XXV. On the molecular theory of refraction, reflexion, and extinction

Ladislav Natanson Ph.D.

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XXV. *On the Molecular Theory of Refraction, Reflexion, and Extinction.* By LADISLAS NATANSON, Ph.D., Professor of Natural Philosophy in the University of Cracow*.

IN two communications brought under the notice of the Cracow Academy of Sciences in January and in June 1914†, I have been concerned with following out the circumstances of the mutual action which must be supposed to take place between an advancing wave of light and the ultimate fragments of matter lying in its path. The leading assumptions adopted in this investigation appeared conducive to clearness and consistency, and I believe they may possibly still commend themselves to the theorist's attention. In the meantime, however, Science has progressed. Owing to excellent researches‡ the treatment of the subject has been notably extended and deepened, so that the interest of the papers published in 1914 may now perhaps be regarded as having passed away. Acceding, nevertheless, to friendly advice I venture to lay before the readers of the Philosophical Magazine the present essay which contains the substance of the papers to which I allude. Having found advantage

* Communicated by the Author.

† *Bulletin Int. de l'Acad. d. Sc. de Cracovie*, Cl. d. Sc. Math. et Nat., Series A, 1914, pp. 1-29 & 335-352.

‡ C. W. Oseen, *Annalen der Physik*, vol. xlviii. pp. 1-56 (1915). P. P. Ewald, *Annalen der Physik*, vol. xlix. pp. 1-38 & 117-143 (1916). Fr. Reiche, *Annalen der Physik*, vol. l. pp. 1-52 & 121-162 (1916). Ragnar Lundblad, *Annalen der Physik*, vol. lvii. pp. 183-202 (1918).

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from a gradual mode of procedure in performing the calculations I have confined myself, in this communication, to the consideration of some simpler aspects of the problems at issue. Thus we shall avoid, as far as we can, analytical intricacies, and the meaning and bearing of the fundamental principles involved will become apparent.

The effect of small particles in diverting portions of incident light from its regular course was investigated by Lord Rayleigh in a series* of admirable papers. It would carry us too far into scientific history to give an account of his work as well as of later inquiries† that fall more or less directly within our province; but I desire to repeat here the acknowledgment, made more than once in the course of the papers, of my indebtedness to Lord Rayleigh's writings. I also desire to take this opportunity of referring to Mr. W. Esmarch's paper cited below; with this author I find myself in closest contact and agreement in respect of many conclusions.

§ 1. Let us consider a linear simple vibrator maintained in vibration by the operation of a train of incident electromagnetic waves. It is natural to begin the calculations by considering the expression of the secondary field that diverges from the vibrator. An effective method of working it out is as follows. Let $V(x_0, y_0, z_0)$ be one of the points occupied by the vibrator, $M(x, y, z)$ any point where the secondary field is to be estimated. If we introduce an auxiliary vector Ψ whose components $\Psi_x(x, y, z, t)$, $\Psi_y(x, y, z, t)$, $\Psi_z(x, y, z, t)$ satisfy equations of type

$$\frac{\partial^2 \Psi_x}{\partial t^2} = c^2 \nabla^2 \Psi_x, \quad \dots \quad (1a)$$

c being the velocity of propagation *in vacuo*, the secondary

* Phil. Mag. (4) vol. xli. pp. 107, 274, 447 (1871); (5) vol. xii. p. 81 (1881); (5) vol. xlvii. p. 375 (1899). 'Scientific Papers,' vol. i. pp. 87, 104, 518; vol. iv. p. 397.

† G. Sagnac, *Recueil de Travaux offerts à H. A. Lorentz*, p. 377 (1900). Lord Kelvin, *Baltimore Lectures*, p. 301 (1904). M. Planck, *Sitz.-Ber. d. Kgl.-Pr. Akad. d. Wiss.* for 1904, p. 748. L. Mandelstam, *Annalen der Physik*, vol. xxiii. p. 626 (1907). A. Schuster, 'Introduction to the Theory of Optics,' 2nd edit., p. 325 (1909). A. Einstein, *Annalen der Physik*, vol. xxxiii. p. 1275 (1910). H. A. Lorentz, *K. Akad. v. Wet. te Amsterdam, Proceedings*, vol. xiii. (1) p. 92 (1910). P. Langevin, *Bulletin d. Séances de la Soc. Franç. de Physique*, p. 80* (1910). *Les Idées Modernes sur la Constitution de la Matière*, p. 97 (1913). L. Vessot King, *Phil. Transactions Roy. Soc. of London*, A. vol. ccxii. p. 375 (1913). W. Esmarch, *Annalen der Physik*, vol. xlii. p. 1257 (1913). (*This paper only came under my notice after my first communication on the present subject was already printed.)

field $\mathbf{E}^{(2)}$, $\mathbf{H}^{(2)}$ initiated by the vibrator is known to be expressible in the form

$$\mathbf{E}^{(2)} = \text{curl}^2 \Psi; \quad (2)$$

$$\mathbf{H}^{(2)} = a \text{curl} (\partial \Psi / \partial t), \quad (3)$$

where $ac=1$. To utilize this principle we have only to adopt for Ψ a suitable form of expression. Let $\mathbf{l}(l_x, l_y, l_z)$ be a direction unit vector belonging to a given vibrator. We will suppose that the direction of $\mathbf{l}$ remains *invariable*; and we will call it, for shortness, the *axis* of the vibrator. If r denotes the distance between $\mathbf{M}$ and $\mathbf{V}$, we will assume

$$\Psi_x = l_x \Psi(r, t) \text{ etc.} \quad (4)$$

where $\Psi(r, t)$ represents the magnitude of Ψ apart from direction.

