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The sine of $\phi - \theta$ is the sum of the projections of OU and OV on OX . Now OX is the sine arm of the angle ϕ along whose second arm lies OU , and also of the angle $\frac{1}{2}\pi + \phi$ along whose second arm lies OV . Hence

$$\begin{aligned}\sin(\phi - \theta) &= OU \sin \phi + OV \sin(\tfrac{1}{2}\pi + \phi) \\ &= \cos \theta \sin \phi - \sin \theta \sin(\tfrac{1}{2}\pi + \phi) \\ &= \cos \theta \sin \phi - \sin \theta \cos \phi.\end{aligned}$$

Negative Angles.

The transition from arithmetic subtraction to algebraic addition is effected by giving to the sine and cosine of a negative angle such meanings as shall make the subtraction formulae particular cases of the addition formulae. It is easily seen that the required convention is simply that when an angle is to be regarded as negative the direction of its sine arm must be reversed.

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A SUGGESTED REARRANGEMENT OF THE BOOK-WORK ON SOME ELEMENTARY SERIES.

IN all the English text-books on Analytical Trigonometry, so far as I know, the power-series for $\cos x$ is obtained as the limit, when n tends towards ∞ , of the finite sum for $\cos x$,

$$\cos^n\left(\frac{x}{n}\right)\left[1 - \left(1 - \frac{1}{n}\right)\frac{x^2}{2!}t^2 + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)\frac{x^4}{4!}t^4 - \dots\right],$$

where $t = \left(\tan \frac{x}{n}\right)/\frac{x}{n}$ and the number of terms is equal to the integer next greater than $\frac{1}{2}n$. The power-series for $\sin x$ is obtained similarly; the two series and the method of finding them being both due to Euler.

Now it appears to me that this method is subject to certain drawbacks, when introduced in a *first* course on Analytical Trigonometry. For there are two ways of calculating the limit; the harder (but accurate) one as given in books such as Chrystal's *Algebra* or Hobson's *Trigonometry*; and the easier (inaccurate) method of the older text-books, which has unfortunately been reproduced in some new texts.[†]

The weakness in the inaccurate form of the proof is found in the assumption that the product

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{2r-1}{n}\right)$$

has the limiting value 1, for *all* values of r which do not exceed $\frac{1}{2}n$ (see for instance pp. 109, 110 of Lachlan and Fletcher's book just quoted). The falsity of this assumption is almost self-evident, when thus stated; but the assumption is, nevertheless, nearly always swallowed by the beginner without a qualm.[‡]

* It is perhaps worth while to enter a plea for the use of the phrase, *from the very beginning*, in dealing with limits. The phrases "the limit when n is ∞ ," "the limit when $n = \infty$ " can do nothing but harm.

† Castle's *Manual of Practical Mathematics* (Macmillan, 1903); Lachlan and Fletcher's *Elements of Trigonometry* (Arnold, 1904). I refer to these as examples, merely; in many other respects both books strike me as good.

‡ I have found a little numerical calculation very useful in convincing the unbeliever that there is an assumption involved. Thus with n even, and $r = \frac{1}{2}n$, we find the product to be less than $1/10^3$ when $n = 10$; less than $1/10^{10}$ for $n = 30$; less than $1/10^{20}$ for $n = 50$.

On the other hand the accurate evaluation of the limit is often found hard to grasp, the essential points of the proof being obscured to a certain extent by the details of the algebra. In my opinion the real difficulty arises from the fact that a *double* limit is introduced in the course of the proof; and that an appeal is made to what is really the principle of *uniform convergence* (in a somewhat disguised form). For what is done amounts to this:—The finite sum for $\cos x$ is divided into two parts, the first m terms $\sigma(m, x, n)$ say, and the remainder $\rho(m, x, n)$. It is then proved that if x lies within certain bounds, $\rho(m, x, n)$ can be made numerically less than any arbitrarily prescribed fraction ϵ , by choice of m only however great n may be (provided that $n > 2m$). In other words $\rho(m, x, n)$ is proved to converge to zero *uniformly* for all values of x, n , subject to the conditions just mentioned. Next, the infinite series $S(x)$ is divided into the first m terms $S(m, x)$ and the remainder $R(m, x)$; and it is proved that for the *same* value of m , $R(m, x)$ is also numerically less than ϵ . Finally, m having been now *fixed*, it is proved that N can be found so that when $n > N$, the difference between $\sigma(m, x, n)$ and $S(m, x)$ is also less than ϵ in numerical value. Hence $\cos x - S(x)$ is numerically less than 3ϵ , provided $n > N$, but $\cos x - S(x)$ is *independent* of n , and consequently $\cos x - S(x) = 0$.

