On the Reduction of a Linear Substitution to a Canonical Form.

By T. J. I'A. Bromwich. Received October 27th, 1899. Read November 9th, 1899.

I. *Introductory Remarks.*

It will be seen at once that the main idea of the following note is the same as that of Herr Netto's paper "Zur Theorie der linearen Substitutionen."* That is to say, we pass from the case of a substitution whose characteristic determinant has a root a repeated p times to the case of a substitution with p roots differing but little from a, but all distinct. The change is made by increasing each coefficient of the substitution by a small arbitrary quantity; and so we reduce the substitution to the limiting case of one with all its roots distinct.

The point of divergence between my work and Herr Netto's is that I have shown that it is unnecessary to retain these small changes in the coefficients when we seek to determine the linear functions which reduce the substitution to a canonical form. This simplification will be found of considerable advantage in cases of numerical calculation; from a purely theoretical point of view, it is probably of less importance. To show that the difficulties of calculation are not great, I have worked out at length the particular example given by Prof. Burnside in illustration of his method for reducing linear substitutions.†

II.

Suppose we have the substitution

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + \ldots \qquad\quad + a_{2n}x_n, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n' &= a_{n1}x_1 + \ldots \qquad\quad + a_{nn}x_n. \end{aligned}$$

* *Acta Mathematica*, Vol. XVII., p. 265.
† *Proc. Lond. Math. Soc.*, Vol. XXX., p. 180.

Then, forming the quantity

$$l_1 x_1' + l_2 x_2' + \ldots + l_n x_n',$$

we shall have that this is θ times

$$l_1 x_1 + l_2 x_2 + \ldots + l_n x_n,$$

provided we have

$$\left.\begin{array}{ll} (a_{11}-\theta)\, l_1 + a_{21} l_2 + \ldots + a_{n1} l_n & = 0 \\ a_{12} l_1 + (a_{22}-\theta)\, l_2 + \ldots + a_{n2} l_n & = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \\ a_{1n} l_1 + a_{2n} l_2 + \ldots \quad + (a_{nn}-\theta)\, l_n & = 0 \end{array}\right\} . \qquad (1)$$

Hence θ is a root of the determinantal equation

$$\Delta = \begin{vmatrix} a_{11}-\theta, & a_{21}, & \ldots, & a_{n1} \\ a_{12}, & a_{22}-\theta, & \ldots, & a_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n}, & a_{2n}, & \ldots, & a_{nn}-\theta \end{vmatrix} = 0.$$

If the roots of this equation are all distinct, we have n values of θ, and n corresponding determinations of the ratios

$$l_1 : l_2 : \ldots : l_n.$$

But this method breaks down if we find that any root of $\Delta = 0$, say $\theta = a$, is repeated. The object of this note is to explain how we can very easily extend our method so as to cover this case. Suppose, then, that $(\theta - a)$ is a p-times repeated factor of Δ and a q-times repeated factor of *every* first minor of Δ; so that $(\theta - a)^{p-q}$ is a first invariant-factor (*Elementartheiler*) of Δ. Solve for the ratios $l_1 : l_2 : \ldots : l_n$ from any $(n-1)$ of the equations (1); we shall suppose the last $(n-1)$ to be selected, to avoid verbal confusion, the method of procedure being the same whatever $(n-1)$ equations are chosen. We now have

$$\frac{l_1}{A_{11}} = \frac{l_2}{A_{21}} = \ldots = \frac{l_n}{A_{n1}},$$

the capital letters being the first minors of the corresponding small letters in Δ. Every one of the quantities $A_{11}, \ldots, A_{n1}$ will contain the factor $(\theta - a)^q$; divide out by this and write $\theta = a + t$. We then have $l_1, l_2, \ldots, l_n$ expressed as polynomials in t; one at least of these polynomials must have a term independent of t, which implies that at least one of the minors $A_{11}, \ldots, A_{n1}$ is *regular* or is not divisible by a higher power than t^q.*

* If this is not the case, we must use another set of $(n-1)$ equations for the l's.

Now write $\xi = (a_{11}-\theta)\, l_1 + a_{21}\, l_2 + \ldots + a_{n1}\, l_n$,

and eliminate $l_1 : l_2 : \ldots : l_n$.