Let $\mathbf{r}$ be a unit vector pointing in the direction of r , from $\mathbf{V}$ to $\mathbf{M}$. Substituting from (4) we get

$$\text{div } \Psi = {}_s(\mathbf{r}\mathbf{l}) \frac{\partial \Psi(r, t)}{\partial r}, \quad (5)$$

${}_s()$ being the symbol of the scalar product. Let us now write

$$\mathbf{A} = \frac{\partial^2 \Psi(r, t)}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi(r, t)}{\partial r}; \quad (6)$$

$$\mathbf{B} = \frac{\partial^2 \Psi(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi(r, t)}{\partial r}; \quad (7)$$

and let (5) be substituted in (2); then this equation becomes

$$\mathbf{E}^{(2)} = -\mathbf{l}\mathbf{B} + r{}_s(\mathbf{r}\mathbf{l})\mathbf{A}. \quad (8)$$

The projections of $\mathbf{E}^{(2)}$ on $\mathbf{l}$ and $\mathbf{r}$ are

$$E_l^{(2)} = -\mathbf{B} + ({}_s(\mathbf{r}\mathbf{l}))^2 \mathbf{A}, \quad (9)$$

$$E_r^{(2)} = {}_s(\mathbf{r}\mathbf{l}) (\mathbf{A} - \mathbf{B}). \quad (10)$$

We now proceed to obtain the expression of $\mathbf{H}^{(2)}$. Combining (3) with (4) we easily find

$$\mathbf{H}^{(2)} = {}_v(\mathbf{r}\mathbf{l})\mathbf{C}, \quad (11)$$

where the vector product is denoted by the symbol ${}_v()$ and

$$\mathbf{C} = a \partial^2 \Psi(r, t) / \partial r \partial t. \quad (12)$$

It appears from (8) and (11) that $\mathbf{E}^{(2)}$ is always in the plane through $\mathbf{r}$ and $\mathbf{l}$, and that $\mathbf{H}^{(2)}$ is perpendicular to the

same; so that $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ are at right angles to each other. The direction of $\mathbf{H}^{(2)}$ being at right angles to $\mathbf{r}$, we are justified in calling $\mathbf{H}^{(2)}$ a transverse vector; with regard to the electric vector, however, equation (10) shows that $\mathbf{E}_r$ cannot vanish unless $s(\mathbf{r}\mathbf{l})=0$, a condition fulfilled only in the equatorial plane containing the vibrator and perpendicular to its axis.

At any point in the equatorial plane the direction of $\mathbf{E}^{(2)}$ is parallel to that of $\pm \mathbf{l}$ and its magnitude is $\mathbf{B}$. For points situated in the (positive or negative) direction of the axis, $\mathbf{H}^{(2)}$ vanishes and the absolute value of $\mathbf{E}^{(2)}$ is $|\mathbf{A}-\mathbf{B}|$; the direction of $\mathbf{E}^{(2)}$ in this case is concurrent with that of $\pm \mathbf{l}$. At any other point

$$(\mathbf{E}^{(2)})^2 = \mathbf{B}^2 + (s(\mathbf{r}\mathbf{l}))^2 \mathbf{A}(\mathbf{A}-2\mathbf{B}) \quad . \quad . \quad . \quad (13)$$

$$(\mathbf{H}^{(2)})^2 = (1-(s(\mathbf{r}\mathbf{l}))^2) \mathbf{C}^2. \quad . \quad . \quad . \quad . \quad (14)$$

so that, in general, the electric and magnetic energy per unit volume have different values.

The following particular relations may be deduced at once from the preceding formulæ:—

	$\mathbf{l} \parallel x.$	$\mathbf{l} \parallel y.$	$\mathbf{l} \parallel z.$
$\mathbf{E}_x^{(2)} =$	$-\mathbf{B} + \mathbf{r}_x^2 \mathbf{A}$	$\mathbf{r}_x \mathbf{r}_y \mathbf{A}$	$\mathbf{r}_x \mathbf{r}_z \mathbf{A}$
$\mathbf{E}_y^{(2)} =$	$\mathbf{r}_y \mathbf{r}_x \mathbf{A}$	$-\mathbf{B} + \mathbf{r}_y^2 \mathbf{A}$	$\mathbf{r}_y \mathbf{r}_z \mathbf{A}$
$\mathbf{E}_z^{(2)} =$	$\mathbf{r}_z \mathbf{r}_x \mathbf{A}$	$\mathbf{r}_z \mathbf{r}_y \mathbf{A}$	$-\mathbf{B} + \mathbf{r}_z^2 \mathbf{A}$
$\mathbf{H}_x^{(2)} =$	0	$-\mathbf{r}_z \mathbf{C}$	$+\mathbf{r}_y \mathbf{C}$
$\mathbf{H}_y^{(2)} =$	$+\mathbf{r}_z \mathbf{C}$	0	$-\mathbf{r}_x \mathbf{C}$
$\mathbf{H}_z^{(2)} =$	$-\mathbf{r}_y \mathbf{C}$	$+\mathbf{r}_x \mathbf{C}$	0

Consider a spherical surface $\mathbf{S}$, of radius r , with centre at $\mathbf{V}(x_0, y_0, z_0)$. Calculate the amount of energy, emitted from the (perfectly isolated) vibrator, which crosses an element $d\mathbf{S}$ of the surface in unit time. Using as before $\mathbf{r}$ to represent a unit vector drawn outwards in the direction of the normal to $d\mathbf{S}$, we have for the flow of electromagnetic energy across $d\mathbf{S}$, in accordance with Poynting's theorem:

$$\frac{c}{4\pi} \frac{d\mathbf{S}}{d\mathbf{t}} \left[-\mathbf{BC}_s(\mathbf{r}_v(\mathbf{l}_v(\mathbf{r}\mathbf{l}))) + s(\mathbf{r}\mathbf{l})\mathbf{AC}_s(\mathbf{r}_v(\mathbf{r}_v(\mathbf{r}\mathbf{l}))) \right]. \quad (15)$$

The value of the second term in square brackets is zero, so that (15) reduces to

$$-\frac{c}{4\pi} \frac{d\mathbf{S}}{d\mathbf{t}} \mathbf{BC} \sin^2(\mathbf{r}\mathbf{l}). \quad . \quad . \quad . \quad . \quad (16)$$

which makes the rate of transmission a maximum for directions contained in the equatorial plane and zero along the (positive or negative) direction of the axis. The amount which crosses the area of S in unit time is $-\frac{2}{3}cr^2BC$.