Now this skeleton proof is, for practical purposes of teaching, further complicated by certain algebraic inequalities which must be used in finding the greatest possible values of ρ , R and $\sigma - S$. And, so far as my experience goes, I have found it sufficiently hard to explain the idea of uniform convergence, even when the simplest examples are used, from which all algebraic difficulties have been removed; and it would therefore seem unreasonable to expect a beginner to follow a proof which depends on uniform convergence and involves certain further difficulties of a purely algebraic character.

For the reasons just explained, I have been led to abandon the use of these limits in a first course; and I am inclined to think that other teachers may have felt the same difficulties. My solution of the problem is to give a short course on the elementary Differential Calculus before touching on the sine and cosine power-series. Having explained that when a (*continuous*) function of x has a positive differential coefficient, the function increases with x , I take the following cases:

- (i) $\frac{d}{dx}(x - \sin x) = 1 - \cos x$,
- (ii) $\frac{d}{dx}[\cos x - (1 - \frac{1}{2}x^2)] = x - \sin x$,
- (iii) $\frac{d}{dx}[\sin x - (x - \frac{1}{6}x^3)] = \cos x - (1 - \frac{1}{2}x^2)$,
- (iv) $\frac{d}{dx}[(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4) - \cos x] = \sin x - (x - \frac{1}{6}x^3)$,

and so on.

Now, in (1), $1 - \cos x$ is positive and consequently $x - \sin x$ increases with x ; it is a continuous function and vanishes for $x=0$; thus $x - \sin x$ is positive for positive values of x . Apply this to (ii); and by a similar argument $\cos x - (1 - \frac{1}{2}x^2)$ is positive for positive values of x ; and this combined with (iii) shows that $\sin x - (x - \frac{1}{6}x^3)$ is positive: passing to (iv), $(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4) - \cos x$ is positive; and so on. Thus we have established that $\sin x$ lies between x and $(x - \frac{1}{6}x^3)$; $\cos x$ between $1 - \frac{1}{2}x^2$ and $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Clearly the number of terms may be increased to any extent; and we may either introduce the infinite series at once, or leave the results in the form of bounds within which $\sin x$ and $\cos x$ are contained.*

*For the purpose of numerical calculation the second form is often more useful than the series.

A similar process can be used in connection with $\arctan x$; for we have

$$\frac{d}{dx}[(x - \frac{1}{3}x^3 + \frac{1}{5}x^5) - \arctan x] = 1 - x^2 + x^4 - \frac{1}{1+x^2} = \frac{x^6}{1+x^2},$$

$$\frac{d}{dx}[\arctan x - (x - \frac{1}{3}x^3)] = \frac{1}{1+x^2} - (1-x^2) = \frac{x^4}{1+x^2},$$

and so, since both of these differential coefficients are positive, we can prove that $\arctan x$ lies between $(x - \frac{1}{3}x^3)$ and $(x - \frac{1}{3}x^3 + \frac{1}{5}x^5)$.

By an exactly similar method we can find bounds for e^{-x} and $\log(1+x)$, x being positive; but there is here a logical difficulty, since the differentiation of x^x (and therefore of $\log x$) depends on the limit

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

The evaluation of this limit by any accurate process leads to difficulties of much the same character as those encountered in the limits for $\cos x$ and $\sin x$. I have generally found it most convincing (in a first course) to approximate to the limit by numerical calculation, using 7- or 8-figure tables. Thus we get the approximate values

$$(1 + \frac{1}{10})^{10} = 2.593, (1 + \frac{1}{100})^{100} = 2.705, (1 + \frac{1}{1000})^{1000} = 2.717,$$

$$(1 - \frac{1}{10})^{10} = 2.868, (1 - \frac{1}{100})^{100} = 2.732, (1 - \frac{1}{1000})^{1000} = 2.719,$$

which indicate† that the limit is in the neighbourhood of 2.718. Taking the limit as known, the differentiation of a^x , $\log x$, etc., is easy by the methods explained in any good text-book on the Calculus (such as Professor Gibson's).

It may be not out of place to record my opinion that, to a beginner, numerical verification of general theorems on series is often more convincing than the algebraic proofs. As a matter of personal experience, I found my own faith greatly strengthened by verifying that, to three places of decimals, the sine-series gave 1.000 and the cosine-series 0.000 for $x = 1.571$; but, as a practical matter, for economy of time smaller values of x will be found preferable, such as $\sin(.5236) = .500$, $\cos(.5236) = .866$, etc.

The sequence of propositions sketched here would, I think, be found to fit in very well with the *Report of the M.A. Committee on Higher School Mathematics* (*Gazette*, vol. 3, July, 1904, p. 52); and it is very likely that the main ideas have already been used by some teachers.‡ The method has the double advantage of giving some elegant illustrations of the elementary rules of the Calculus and, at the same time, of lightening the work in Trigonometry.