We obtain a determinant equal to zero, which only differs from Δ in having a_{k1} replaced by $a_{k1} - \xi/l_k$, where k is any one of the numbers $1, 2, \ldots, n$ which satisfies the condition that A_{k1} is regular.

Thus we have $$\Delta - (\xi A_{k1}/l_k) = 0,$$

and, replacing θ by $(a+t)$, we see that Δ contains t^p as a factor, and A_{k1}/l_k contains t^q. Hence ξ contains t^{p-q} as a factor. the remaining factor not vanishing with t.

Thus we have the equations

$$a_{11}\, l_1 + a_{21}\, l_2 + \ldots + a_{n1}\, l_n = (a+t)\, l_1$$
$$+ \text{terms of order } t^{p-q} \text{ and higher orders},$$

$$a_{12}\, l_1 + a_{22}\, l_2 + \ldots + a_{n2}\, l_n = (a+t)\, l_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_{1n} l_1 + a_{2n}\, l_2 + \ldots + a_{nn} l_n = (a+t)\, l_n.$$

Hence $$l_1 x_1' + l_2 x_2' + \ldots + l_n x_n' = (a+t)(l_1 x_1 + l_2 x_2 + \ldots + l_n x_n)$$
$$+ \text{terms of order } t^{p-q},$$

or, if we expand $l_1 x_1 + \ldots + l_n x_n$ in the form

$$X_1 + X_2\, t + X_3\, t^2 + \ldots,$$

we shall have

$$X_1' + X_2' t + X_3' t^2 + \ldots = (a+t)(X_1 + X_2\, t + X_3\, t^2 + \ldots)$$
$$+ \text{terms of order } t^{p-q}.$$

Thus, equating coefficients of corresponding powers of t,* we have

$$X_1' = a X_1,$$
$$X_2' = a X_2 + X_1,$$
$$X_3' = a X_3 + X_2,$$
$$\ldots \quad \ldots \quad \ldots$$
$$X_{p-q}' = a X_{p-q} + X_{p-q-1}.$$

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* This step is legitimate, for t is arbitrary, and the series on the two sides will be terminated.

It should be observed that the values of the X's so determined are not unique; for we need not take

$$l_1 = \frac{A_{11}}{t^q}, \text{ \&c.,}$$

and it is only necessary to put

$$l_1 = \frac{A_{11}}{t^q}(\beta_1 + \beta_2 t + \beta_3 t^2 + \ldots),$$

$$l_2 = \frac{A_{21}}{t^q}(\beta_1 + \beta_2 t + \beta_3 t^2 + \ldots),$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$l_n = \frac{A_{n1}}{t^q}(\beta_1 + \beta_2 t + \beta_3 t^2 + \ldots).$$

Then we see that, if

$$(X_1 + X_2 t + X_3 t^2 + \ldots)(\beta_1 + \beta_2 t + \beta_3 t^2 + \ldots) = Y_1 + Y_2 t + Y_3 t^2 + \ldots,$$

so that
$$Y_m = \beta_1 X_m + \beta_2 X_{m-1} + \ldots + \beta_m X_1,$$

the series of Y's will satisfy the same equations as the X's; and the Y's contain $(p-q)$ arbitrary constants. There may be a further indeterminacy, according to the particular line in equations (1) whose minors give the l's.

We now proceed to get more linear functions of $x_1, \ldots, x_n$ which possess similar properties. Suppose that $(\theta - a)^r$ is a factor of every second minor of Δ, and, further, that this is the highest power of $(\theta - a)$ that does occur in every second minor.

Solve for the ratios $l_1 : l_2 : \ldots : l_n$ from the last $(n-2)$ equations* of (1); the results will involve one arbitrary quantity (such as the ratio $l_1 : l_2$). We write $\theta = a + t$, and express the ratios in powers of t, dividing by t^r; now put

$$\xi = (a_{11} - \theta)\, l_1 + a_{21} l_2 + \ldots + a_{n1} l_n,$$

$$\eta = a_{12} l_1 + (a_{22} - \theta)\, l_2 + \ldots + a_{n2} l_n.$$

* We might use equally well any other $(n-2)$ of the equations; the process of selection adopted here is to cancel successively the first, second, third, &c., of equations (1).