§ 2. To put our results in a form suitable for use in further deductions, we follow a well-known method of procedure, due to Rowland and Hertz. We begin by assuming

$$\Psi(r, t) = -\frac{iA^{(2)}}{r} \epsilon^{in(t-ar-bz_0+q)}, \quad . \quad . \quad . \quad (1)$$

where n is the frequency of transmitted vibrations and b , $A^{(2)}$, and q are new constants, characteristic of the secondary radiation. Evidently (1) is an admissible solution. Let us first consider the part of the field that lies nearest to the vibrator; in this region anr may be supposed to be a very small fraction and (2), § 1, reduces to

$$E^{(2)} = \nabla \operatorname{div} \Psi. \quad . \quad . \quad . \quad (2)$$

In order to write out the full expression take (5), § 1, and substitute from (1) above. If we write

$$\mathbf{L} = -iA^{(2)} \epsilon^{in(t-bz_0+q)} \mathbf{l}, \quad . \quad . \quad . \quad (3)$$

the resulting formula, holding true in the immediate proximity of the vibrator, is as follows:

$$E^{(2)} = \nabla_s \left(\mathbf{L} \cdot \nabla \left(\frac{1}{r} \right) \right); \quad . \quad . \quad . \quad (4)$$

and from this we conclude that the electric moment of the vibrator is correctly represented by the vector $\mathbf{L}$ localized in the axis l .

Reverting now to the general case when $M(x, y, z)$ is an arbitrary point, situated anywhere in the field of the vibrator, we propose to calculate the values of the quantities $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ on which, as we have seen, the determination of the secondary field ultimately depends. Using R to denote $1/anr$ we easily deduce

$$+\mathbf{A} = \frac{a^2 n^2 A^{(2)}}{r} \epsilon^{in(t-ar-bz_0+q)} \{ (1-3R^2)i + 3R \}, \quad . \quad (5)$$

$$+\mathbf{B} = \frac{a^2 n^2 A^{(2)}}{r} \epsilon^{in(t-ar-bz_0+q)} \{ (1-R^2)i + R \}, \quad . \quad (6)$$

$$-\mathbf{C} = \frac{a^2 n^2 A^{(2)}}{r} \epsilon^{in(t-ar-bz_0+q)} \{ R + i \}. \quad . \quad . \quad (7)$$

It will save needless repetition if it is understood that in obtaining these results no approximation has been employed; they are exact consequences of the initial equations from which they are derived. The importance of a distinction between exact relations and those which (at great distances from the vibrator) hold true approximately does not appear to be always fully appreciated.

§ 3. Suppose we had a *material medium*, that is an immense collection of very minute, equal, simple linear vibrators distributed at random through space. Consider the surface of the medium as plane; taking it for the plane $z=0$, refer everything to rectangular axes x, y, z , the positive part of z being measured vertically upwards through the medium. Below the boundary plane conceive a void region extending to infinity in the negative direction of the axis of z . Suppose also that the material medium is unlimited in the positive direction of the axis of z .

Let an infinite train of homogeneous electromagnetic waves (proceeding from vacuum in the positive direction of the axis of z) be externally incident on the surface of the medium. Suppose the components of its electric and magnetic field to be simply periodic with respect to z and t and write them in the form

$$E_x^{(1)} = A^{(1)} \epsilon^{in(t-az+p)}, \quad E_y^{(1)} = 0, \quad E_z^{(1)} = 0, \quad (1)$$

$$H_x^{(1)} = 0, \quad H_y^{(1)} = A^{(1)} \epsilon^{in(t-az+p)}, \quad H_z^{(1)} = 0. \quad (2)$$

Here p and $A^{(1)}$ are constants; n represents as before the frequency of transmitted vibrations. In what follows we propose to give, to the disturbance defined by (1) and (2), the name of *incident* or *primary wave*.

According to the present view, the primary wave is conceived to get *undisturbed* across the aggregation of molecular vibrators; its progress continues in fact over indefinite depths in the medium without any mitigation or attenuation whatever. The only effect of the medium is to superimpose, on the field propagated with the incident wave, a secondary field arising under the joint operation of the vibrators. From each vibrator that is struck by waves travelling along in the medium, an infinite sequence of secondary waves is proceeding: when all these combine, a resultant tumultuous secondary radiation becomes established. The steady state is regarded as that which will not be further disturbed owing to adjustment and readjustment of the primary wave

and the extreme variety of innumerable secondary disturbances. The nature of this ultimate state of equilibrium must now be elucidated,

We commence by briefly recapitulating the most important assumptions which we shall adopt. We will suppose, in the first place, that the greatest linear dimensions of the vibrators are exceedingly small compared with the mean distance between neighbouring vibrators or $N^{-\frac{1}{3}}$, N being the number of vibrators per unit volume. It follows from this that the vibrators, being outside each other's sphere of action, are free to vibrate with practically perfect independence. On the other hand, in order to arrive at definite results we have to admit (inasmuch as our mode of argument essentially involves spacial averaging) that the mean distance between neighbouring vibrators is negligible in comparison with the shortest wave-length with which in the applications of theory we shall be concerned. In speaking in future of material media we shall suppose that these conditions are amply fulfilled.

§ 4. In the interior of the medium imagine a stratum bounded by the planes $z=z_0$ and $z=z_0+dz_0$, so that dz_0 represents the thickness of the stratum. By what has gone before, we have to consider the stratum as permeated by vibrators each of which determines a secondary field around it; hence, at a point $M(x, y, z)$, a resultant electric field is originated under the operation of the stratum. We now intend to evaluate its components $[E_x^{(2)}]$ etc. For this purpose we have to effect a summation over the vibrators, say $V, V_1, V_2, \dots$ contained in the stratum. In lieu of this we shall find it convenient to follow a different course. From the point V draw lines $VM_1, VM_2, \dots$ parallel respectively to $V_1M, V_2M, \dots$, and such that $VM_1 = V_1M$; $VM_2 = V_2M$, etc. The points $M, M_1, M_2, \dots$ fill a second stratum parallel to, and situated at a distance $z-z_0$ from the first. Instead of adding the component fields arising at M under the joint operation of $V, V_1, V_2, \dots$, we obtain the desired result if we combine the fields which V determines at $M, M_1, M_2, \dots$. The assumption on which we proceed is here evidently that every wave under consideration must be exactly plane. When this condition is fulfilled, the advantage of introducing the auxiliary points $M_1, M_2, \dots$ is obvious.