In conclusion, I venture to express the opinion that, as a general rule, any work which involves double limits or uniform convergence should be omitted from the school course; or practically speaking, that "ε-proofs" should be left out. Such subjects appear to be altogether better suited to a later stage in mathematical training; and the more so because they are entirely unnecessary to anyone but professional mathematicians.§ If this suggestion is accepted, as a matter of course the infinite products for $\sin x$ and $\cos x$ would

*The angle is taken (as usual in the Calculus) to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

†They also serve to show that the convergence to the limit is very slow, compared with that of the exponential series.

‡Some of the results (perhaps all of them) are given as exercises in Prof. Gibson's excellent *Calculus*; I had, however, obtained them independently before his book was published.

§It is perhaps going too far at present to suggest that the theory of double limits and uniform convergence is essential for a professed mathematician in England; just as, for the present, it seems hopeless to expect even an elementary knowledge of the foundations of geometry from our "up-to-date" Euclids.

be omitted (compare the last paragraph of the *Report* quoted above). But, so long as such proofs must be taught, I would strongly recommend the arrangement given in J. Tannery's admirable *Introduction à la Théorie des Fonctions d'une Variable*, a text-book which deserves to be better known in this country than it appears to be.*

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MATHEMATICAL NOTES.

153. [P. 3. b.] *On a Fundamental Theorem in Inversion.*

If two figures F, F' be inverses of each other, their inverses with respect to any pair of corresponding points P, P' are similar.

Demonstration. Let O be the centre, and ρ the radius of inversion of F, F' . Let Q, R be any two points of the figure F ; and Q', R' the corresponding points of the figure F' . Also let q, r be the inverses of Q, R with respect to centre P , radius of inversion λ ; and q', r' the inverses of Q', R' with respect to centre P' , radius of inversion λ' . The theorem will be proved if we show that the ratio of qr to $q'r'$ is constant.

Use the following lemma: If A, B be the inverses of A', B' with respect to a centre O , radius of inversion ρ , then $A'B'/AB = \rho^2/OA \cdot OB$.

We have by the lemma

$$\begin{aligned} \frac{qr}{QR} &= \frac{\lambda^2}{PQ \cdot PR}; \quad \text{and} \quad \frac{q'r'}{Q'R'} = \frac{\lambda'^2}{P'Q' \cdot P'R'}; \\ \therefore \frac{qr}{q'r'} &= \frac{\lambda^2}{\lambda'^2} \cdot \frac{QR}{Q'R'} \cdot \frac{P'Q'}{PQ} \cdot \frac{PR}{P'R'} \\ &= \frac{\lambda^2}{\lambda'^2} \cdot \frac{\rho^2}{OQ' \cdot OR'} \cdot \frac{OP' \cdot OQ'}{\rho^2} \cdot \frac{OP \cdot OR}{\rho^2} \\ &= \frac{\lambda^2}{\lambda'^2} \cdot \frac{OP'^2}{\rho^2} = \frac{\lambda^2}{\lambda'^2} \cdot \frac{OP'}{OP} \quad \text{since } \rho^2 = OP \cdot OP' \\ &= \frac{\lambda^2}{OP} \div \frac{\lambda'^2}{OP'} \end{aligned}$$

Thus the ratio of qr to $q'r'$ is constant and therefore the theorem is proved.

A second demonstration will be given later on. The metrical relation obtained above, namely, that, if λ, λ' be the radii of inversion at P, P' the corresponding lengths in the inverses of F, F' with respect to P, P' are to one another as $\frac{\lambda^2}{OP} = \frac{\lambda'^2}{OP'}$, should be noted, as it is of importance in the applications

of the theorem to recent Geometry.

From the theorem it follows that if a figure F undergoes n successive inversions and is thereby converted into F_n , and to the point P in F corresponds the point P_n in F_n , then the inverse of F_n with respect to P_n is similar to the inverse of F with respect to P .

154. [P. 3. b.] *Continued Inversion by Coaxial Circles.*

An interesting way of obtaining some of Mr. C. E. Youngman's results (p. 7) is by the stereographic projection. Let HK be the points common to the coaxial system. Bisect HK at O , and draw OV equal to OH and perpendicular to the plane of the system. Draw a sphere through V with O as

*It may not be out of place to refer to the fact that the real difficulty in finding the infinite product is not to prove that $x(1-x^2/\pi^2)$, etc. are factors; but to prove that no factor of the form e^{ax} is present. This remark is due to Stolz, but is not made in any of the English text-books.