Then we see that

$$\begin{vmatrix} a_{12}l_1 + a_{k2}l_k - \eta, & a_{22} - \theta, & \dots, & a_{n2} \\ a_{13}l_1 + a_{k3}l_k, & a_{23}, & \dots, & a_{n3} \\ \dots & \dots & \dots & \dots \\ a_{1n}l_1 + a_{kn}l_k, & a_{2n}, & \dots, & a_{nn} - \theta \end{vmatrix} = 0,$$

where the two l's in the first column may be any two we please, although here one has been taken to be l_1, simply to save multiplicity of distinguishing suffixes.

Expanding out this determinant, we have

$$\eta \text{ (a second minor of } \Delta) = l_1 \text{ (a first minor)} + l_k \text{ (a first minor)}.$$

Now, at least one of the second minors contains as a factor t raised to no higher power than t^r (or is regular); we here suppose this to be true for the minor obtained by deleting the first and second rows and columns of Δ. All the first minors contain t^q as a factor, while none of the l's contain negative powers of t. Hence the lowest power of t in η is at least t^{q-r}; the same result holds for ξ.

Accordingly we have the equations

$$\begin{aligned} a_{11}l_1 + a_{21}l_2 + \dots + a_{n1}l_n &= (a+t)\, l_1 + \text{terms of order } t^{q-r}, \\ a_{12}l_1 + a_{22}l_2 + \dots + a_{n2}l_n &= (a+t)\, l_2 + \text{terms of order } t^{q-r}, \\ a_{13}l_1 + a_{23}l_2 + \dots + a_{n3}l_n &= (a+t)\, l_3, \\ \dots \quad \dots \quad &\dots \quad \dots \quad \dots \\ a_{1n}l_1 + a_{2n}l_2 + \dots + a_{nn}l_n &= (a+t)\, l_n. \end{aligned}$$

Thus, if with these values of $l_1, l_2, \dots, l_n$, we write

$$l_1x_1 + \dots + l_nx_n = X_{p-q+1} + tX_{p-q+2} + t^2X_{p-q+3} + \dots + t^{q-r-1}X_{p-r} + \dots,$$

we shall have

$$X'_{p-q+1} + tX'_{p-q+2} + t^2X'_{p-q+3} + \dots = (a+t)(X_{p-q+1} + tX_{p-q+2} + \dots) \\ + \text{terms of order } t^{q-r}.$$

Hence

$$\begin{aligned} X'_{p-q+1} &= aX_{p-q+1}, \\ X'_{p-q+2} &= aX_{p-q+2} + X_{p-q+1}, \\ \dots \quad &\dots \quad \dots \quad \dots \\ X'_{p-r} &= aX_{p-r} + X_{p-r-1}. \end{aligned}$$

Just as before, we can construct a set of Y's which satisfy these equations and contain $(q-r)$ arbitrary constants. But we may add to Y_{p-q+k} any term such as

$$\gamma_1 X_k + \gamma_2 X_{k-1} + \ldots + \gamma_k X,$$

for $X_1, X_2, \ldots, X_k, \ldots$ satisfy the same relations as $X_{p-q+1}, X_{p-q+2}, \ldots, X_{p-q+k} \ldots$, and so we get $(q-r)$ more arbitrary constants. It should be observed that the last $(q-r)$ constants should include the one arbitrary constant that appears in the original solution for $l_1 : l_2 : \ldots : l_n$. We have thus in the general reduction of these $(q-r)$ terms $2(q-r)$ arbitrary constants.

