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GEORGE ROBERT STEPHENSON, President,
in the Chair.

No. 1,457.—“On the Probable Errors of Levelling; with Rules for the Treatment of Accumulated Errors.” By WILFRID AIRY, B.A., M. Inst. C.E.

As the members of the Institution are no doubt familiar with the processes of levelling, it is not intended to do more than glance at the various precautions taken to provide against accidental errors. In most engineering works it is unnecessary to give much attention to the small discrepancies between the results arrived at by different observers; but in the case of a complicated system of long lines of levelling, such as is required for the levels of an entire country, the aggregation of small errors becomes formidable, and the proper treatment of them requires considerable method and skill. In the present Paper the Author has dealt with errors of levelling under the following heads:—

- A. The different classes of errors which occur in all levelling operations.
- B. The probable errors of levelling on ground of different degrees of inclination, as deduced from standard levelling in different countries.
- C. The treatment of probable errors in the case of a complex system of levelled lines.
- D. Remarks on the instruments and methods of working used in accurate levelling.

A. THE DIFFERENT CLASSES OF ERRORS WHICH OCCUR IN ALL
LEVELLING OPERATIONS.

These are of three kinds, viz. :—

- (1) Mistakes.
- (2) Instrumental errors.
- (3) Errors of observation.

The first two of these are not of the nature of graduated errors. ‘Mistakes’ arise from want of care in reading the staff, or the bubble, or in booking the readings: as well as from the foot of the staff

being shifted between the readings, and from want of care in holding the staff vertical. The only effectual safeguard is to check the work throughout by independent observers, and to divide the distance into short sections defined by benchmarks. As a check on individual levelling, it has been customary in Great Britain and in India to graduate the staves differently on the two sides, and always to read both sides of the staff: the readings of the two sides ought, of course, to differ always by a constant quantity. To secure the foot of the staff from being shifted between the readings, it is usual to place the staff either upon a peg driven into the ground, or upon a three-cornered iron shoe jammed down firmly on the ground. To insure, as far as possible, the verticality of the staves, it is usual to have small levels, or plummets, let into them at one side, and in India the staves were guyed upright by ropes.

(2) Instrumental errors are either errors of collimation, of level, or of form of the telescope, the last-mentioned only coming into question when the spirit-level is not permanently attached. None of these will affect the accuracy of the levelling, so long as the instrument is always placed exactly midway between the staves: this condition is essential to all exact levelling, and has been observed in all operations of a standard character. Along with instrumental errors must be classed those of length and graduation of the staves: in general the graduation of the staves is very uniform, but errors of absolute length are not uncommon, and in all exact operations a standard bar should be kept at hand for testing the length of the staves.

(3) Errors of observation are the small graduated errors inseparable from all operations which depend upon human nerves and judgment. For any single operation they are very small; but, as they are repeated with every operation, they may amount to a considerable aggregate at the termination of a long course of work. For different persons the errors will, in all probability, have different values, as they depend upon natural and acquired keenness of estimation. They are, in fact, due to the degree of uncertainty existing in the mind of the observer as to the exact value of the several quantities which he has to estimate, and it is to be presumed that his estimate of these quantities is as likely to exceed as to fall short of the true value. Errors of judgment (but not 'mistakes') also occur in reading the staves, and the bubble; and errors also arise from small oscillations of the staves from the vertical, but not from carelessness or any determined bias on the part of the staff-holder. To these must be

added slight alterations in length of the staves caused by changes of temperature or moisture: which, though sensible in amount, were found in the Swiss levelling to be very slow, small, and irregular, so that no law could be safely deduced. The combination of these produces the final error, and it is with these that the present Paper is chiefly intended to deal. There is no safeguard against errors of observation, but their amount may be reduced by the selection of observers of natural quickness of sight and decision, well trained to the work.

Dr. Whewell and Colonel Walker have noticed that, in general, the results of levelling any given line differ considerably according as the line is levelled in one direction or the other. Colonel Walker attributes this mainly to the dislevelling effect of the sun's heat upon the instruments, and to errors of reading the bubble caused by the oblique direction in which the light falls upon it. To obtain reliable results it is essential that every line should be levelled both ways, and the mean of the results adopted as the true value, as the various circumstances of heat, light, ground, weather, and indeed everything that would tend to give a bias to the levelling, would stand a chance of being reversed or modified.

B. THE PROBABLE ERRORS OF LEVELLING ON GROUND OF DIFFERENT DEGREES OF INCLINATION, AS DEDUCED FROM STANDARD LEVELLING IN DIFFERENT COUNTRIES.

In general the results obtained by different observers on a long line of levelling are different. But there are not unfrequently cases, even on very long lines and on steep ground, in which the results of the different observers agree closely, and it is usual to quote such close agreement as a proof of the accuracy of the common result. Nevertheless it is always highly probable that the result arrived at is incorrect: for no observer could reckon on arriving at the same result each time, if he re-levelled the section several times over, and yet he would probably be unable to say that one result was more correct than another. Therefore the chances are greatly against the probability of the common result being correct, and it may be assumed that the result is in error, but that the errors of the different observers have all run in the same direction, and are of equal amount.

