumption,—that the pressure on the sides is the hydrostatic pressure ρgz , whereas it must be diminished by $\frac{1}{2}\rho v^2$ wherever there is a velocity v .

Now v will evidently be greatest near the hole, so that the pressure on the side next the hole will be diminished for this reason, as well as by the removal of the stopper of the hole, and the value of F' will be greater than $g\rho \bar{z}'S'$, and therefore S will be greater than $\cdot 5S'$.

If the fluid is a perfect liquid originally at rest, it will have a velocity-potential ϕ , and the pressure is given by the equation

$$p = \rho g z - \rho \frac{\delta \phi}{\delta t} = \rho g z - \rho \frac{d\phi}{dt} - \frac{1}{2} \rho V^2,$$

where $\frac{\delta}{\delta t}$ denotes the rate of change in a moving particle, and $\frac{d}{dt}$ denotes rate of change at a point of space. V is the velocity of the fluid.

We must also have at the surface of the vessel $\frac{d\phi}{dv} = 0$, where v is the normal to the surface, and ϕ satisfies Laplace's equation

$$\frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} + \frac{d^2 \phi}{dz^2} = 0.$$

Now consider the case of a body which conducts electricity,—a mass of copper of the same form as the vessel; and let electricity be supplied at the upper surface, and carried off at the point corresponding to the hole, so that the current of electricity entering or escaping is everywhere proportional and parallel to the current of water. Then the electric-potential everywhere will be equal to ϕ , and its value is perfectly definite for a given problem, and problems can be constructed for which the solution can be found by our present methods.

Let ψ be the electric-potential for the copper conductor; then, if the form of the vessel and the motions of the surface remain similar, the velocity-potential $\phi = \psi T$, where T is a function of the time.

If we assume $\psi = 0$ at the free surface, then, since ψ increases along every line of motion, the value of ψ will be positive throughout the vessel, and will have its greatest values at the place of the issuing stream.

Let us first consider the *immediate* effect of opening the hole. In this case we may neglect the square of the velocity; and since at the orifice the pressure becomes zero, we have

$$p = 0 = \rho g \bar{z} - \rho \frac{d\bar{\phi}}{dt},$$

where $\bar{\phi}$ is the value of ϕ at the orifice. If $\bar{\psi}$ is the value of ψ at the orifice, then

$$\bar{\phi} = \bar{\psi} g \bar{z} t;$$

and if ψ is the value of ψ at any other point, then

$$\phi = \psi g \bar{z} t,$$

and

$$p = \rho g z - \rho g \bar{z} \frac{\psi}{\bar{\psi}}.$$

Hence the assertion that the pressure is everywhere diminished is correct, and the amount of diminution is everywhere proportional to ψ , the electric-potential of a conducting mass similar to the water in the vessel, of which the upper surface is maintained at potential zero and the orifice at potential unity, the sides being supposed coated with insulating material.

When the motion has become constant, we may neglect $\frac{d\phi}{dt}$, and we

have

$$p = \rho g z - \frac{1}{2} \rho V^2,$$

where V is the velocity at any point. At the orifice, $V^2 = 2g\bar{z}$. It appears, therefore, that in this case also the pressure at every point is less than the hydrostatic pressure.

Mr. Jenkins read a letter from Mr. Clerk-Maxwell, asking the following question :—

“Can the potential of a uniform circular disc at any point be expressed by means of elliptic integrals? Suppose V is the potential of the disc bounded by the circle

$$z = 0, \quad x^2 + y^2 = a^2;$$

$$\text{then} \quad \frac{dV}{dx} = 2x \left(\frac{a}{r} \right)^{\frac{1}{2}} \frac{1}{\sqrt{c}} (E - F),$$

$$\text{where} \quad r^2 = x^2 + y^2;$$

and if AB be a diameter parallel to r ,

$$c = \frac{PB - PA}{PB + PA},$$

and E, F are complete elliptic functions for modulus c ;

$$\text{also} \quad \frac{dV}{dy} = 2y \left(\frac{a}{r} \right)^{\frac{1}{2}} \frac{1}{\sqrt{c}} (E - F).$$

“But $\frac{dV}{dz} = w$, where w is the solid angle subtended at P by the circle; that is, the area of the spherical ellipse on a sphere of unit radius cut off by the cone whose vertex is P and base the circle. We have expressions for $\frac{dV}{dx}$ and $\frac{dV}{dy}$. Can $\frac{dV}{dz}$ also be expressed by elliptic functions? and if so, can V itself be so expressed?

“I am writing out the theory of circular electric currents, in which these quantities occur. The expression $\frac{dV}{dz}$ for an elliptic disc can be found if we know it for a circular one, for the spherical ellipses in the one case are no more complicated than in the other. Can $\frac{dV}{dx}$, or V itself, be found for the elliptic disc?”

The President and Mr. Merrifield spoke on the question without deciding it.

Mr T. Cotterill read a paper on “Opposite Points on a Curve.” Mr. Roberts and Mr. Clifford joined in a discussion on the paper.

The following presents were made to the Society :—

“Annali di Matematica,” serie ii^a, tomo iii^o, Fasc. 1^o Ottobre, 1869.

"Monatsbericht," Juli, August, 1869.

"Nature," Nos. 1 and 2.

"Royal Society Catalogue of Scientific Papers," vol. iii.

November 25th, 1869.

Prof. HIRST, and subsequently Prof. SYLVESTER, V.P., in the Chair.

The Auditor (Mr. Merrifield) certified that he had examined the Treasurer's accounts, and found them perfectly correct. The Treasurer then suggested that a statement of the financial position of the Society should be drawn up and circulated amongst the Members. This proposal was unanimously agreed to by the Meeting.

The Rev. James White was elected a Member, and admitted into the Society. The Rev. Percival Frost, M.A., was proposed for election.

Dr. Henrici exhibited a model of the cubic surface

$$xyz - \left(\frac{1}{2}\right)^3 (x+y+z-1)^3 = 0,$$

which has three biplanar nodes (to a scale of $2\frac{1}{2}$ inches as unit). A sufficient number (eleven) of sections, $x+y+z-1 = \text{a constant}$, cut out in cardboard, are connected in a horizontal position, and kept at their proper distance by three vertical sections $y=z$, $z=x$, $x=y$, with regard to which the surface is symmetrical. The model contains the central part of the surface with the three nodes, and is bounded by a sphere of eight inches' radius, with its centre at the origin, large enough to show the position of the three straight lines in the surface (each counting for nine), and to give an idea how the surface extends to infinity. The interstices between the cardboard are intended to be filled up with plaster of Paris, so as to form a solid model.

Mr. W. K. Clifford then read the following paper:—

On Syzygetic Relations among the Powers of Linear Quantics.

In his *Géométrie de Direction* (Paris, 1869), M. Paul Serret makes very beautiful use of a principle which he states nearly as follows (p. 138):—

"In order that a system of points (in a plane) may be so related that every curve of order m passing through all but one of them must pass through the remaining one, it is necessary and sufficient that the m^{th} powers of their distances from an arbitrary line should satisfy a linear homogeneous relation

$$\lambda_1 P_1^m + \lambda_2 P_2^m + \lambda_3 P_3^m + \dots \equiv 0."$$

There is, of course, an analogous theorem for surfaces, and in fact M. Serret combines the two enunciations into one; he states also the correlative theorems concerning a system of lines or planes such that

* In the *Bulletin des Sciences Mathématiques et Astronomiques*, January, 1870, M. Darboux observes that this theorem, for the special case $m=2$, had been given by Hesse, *Vier Vorlesungen aus der analytischen Geometrie*, Leipzig, 1866.