It is now easy to see how we can extend the method proposed so as to deal with terms which arise from minors of higher orders. Suppose h denotes the difference between the index of $(\theta-a)$ in the greatest common measure of all the $(k-1)^{\text{th}}$ minors of Δ, and the corresponding index for the k^{th} minors; so that $(\theta-a)^h$ is the k^{th} invariant-factor (*Elementartheiler*) of Δ. We solve for $l_1 : l_2 : \ldots : l_n$ from the last $(n-k)$ of equations (1)*; then we write $\theta = a+t$, and divide by the extraneous power of t; the values of $l_1 : \ldots : l_n$ so found will satisfy the first k of equations (1) up to terms in t^h. Then we expand $l_1 x_1 + \ldots + l_n x_n$ up to terms in t^{h-1}, and so obtain h linear functions of $x_1, \ldots, x_n$, say $X_{v+1}, X_{v+2}, \ldots, X_{v+h}$. These will satisfy

$$\begin{aligned} X'_{v+1} &= aX_{v+1}, \\ X'_{v+2} &= aX_{v+2} + X_{v+1}, \\ \ldots & \quad \ldots \quad \ldots \\ X'_{v+h} &= aX_{v+h} + X_{v+h-1}. \end{aligned}$$

The most general values which satisfy these equations can be constructed as before explained; and we see that they will contain hk arbitraries, h from each of the k groups of linear functions $(X_1, \ldots, X_{p-q}), (X_{p-q+1}, \ldots, X_{p-r}), \ldots, (X_{v+1}, \ldots, X_{v+h})$.

Our process goes on until we reach a minor of Δ which does not contain $(\theta-a)$ as a factor. We shall then have found

$$(p-q)+(q-r)+(r-s)+\ldots = p$$

linear functions of $x_1, \ldots, x_n$ which reduce to canonical forms those parts of the substitution which correspond to the root $\theta = a$ of $\Delta = 0$.

Proceeding in this way for each root of $\Delta = 0$, we finally obtain n linear functions which will reduce the given substitution to its canonical form.

* We assume that one at least of the minors formed from these $(n-k)$ equations is regular.

III.

To give a numerical illustration, and to compare the method with Prof. Burnside's (*Proc. Lond. Math. Soc.*, Vol. xxx., p. 191), let us take his example,

$$\begin{aligned} x_1' &= -2x_1 - x_2 - x_3 + 3x_4 + 2x_5, \\ x_2' &= -4x_1 + x_2 - x_3 + 3x_4 + 2x_5, \\ x_3' &= x_1 + x_2 - 3x_4 - 2x_5, \\ x_4' &= -4x_1 - 2x_2 - x_3 + 5x_4 + x_5, \\ x_5' &= 4x_1 + x_2 + x_3 - 3x_4. \end{aligned}$$

We find the equations for the l's,

$$\begin{aligned} -(2+\theta)\, l_1 - 4l_2 + l_3 - 4l_4 + 4l_5 &= 0, \\ -l_1 + (1-\theta)\, l_2 + l_3 - 2l_4 + l_5 &= 0, \\ -l_1 - l_2 - \theta l_3 - l_4 + l_5 &= 0, \\ 3l_1 + 3l_2 - 3l_3 + (5-\theta)\, l_4 - 3l_5 &= 0, \\ 2l_1 + 2l_2 - 2l_3 + l_4 - \theta l_5 &= 0, \end{aligned}$$

and these give $\Delta = -(\theta+1)^2(\theta-2)^3.$

The minors of the elements of the first row are found to be, in order,

$$\begin{aligned} &\theta(\theta-2)^3, \\ &-(\theta+1)^2(\theta-2), \\ &-(\theta-2)^3, \\ &3(\theta+1)(\theta-2)^2, \\ &(\theta+1)(\theta-2)(2\theta-7), \end{aligned}$$

so that $(\theta-2)$ is a factor of all these, but $(\theta+1)$ is not. As a matter of fact $(\theta-2)$ is a factor of every first minor; so that for $(\theta-2)$ the second and fifth of the above are regular; for $(\theta+1)$, the first and third. To proceed put $\theta = -1+t$, and, the corresponding difference $(p-q)$ being equal to 2, we have to expand only as far as the terms in t. After dividing by $-3(\theta-2)$, we find, neglecting terms in t^2 and higher powers of t, in agreement with our rule,

$$\begin{aligned} l_1 &= \tfrac{1}{3}(1-t)(3-t)^2 = 3-5t, \\ l_2 &= \tfrac{1}{3}t^2 \qquad\qquad\quad = 0, \\ l_3 &= \tfrac{1}{3}(3-t)^2 \qquad\quad = 3-2t, \\ l_4 &= t(3-t) \qquad\qquad = 3t, \\ l_5 &= \tfrac{1}{3}t(9-2t) \qquad = 3t. \end{aligned}$$

Hence
$$X_1 = 3(x_1+x_5),$$
$$X_2 = -5x_1-2x_3+3(x_4+x_5).$$

We shall not have to proceed to the second minors, as $(\theta+1)$ is not a factor of all the first minors.