To find $[E_x^{(2)}]$ we now refer to cylindrical coordinates whose axis coincides in direction with the former axis of z . Let ρ be the radius vector to the point M drawn perpendicular

to the axis, and let γ represent the angle between the plane (ρz) and (xz) , a fixed plane through the axis. Using r to denote as before the distance between M and V, we have

$$[E_x^{(2)}] = N dz_0 \int_0^{2\pi} \int_0^\infty d\gamma \rho d\rho E_x^{(2)} = N dz_0 \int_0^{2\pi} \int_{|z-z_0|}^\infty d\gamma r dr E_x^{(2)}. \quad (1)$$

$|z-z_0|$ stands here for $\pm(z-z_0)$, the plus or minus sign being taken according as $z-z_0$ is positive or negative. The axes of the vibrators are parallel to the direction of x so that, by § 1,

$$E_x^{(2)} = -\mathbf{B} + \mathbf{r}_x^2 \mathbf{A}. \quad (2)$$

From (1) and (2) we deduce

$$[E_x^{(2)}] = 2\pi N dz_0 (\mathbf{I}_A - \mathbf{I}_B) \quad (3)$$

where

$$\mathbf{I}_A = \frac{1}{2} \int_{|z-z_0|}^\infty r dr \left\{ 1 - \frac{(z-z_0)^2}{r^2} \right\} \mathbf{A}; \quad (4a)$$

$$\mathbf{I}_B = \int_{|z-z_0|}^\infty r dr \mathbf{B}. \quad (4b)$$

In calculating $\mathbf{I}_A$ and $\mathbf{I}_B$ we must use in (4), instead of $\mathbf{A}$ and $\mathbf{B}$, the expressions (5) and (6), § 2. A difficulty now emerges because, unless some further conditions are assigned, the integrals (4) are really ambiguous. This difficulty is not peculiar to the present discussion; it arises in a similar manner, for instance, in the Theory of Diffraction. The ambiguity is usually avoided by supposing* that the integrated term at the upper limit may be taken to vanish. [An excellent proof of the legitimacy of this mode of procedure will be found in Dr. Fr. Reiche's paper cited above.] On this understanding the values of $\mathbf{I}_A$ and $\mathbf{I}_B$ are at once obtainable. We find

$$\mathbf{I}_A = \frac{an\mathbf{A}^{(2)}\epsilon^{in[t-a|z-z_0|-bz_0+q]}}{ian|z-z_0|}, \quad (5)$$

$$\mathbf{I}_B = \frac{an\mathbf{A}^{(2)}\epsilon^{in[t-a|z-z_0|-bz_0+q]}}{ian|z-z_0|} \{1 + ian|z-z_0|\}. \quad (6)$$

Going back to (3) we obtain $[E_x^{(2)}]$ in the form

$$[E_x^{(2)}] = -2\pi an dz_0 N \mathbf{A}^{(2)} \epsilon^{in[t-a|z-z_0|-bz_0+q]}. \quad (7)$$

It may be easily verified that $[E_y^{(2)}] = 0$ and $[E_z^{(2)}] = 0$.

* Cf. Lord Rayleigh's Scientific Papers, vol. iii. pp. 76-77.

The evaluation of the magnetic field excited by the stratum follows a similar course ; its component along y is

$$[H_y^{(2)}] = 2\pi(z-z_0)dz_0N \int_{|z-z_0|}^{\infty} drC. \quad . \quad . \quad . \quad (8)$$

Applying (7), § 2, we obtain as before

$$[H_y^{(2)}] = \mp 2\pi an dz_0 NA^{(2)}\epsilon^{in}[t-a|z-z_0|-bz_0+q], \quad . \quad (9)$$

where the upper sign is to be taken if the difference $z-z_0$ is positive and the lower if it is negative.

We shall have in a similar manner :

$$[H_z^{(2)}] = 0, \quad [H_x^{(2)}] = 0.$$

§ 5. The effect of a stratum having been ascertained, we are enabled to evaluate the aggregate secondary field arising from the presence of matter in bulk. Let us suppose for clearness of conception that the medium under consideration extends from $z=0$ to $z=\infty$. Consider a point M in the medium, situated in a plane $z=z$ where $z>0$. Secondary electromagnetic waves, emanating from various strata of the substance, are uninterruptedly transmitted through the plane. Let us calculate the electric field (measured along x) which arises at M under the simultaneous operation of secondary waves travelling in the *positive* direction of the axis of z . Things having become steady, this field, in accordance with § 4 above, is

$$-2\pi an NA^{(2)} \int_0^z dz_0 \epsilon^{in}[t-bz_0-a(z-z_0)+q]. \quad . \quad . \quad (1)$$

A convenient way of explaining the meaning of the expression (1) is to remark that it represents the cumulative effect of *concurrent* secondary waves, advancing in the direction in which the incident wave traverses the medium. For simplicity and definiteness, we shall use the terms *positive* and *negative* to distinguish effects due to secondary waves which are propagated, relatively to the medium, in the same direction as the incident wave and in the opposite direction. Putting then $(E_x^{(2)})_+$ for the secondary *positive* electric effect, we have

$$(E_x^{(2)})_+ = \frac{2\pi NA^{(2)}}{i(bc-1)} \{ \epsilon^{in}(t-bz+q) - \epsilon^{in}(t-az+q) \}. \quad . \quad (2)$$

A similar argument may be applied to find the value of the secondary *negative* electric effect $(E_x^{(2)})_-$ which is generated, in the plane $z=z$ in the medium, by the operation of

secondary waves crossing the plane in the negative direction of the axis of z :

$$(E_x^{(2)})_- = -\frac{2\pi N A^{(2)}}{i(bc+1)} \epsilon^{in(t-bz+q)}. \quad (3)$$

The actual electric field induced at M, say (E_x) , may be considered as compounded of $(E_x^{(2)})_+$, $(E_x^{(2)})_-$, and the primary effect $E_x^{(1)}$ directly due to the incident wave :

$$(E_x) = \frac{4\pi N A^{(2)}}{i(b^2c^2-1)} \epsilon^{in(t-bz+q)} + \left[A^{(1)} \epsilon^{in(p-q)} - \frac{2\pi N A^{(2)}}{i(bc-1)} \right] \epsilon^{in(t-az+q)}. \quad (4)$$

In this equation it will be observed that the occurrence of the term containing $\epsilon^{in(t-az+q)}$ points to a train of waves transmitted through the medium with vacuum phase-velocity c and without any loss or enfeeblement whatever. Among the observed phenomena of light an effect of this kind is unknown; and if we confine ourselves to the consideration of *steady states*, we are justified in denying the possibility of its realization. In the subsequent treatment we will assume that electromagnetic waves are always obliterated as they advance in the medium. To satisfy this condition it is requisite that terms such as those to which we allude should remain *inoperative* in our solutions. The word is so significant of our meaning that there is presumably no need for apology in using it for the sake of brief terminology.

We have accordingly

$$A^{(1)} \epsilon^{in(p-q)} = \frac{2\pi N A^{(2)}}{i(bc-1)}, \quad (5)$$

and

$$(E_x) = \frac{4\pi N A^{(2)}}{i(b^2c^2-1)} \epsilon^{in(t-bz+q)}. \quad (6)$$

Let us determine the corresponding magnetic effects generated at the point M, supposing as before that the state of the system has become thoroughly steady. Reverting to § 4 we easily find, for the secondary *positive* magnetic effect

$$(H_y^{(2)})_+ = \frac{2\pi N A^{(2)}}{i(bc-1)} \{ \epsilon^{in(t-bz+q)} - \epsilon^{in(t-az+q)} \}, \quad (7)$$

and for the secondary *negative* magnetic effect

$$(H_y^{(2)})_- = \frac{2\pi N A^{(2)}}{i(bc+1)} \epsilon^{in(t-bz+q)}. \quad (8)$$

Hence the expression of the component along y of the actual magnetic field at M becomes

$$(H_y) = \frac{4\pi N A^{(2)} \cdot bc}{i(b^2 c^2 - 1)} e^{in(t-bz+q)} + \left[A^{(1)} e^{in(p-q)} - \frac{2\pi N A^{(2)}}{i(bc-1)} \right] e^{in(t-az+q)}. \quad (9)$$

To prevent the occurrence in (9) of terms representing "vacuum-waves" (which the medium is incapable of propagating), precisely the same condition must be fulfilled which is already satisfied in consequence of equation (5) above. This should be observed in corroboration of our previous result. Finally we have

$$(H_y) = \frac{4\pi N A^{(2)} \cdot bc}{i(b^2 c^2 - 1)} e^{in(t-bz+q)}. \quad (10)$$

§ 6. Let us consider now an electron which is set vibrating to and fro under the operation of an electric force $\mathbf{E}$. Write e for the charge on the electron and m for its effective mass. Let ξ be the component, in the direction of x , of the displacement of the electron, reckoned from the position of equilibrium; let n_0 represent a constant and $\mathfrak{S}$ a definite period of time of very short duration which in most cases of interest may be put $= 2e^2/3mc^3$. Confining ourselves to vibrations in which $\mathfrak{S}$ is a very small fraction of the periodic time $2\pi/n$, we assume as the equation of motion in the direction of x :

$$m(\ddot{\xi} - \mathfrak{S}\ddot{\ddot{\xi}} + n_0^2 \xi) = e\mathbf{E}_x. \quad (1)$$

[The second term of the left-hand member arises here from damping due to radiation. In a celebrated paper, Professor H. A. Lorentz has shown (*l. c.*, pp. 103, 104) that this simple form of expression, $-\mathfrak{S}\ddot{\ddot{\xi}}$, is adequate whenever the molecules of the substance, as in gases and liquids, can be supposed to be irregularly distributed. The third term is generally regarded as representing the component along x of the "quasi-elastic" force supposed to connect the electron to the system to which it belongs. The introduction of this concept has not escaped criticism; by some writers it is considered as a gratuitous supposition, devoid of physical significance. Arguing from analogy we might perhaps invoke in its favour the well-known propositions deducible from the theory of small vibrations performed, about an equilibrium configuration, by a dynamical (holonomic)

system. But from the point of view of the present paper the question evidently is of secondary importance.]

It remains for us (with the view of investigating forced vibrations) to evaluate the force $\mathbf{E}$ acting on the electron. The component $\mathbf{E}_x$ of this *efficacious* force is made up of $(\mathbf{E}_x)$, the electric field calculated in § 5 and a further additional part whose importance was insisted upon, many years ago, by H. A. Lorentz and Sir Joseph Larmor. In order to express this term in our notation, we must recur to (3), § 2. With use of this relation we can write *

$$\mathbf{E}_x = \mathbf{a}\epsilon^{in(t-bz_0+p)} - \frac{4}{3}\pi\mathbf{N} \cdot i\mathbf{A}^{(2)}\epsilon^{in(t-bz_0+q)}, \quad (2)$$

where $\mathbf{a}$ is a constant. When we bear in mind that the expression (6), § 5, with z_0 written instead of z , will exactly represent what in the foregoing equation (2) is denoted by $\mathbf{a}\epsilon^{in(t-bz_0+p)}$, we find

$$4\pi\mathbf{N}\mathbf{A}^{(2)} = i(b^2c^2 - 1)\mathbf{a}\epsilon^{in(p-q)}. \quad (3)$$

We now require another equation connecting $\mathbf{A}^{(2)}$ with $\mathbf{a}$; for this purpose recourse must be had to the equation of motion. From (1) and (2) we deduce

$$\{n_0^2 - n^2 + i\mathfrak{S}n^3\}e\xi = \frac{e^2}{m} \{ \mathbf{a}\epsilon^{in(p-q)} - \frac{4}{3}\pi\mathbf{N} \cdot i\mathbf{A}^{(2)} \} \epsilon^{in(t-bz_0+q)}, \quad (4)$$

and substituting from (3), § 2, we have the result

$$\left(n_0^2 - n^2 - \frac{4}{3}\pi\mathbf{N} \frac{e^2}{m} + i\mathfrak{S}n^3 \right) \mathbf{A}^{(2)} = i \frac{e^2}{m} \mathbf{a}\epsilon^{in(p-q)}. \quad (5)$$

We come therefore to an important conclusion. If we assume

$$bc = \nu - i\kappa, \quad (6)$$

ν will represent the refractive index, κ the coefficient of extinction of the medium. Equations (3), (5), (6) now give

$$\left(n_0^2 - n^2 - \frac{4}{3}\pi\mathbf{N} \frac{e^2}{m} + i\mathfrak{S}n^3 \right) \left((\nu - i\kappa)^2 - 1 \right) = 4\pi\mathbf{N} \frac{e^2}{m}. \quad (7)$$

This is in agreement with experimental evidence and has been known to hold true ever since the Theory of Dispersion in its contemporary form was elaborated.