We next put $\theta = 2+t$, and divide by $(-3t)$; then

$$\begin{aligned} l_1 &= -\tfrac{1}{3}t^2(2+t) &&= 0,\\ l_2 &= \tfrac{1}{3}(3+t)^2 &&= 3+2t,\\ l_3 &= \tfrac{1}{3}t^2 &&= 0,\\ l_4 &= -t(3+t) &&= -3t,\\ l_5 &= \tfrac{1}{3}(3+t)(3-2t) &&= 3-t, \end{aligned}$$

where we expand only to terms in t, because the difference $(p-q)$ is 2.

Thus
$$X_3 = 3(x_2+x_5),$$
$$X_4 = 2x_2-3x_4-x_5.$$

In this case we have to go on to solve the last three equations in the l's; but we may simply write $\theta = 2$, and drop the t, for the difference $(q-r)$ is 1; so that the l's need only be calculated to the term independent of t. Hence we take

$$-(l_1+l_2)-2l_3-l_4+l_5 = 0,$$
$$(l_1+l_2)-l_3+l_4-l_5 = 0,$$
$$2(l_1+l_2)+2l_3+l_4-2l_5 = 0,$$

giving
$$l_3 = 0,\quad l_4 = 0,\quad l_1+l_2 = l_5.$$

Hence
$$X_5 = l_1(x_1+x_5)+l_2(x_2+x_5),$$

and l_1 must not vanish, for, if so,
$$3X_5 = l_2 X_3.$$

We now have the given substitution reduced to

$$X_1' = -X_1,$$
$$X_2' = -X_2+X_1,$$
$$X_3' = 2X_3,$$
$$X_4' = 2X_4+X_3,$$
$$X_5' = 2X_5,$$

which is a canonical form of the substitution. We see that the classification of any substitution is given by the indices of the invariant-factors, and hence this substitution is typified by [(2. 1), 2].

We may take the generalized reducing functions

$$\begin{aligned} Y_1 &= \beta_1 X_1, \\ Y_2 &= \beta_1 X_2 + \beta_2 X_1, \\ Y_3 &= \gamma_1 X_3, \\ Y_4 &= \gamma_1 X_4 + \gamma_2 X_3, \\ Y_5 &= \delta_1 X_5, \end{aligned}$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1$ are arbitrary.

Prof. Burnside's two reductions are given by

(i.) $\beta_1 = -3, \quad \beta_2 = 0, \quad \gamma_1 = -3, \quad \gamma_2 = 0, \quad l_2 = -l_1.$

(ii.) $\beta_1 = \frac{1}{3}, \quad \beta_2 = \frac{5}{9}, \quad \gamma_1 = \frac{1}{3}, \quad \gamma_2 = -\frac{8}{9}, \quad l_2 = -l_1.$

[*February* 15*th*.—Since completing the above I have seen Mr. A. C. Dixon's paper (p. 170 of this volume). His method is quite different from mine.]

Notes on the Theory of Automorphic Functions. *By* A. C. Dixon.

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Under the above heading I propose to make some remarks on certain points in the theory of automorphic functions, from the point of view taken by Poincaré in his memoirs (*Acta Mathematica*, Vols. I., III., IV., V.).

In the first place, I show how the theorem given by him that a Fuchsian function exists of the second family and of class 0, and taking assigned values at singular points, may be used to establish the existence theorem on a Riemann surface, so far at least as that theorem relates to uniform functions of position on the surface.

Next I give expressions for Abelian integrals of the first two kinds in terms of series of the type used by Poincaré. Series of the same type are also used to form factorial functions.

It is also shown that a uniform function of the automorphic class exists which will serve as a prime function in the expression of Fuchsian functions as the product of factors. Such have been constructed for automorphic functions existing all over the plane. That which is here given serves for the other class.