In the domain of Optical Dispersion, the basis of the usual

* A slight change in the original calculation is introduced here; cf. *l. c.* pp. 10 and 22.

procedure rests on a scheme of differential equations determining the laws of propagation of light in material media. On the view here being developed, these laws of course continue to apply and therefore must be derivable from the molecular standpoint. It will then serve to bring out more clearly the scope of the foregoing calculation if we show that our solutions are consistent with the fundamental Maxwell-Lorentz equations. To examine this point it will be convenient to recapitulate some of the results previously obtained. We have :

$$(E_x) = \frac{4\pi N A^{(2)}}{i(b^2 c^2 - 1)} e^{in(t-bz+q)}, \quad (E_y) = 0, \quad (E_z) = 0, \quad (8)$$

$$(H_x) = 0, \quad (H_y) = \frac{4\pi N A^{(2)} b c}{i(b^2 c^2 - 1)} e^{in(t-bz+q)}, \quad (H_z) = 0, \quad (9)$$

$$(P_x) = -i N A^{(2)} e^{in(t-bz+q)}, \quad (P_y) = 0, \quad (P_z) = 0. \quad (10)$$

These equations merely reiterate our foregoing results : (6) and (10) of § 5 and (3), § 2 ; (E_x) , (H_x) etc. represent as before the components of the actual electric and magnetic field at a point in the medium and (P_x) etc. are the components of the polarization electrically induced. It may now be immediately verified that (8), (9), and (10) satisfy the well-known Maxwell-Lorentz equations as usually formulated.

§ 7. A slightly different form may be given to the foregoing equation (7), § 6, namely :

$$\frac{n_0^2 - n^2 - \frac{4}{3}\pi N \frac{e^2}{m}}{4\pi N \frac{e^2}{m}} = \frac{\nu^2 - \kappa^2 - 1}{(\nu^2 - \kappa^2 - 1)^2 + 4\nu^2 \kappa^2}, \quad \dots \quad (1).$$

$$\frac{3n^3}{4\pi N \frac{e^2}{m}} = \frac{2\nu\kappa}{(\nu^2 - \kappa^2 - 1)^2 + 4\nu^2 \kappa^2}, \quad \dots \quad (2).$$

Equation (2) includes as a particular case Lord Rayleigh's celebrated formula of extinction ; for assuming κ to be very small in comparison with $\nu - 1$ we have from (2)*, if we retain the leading term only :

$$\frac{\nu h}{(\nu^2 - 1)^2} = \frac{3n^4}{4\pi c N \frac{e^2}{m}} = \frac{8\pi^3}{3\lambda^4 N}, \quad \dots \quad (3).$$

* Cf. *Bulletin Int. de l'Acad. d. Sc. de Cracovie* for 1909, p. 9.0.

where λ is the wave-length and $h=4\pi\kappa/\lambda$ represents Lord Rayleigh's "coefficient of transmission." Thus divesting our equations of perhaps unnecessary generality (for it is doubtful in how far we may consider the calculation applicable unless κ is small) we revert to Lord Rayleigh's fundamental theorem.

It may be well to observe, by way of caution, that (2) and (3) are quite independent of the form assumed by the term which appears in the numerator of the left-hand member of (1). It must be borne in mind, in this connexion, that we can adopt a view of the damping action (experienced by the electron) which departs widely from that here contemplated without necessarily invalidating the general form of (1) and (2). If we do so, the expression for the left-hand member of (1) may not become substantially modified, whereas (2) will in general be quite different. This consideration is serviceable in illustrating the intimate connexion which exists between Lord Rayleigh's equation and the hypothesis from which we have started as to the nature of the frictional effect generated during the vibration of the electron. If we were to adopt, for instance, Helmholtz's tentative supposition (beautifully exemplified in H. A. Lorentz's theory of impacts) according to which the vibrating electron experiences a resisting force in simple proportion to its instantaneous velocity of motion, we should have to put λ^2 , instead of λ^4 , in the denominator of the right-hand member of (3) (and to modify besides the value of the constant).

§ 8. We now proceed to consider the electric field *in vacuo* where $z < 0$. This field consists of the effect of the incident wave

$$E_x^{(1)} = A^{(1)} e^{in(t-az+p)} \quad . \quad . \quad . \quad (1)$$

and the cumulative electric effect

$$\begin{aligned} -2\pi anNA^{(2)} \int_0^\infty dz_0 e^{in[t-hz_0+a(z-z_0)+q]} \\ = -\frac{2\pi NA^{(2)}}{i(bc+1)} e^{in(t+az+q)}, \quad . \quad . \quad (2) \end{aligned}$$

produced by *negative* secondary waves which are emitted, in the negative direction of the axis of z , by every stratum of the material medium, from $z_0=0$ to $z_0=\infty$. We will assume that (2) simply represents the *reflected* wave thrown back

into vacuum, *seemingly* from the surface of the body. Hence, if the reflected wave be

$$E_x^* = A^* \epsilon^{in(t+az+q)} \quad . \quad . \quad . \quad (3)$$

(where A^* will be supposed to be real), we get

$$A^* = -\frac{2\pi N A^{(2)}}{i(bc+1)} \quad . \quad . \quad . \quad (4)$$

To connect A^* with $A^{(1)}$ it is only necessary to resort to equation (5) of § 5; the result may be expressed

$$A^* = -\frac{bc-1}{bc+1} A^{(1)} \epsilon^{in(p-q)}, \quad . \quad . \quad . \quad (5)$$

which gives the well-known relations, applicable in the case of external reflexion:

$$\frac{A^*}{A^{(1)}} \cos n(q-p) = -\frac{\nu^2-1+\kappa^2}{(\nu+1)^2+\kappa^2}, \quad . \quad . \quad (6a)$$

$$\frac{A^*}{A^{(1)}} \sin n(q-p) = +\frac{2\kappa}{(\nu+1)^2+\kappa^2}. \quad . \quad . \quad (6b)$$

It may be easily verified that the component along y of the magnetic field of the reflected wave is

$$-A^* \epsilon^{in(t+az+q)}. \quad . \quad . \quad . \quad (7)$$

In concluding this subject it may be well to consider in a similar manner the *refracted* wave, propagated within the material medium ($z > 0$). In order to represent this wave, we write

$$E_x^{**} = A^{**} \epsilon^{in(t-bz+r)} \quad . \quad . \quad . \quad (8)$$

supposing A^{**} to be real; having regard to (6), § 5, we obtain

$$A^{**} = \frac{4\pi N A^{(2)}}{i(b^2c^2-1)} \epsilon^{in(q-r)}. \quad . \quad . \quad . \quad (9)$$

By (5), § 5, used in (9) we deduce

$$A^{**} = \frac{2A^{(1)}}{bc+1} \epsilon^{in(p-r)}, \quad . \quad . \quad . \quad (10)$$

and therefore we find

$$\frac{A^{**}}{A^{(1)}} \cos n(r-p) = \frac{2(\nu+1)}{(\nu+1)^2+\kappa^2}, \quad . \quad . \quad (11a)$$

$$\frac{A^{**}}{A^{(1)}} \sin n(r-p) = \frac{2\kappa}{(\nu+1)^2+\kappa^2}. \quad . \quad . \quad (11b)$$

For the magnetic field of the refracted wave let us assume

$$B^{**} \epsilon^{in(t-hz+s)}, \quad . \quad . \quad . \quad . \quad (12)$$

where B^{**} is real; the application of (10), § 5, and the preceding equation (9) gives

$$B^{**} = bcA^{**} \epsilon^{iu(r-s)}. \quad . \quad . \quad . \quad . \quad (13)$$

This shows that

$$\frac{B^{**}}{A^{**}} \cos n(s-r) = \nu; \quad . \quad . \quad . \quad . \quad (14a)$$

$$\frac{B^{**}}{A^{**}} \sin n(s-r) = -\kappa. \quad . \quad . \quad . \quad . \quad (14b)$$

Our discussion has thus far been confined to the case of external reflexion. The general conclusion at which we arrive is that, availing ourselves of the principles established in §§ 1-5 of the present communication, we are led in all cases to correct results, although we have taken no account whatever of the requirements of *boundary conditions*. It should be remarked that the merit of having called attention to the advantage that can thus be secured belongs entirely to Mr. W. Esmarch.

§ 9. Let us imagine a plate of thickness Z and of optical behaviour (ν, κ) different from that of the void region on each side of it. As before, let the axis of z be measured vertically upwards. The plate is supposed to be bounded by two parallel planes which for distinctness of conception we suppose horizontal; let the lower plane be taken for the plane $z=0$ and the upper one for the plane $z=Z$. Conceive a series of plane waves of linearly-polarized light propagated vertically upwards through the region of negative values of z :

$$E_x^{(1)} = A^{(1)} \epsilon^{in(t-az+d)}, \quad E_y^{(1)} = 0, \quad E_z^{(1)} = 0. \quad (1)$$

$$H_x^{(1)} = 0, \quad H_y^{(1)} = A^{(1)} \epsilon^{in(t-az+d)}, \quad H_z^{(1)} = 0. \quad (2)$$

Under the influence of the penetrating wave the molecules constituting the substance of the plate are thrown into sympathetic vibration and may be regarded, at least very approximately, as centres of diverging disturbances. Our aim and object is now to specify a system of these secondary wave disturbances capable of maintaining themselves steadily in the plate. To effect this, let us consider in how far, for

the purposes of the present problem, the scheme of relations laid down in our preceding calculations requires modification. It may be readily seen that all the results established in § 1 above will here still hold true ; but in order that they may thus preserve their generality, we must replace (1), § 2, by the following equation of definition :

$$\Psi(r, t) = -\frac{i}{r} (P\epsilon^{-inbz_0} + Q\epsilon^{inbz_0})\epsilon^{in(t-ar)}, \quad . \quad . \quad (3)$$

where P and Q are new constants, to be determined hereafter. It follows that the quantities **A**, **B**, **C** (on which the calculation of secondary effects ultimately depends) are expressible in the form

$$+ \mathbf{A} = \frac{a^2 n^2}{r} (P\epsilon^{-inbz_0} + Q\epsilon^{inbz_0})\epsilon^{in(t-ar)} \{ (1 - 3R^2)i + 3R \}, \quad (4)$$

$$+ \mathbf{B} = \frac{a^2 n^2}{r} (P\epsilon^{-inbz_0} + Q\epsilon^{inbz_0})\epsilon^{in(t-ar)} \{ (1 - R^2)i + R \}, \quad . \quad (5)$$

$$- \mathbf{C} = \frac{a^2 n^2}{r} (P\epsilon^{-inbz_0} + Q\epsilon^{inbz_0})\epsilon^{in(t-ar)} \{ i + R \}, \quad . \quad . \quad (6)$$

where $R = 1/anr$ as before.

We now pass on to the calculation of the secondary effects to be expected, in the present case, upon the principles established in §§ 4 and 5 of this communication. Imagine a thin stratum in the plate, at right angles to the axis of z ; let dz_0 be the thickness of the stratum. We have to evaluate the resultant electric field, at a point $M(x, y, z)$ of the field, due to the presence of vibrators occupying the stratum. Let $[E_x^{(2)}]$ represent the component along x of this resultant field ; proceeding as in § 4 we easily find :

$$[E_x^{(2)}] = -2\pi an dz_0 N (P\epsilon^{-inbz_0} + Q\epsilon^{inbz_0})\epsilon^{in[t-a|z-z_0|]}. \quad (7)$$

By a like process, for the component along y of the resultant magnetic field excited by the stratum, we obtain

$$[H_y^{(2)}] = \mp 2\pi an dz_0 N (P\epsilon^{-inbz_0} + Q\epsilon^{inbz_0})\epsilon^{in[t-a|z-z_0|]}; \quad (8)$$

the upper sign is to be taken when $z - z_0$ is positive and the lower sign when this difference is negative.

Let the point M be chosen within the substance of the plate, in a plane $z = z$ at right angles to the axis of z . We proceed to consider the electric field (measured along x) which arises at M under the simultaneous operation of secondary waves travelling across the plane in the *positive*

direction of the axis of z ; or, which is the same thing, of concurrent secondary waves advancing in the direction in which the incident wave traverses the plate. Using the symbol $(E_x^{(2)})_+$ to denote this quantity (which, as in § 5 above, we propose to call "the secondary positive electric effect") we shall have by (7):

$$(E_x^{(2)})_+ = -2\pi N \left[\frac{P}{i(bc-1)} - \frac{Q}{i(bc+1)} \right] e^{in(t-az)} \\ + \frac{2\pi NP}{i(bc-1)} e^{in(t-bz)} - \frac{2\pi NQ}{i(bc+1)} e^{in(t+bz)}. \quad (9)$$

It remains to find the secondary *negative* electric effect $(E_x^{(2)})_-$ which arises at M under the operation of secondary waves crossing the plane under consideration in the negative direction of the axis of z . Going back again to (7) we find

$$(E_x^{(2)})_- = 2\pi N \left[\frac{P e^{-inbZ}}{i(bc+1)} - \frac{Q e^{inbZ}}{i(bc-1)} \right] e^{-inaZ} e^{in(t+az)} \\ - \frac{2\pi NP}{i(bc+1)} e^{in(t-bz)} + \frac{2\pi NQ}{i(bc-1)} e^{in(t+bz)}. \quad (10)$$

For the actual electric field at M we obtain, by means of (9), (10) and with the assistance of (1) and (2), an expression which represents several superimposed sets of waves travelling in the plate along the positive and negative direction of z . But if we assume (as we do throughout these calculations) that the state of radiation attained by the system is already thoroughly steady, we have to remember that the substance of the plate is incapable of transmitting electromagnetic disturbances without enfeeblement and with velocity c ; to secure the inexistence of such waves the "inoperative" terms in the expression for the resultant secondary effect must be balanced by the field of the incident wave. We therefore have the equations

$$2\pi N \left[\frac{P}{i(bc-1)} - \frac{Q}{i(bc+1)} \right] = A^{(1)} e^{ind}. \quad (11)$$

and

$$\frac{P e^{-inbZ}}{bc+1} - \frac{Q e^{inbZ}}{bc-1} = 0, \quad (12)$$

and the component in the direction of x of the electric field at M is simply

$$\frac{4\pi N}{i(b^2c^2-1)} (P e^{in(t-bz)} + Q e^{in(t+bz)}). \quad (13)$$

From (11) and (12) it appears that

$$2\pi NP = \frac{i(bc+1)(b^2c^2-1)A^{(1)}\epsilon^{in(d+bZ)}}{(bc+1)^2\epsilon^{inbZ}-(bc-1)^2\epsilon^{-inbZ}}, \quad . \quad . \quad (14)$$

$$2\pi NQ = \frac{i(bc-1)(b^2c^2-1)A^{(1)}\epsilon^{in(d-bZ)}}{(bc+1)^2\epsilon^{inbZ}-(bc-1)^2\epsilon^{-inbZ}}. \quad . \quad . \quad (15)$$

All this may be corroborated, as a little consideration will show, when our knowledge of magnetic effects is turned to account. We have indeed to assume that the "inoperative" terms in the expression of the resultant secondary magnetic effect are destroyed owing to the cooperation of the primary magnetic field; clearly this is a condition of things which is consistent with steadiness and permanency. To attain our object we need only evaluate $(H_y^{(2)})_+$ and $(H_y^{(2)})_-$. We find:

$$\begin{aligned} (H_y^{(2)})_+ = & -2\pi N \left[\frac{P}{i(bc-1)} - \frac{Q}{i(bc+1)} \right] \epsilon^{in(t-az)} \\ & + \frac{2\pi NP}{i(bc-1)} \epsilon^{in(t-bz)} - \frac{2\pi NQ}{i(bc+1)} \epsilon^{in(t+bz)}, \quad . \quad . \quad (16) \end{aligned}$$

$$\begin{aligned} (H_y^{(2)})_- = & -2\pi N \left[\frac{P\epsilon^{-inbZ}}{i(bc+1)} - \frac{Q\epsilon^{inbZ}}{i(bc-1)} \right] \epsilon^{in[t+a(z-Z)]} \\ & + \frac{2\pi NP}{i(bc+1)} \epsilon^{in(t-bz)} - \frac{2\pi NQ}{i(bc-1)} \epsilon^{in(t+bz)}. \quad . \quad . \quad (17) \end{aligned}$$

By application of (2) we conclude that the component along y of the actual magnetic field at M is as follows:

$$\frac{4\pi Nbc}{i(b^2c^2-1)} (P\epsilon^{in(t-bz)} - Q\epsilon^{in(t+bz)}), \quad . \quad . \quad (18)$$

and, on the other hand, that our present equations (11) and (12) are fulfilled. Thus, whether electric or magnetic effects are considered, the inference is, as before, that (11), (12), and accordingly (14), (15), must hold true.

It is of importance to assure ourselves that the intensity of the wave apparently *reflected* from the interface $z=0$ can be correctly deduced from our investigation. For this purpose we calculate the cumulative electric effect of secondary waves emitted downwards, into the region of

negative values of z , by all the vibrators contained in the plate. If this effect be represented by E_x^* , we have by (7):

$$E_x^* = -2\pi anN \left[P \int_0^Z dz_0 \epsilon^{in[t-bz_0+a(z-z_0)]} + Q \int_0^Z dz_0 \epsilon^{in[t+bz_0+a(z-z_0)]} \right]. \quad (19)$$

From this we easily obtain, making use of (12), (14), (15),

$$E_x^* = - \frac{(b^2c^2-1)(\epsilon^{inbZ} - \epsilon^{-inbZ})A^{(1)}\epsilon^{in(t+az+d)}}{(bc+1)^2\epsilon^{inbZ} - (bc-1)^2\epsilon^{-inbZ}} \quad (20)$$

The last application that we shall make of the results obtained is to investigate the intensity of the *transmitted* beam of light. We begin by calculating, at a point M past the plate ($z > Z$), the cumulative electric effect that arises under the influence of our system of vibrators. By means of (7) we get the result

$$2\pi N \left[\frac{Q}{i(bc+1)} - \frac{P}{i(bc-1)} + \left(\frac{P\epsilon^{-inbZ}}{i(bc-1)} - \frac{Q\epsilon^{inbZ}}{i(bc+1)} \right) \epsilon^{inaZ} \right] \epsilon^{in(t-az)}. \quad (21)$$

The effect of the vibrators in the plate is thus to increase by the quantity (21) the component along x of the electric field of the incident wave (1) conceived to pursue its course undisturbed. Accordingly, for the complete expression of the actual electric field at the point M past the plate, we have with use of (11), (14), and (15):

$$E_x^{**} = \frac{4bcA^{(1)}\epsilon^{in[t-a(z-Z)+d]}}{(bc+1)^2\epsilon^{inbZ} - (bc-1)^2\epsilon^{-inbZ}} \quad (22)$$

It may serve as a confirmation of the legitimacy of the method employed to mention that (21) and (22) are in accord with the results which it is customary to obtain by the aid of certain assumptions respecting boundary conditions. While constructing our argument we have convinced ourselves that, on the view here advocated, boundary conditions could be altogether ignored.