Since then it would be quite unsafe to form conclusions as to the probable error of the levelling on any line from the actual discordance of the observations on that particular line, and since it is

necessary to be furnished with values of the probable errors in order to combine the results on different lines in the most probable manner, it becomes necessary to form rules or tables for arriving at the probable errors either by calculation or inspection. This subject has received much consideration from Messrs. Hirsch and Plantamour at pages 108-121 of the "Nivellement de la Suisse."

Obviously the amount of the error on any given line will in general depend upon the length of the line and the steepness of the ground. If length of line only were considered as affecting the error, and (e) be taken to represent the amount of the error which may be admitted as fairly due to a single mile of the levelling, then the probable error on a line of length (k) miles will be $e\sqrt{k}$. (See Airy's "Theory of Errors of Observation," art. 53.) But when there is much difference in the nature of the ground, it is certain that the error for any line will greatly depend upon the sum of the ups and downs on that line. Errors in the lengths of the staves and in small oscillations of the staves will certainly be proportional to the sum of the ups and downs: also, in all probability, steep ground will lead to high elevations where the effect of the wind and rain will be more severely felt, and the uncertainty of the readings will in consequence be greater. If the sum of the ups and downs only were considered as affecting the error, and (f) be taken to represent the amount of the error which may be admitted as due to a rise or fall of 1 foot, then the probable error on any line will be (f) times the sum of the ups and downs on that line.

In the application of these formulæ to the case of a 'polygon,' Messrs. Hirsch and Plantamour have apparently made no attempt to combine their effects. They have deduced the values of (e) and (f) from the closing error of the polygon, by assuming that the closing error is uniformly distributed over the entire length of the polygon, and have applied either one or other of the formulæ to determine the corrections to the several sides of the polygon, as a first approximation. Then if, as was usually the case, some of the sides of the polygon went to form adjacent polygons, new values were ascertained for the corrections to those common sides from similar operations with the adjacent polygons. These new values were then imported into the original polygon, and a new closing error obtained for that polygon, from which in the same way as before second approximations to the values for the several sides were established. And a similar method was pursued for third and fourth approximations.

This plan appears open to several objections. First, it is certain that the error due to any given line depends both upon the length of the line and the contour of the ground, and whatever formula is used for the corrections should have regard to both. Secondly, it is very unsafe to use the actual closing error of a polygon as the basis of the corrections which are to be applied to the several sides: for it might well happen that the closing error was exceedingly small, while the error at intermediate points was large. Thirdly, the method of successive approximations, as described, is laborious and unsatisfactory. As soon as any network of lines has been completely levelled, no matter how complicated it may be, there is no difficulty in applying the Theory of Probable Errors so as to obtain directly and simultaneously the most probable values for the levels of all the points of the system.

In considering the question of dealing, in a satisfactory way, with the adjustment of the levels on a complicated system, it occurred to the Author to frame Tables of Probable Errors, such as would be applicable to the circumstances of the different lines of the system. The data for framing such tables would be drawn from the official records of standard levelling in different countries, and the value of the tables when framed would depend in a great degree upon the mass of statistics reduced in order to compile them. Through the courtesy of Colonel Clarke, R.E., of the Ordnance Survey Department, and of Colonel Walker, of the Survey Department of India, the Author has obtained access to the original records of the check levelling executed on long lines in Great Britain and in India, and he has extracted similar information with reference to the Swiss levelling from the "*Nivellement de la Suisse*."

A critical examination of the discrepancies between observers on the same lines of levels, showed that the average amount of the discrepancies depended very much upon the inclination of the ground. If the average inclination between two benchmarks was steep, no matter whether parts of the distance were uphill and other parts downhill, then the discrepancy was greater for that section than for another section on which the ground was flatter, and this rule was found to hold with great regularity. It was therefore possible to frame a Table of Errors suitable to different inclinations of ground, and the application of the table is as follows:—Suppose that a line of (r) miles in length has been levelled, and that the character of the ground corresponds with one or other of the classes mentioned in the Table. The probable error due to a single mile of levelling on such ground would be taken

from the Table by simple inspection, and this probable error multiplied by $\sqrt{r}$ would give the probable error due to the whole line. By this method due account would be taken of all the principal causes of error, as depending upon the contour of the ground and the length of the line, and the probable error for each line of the system would be determined on a broad and uniform basis.

The results of the reduction of a large mass of observations in Great Britain, India, and Switzerland are given in Table I. There appeared to be no practical advantage in quoting the average differences for very small changes of inclination of the ground, and the differences are therefore given for the gradients included between broad and well-defined limits of inclination. Whenever the differences were tested for gradients not quoted, the law of increase of differences with increase of inclination was observed to hold good with considerable uniformity. Thus, for instance, in the English levelling the average differences in a single mile were—

For gradients below 1 in 100	Foot. 0·024
Do. between 1 in 100 and 1 in 50	0·035
Do. do. 1 in 50 and 1 in 20	0·042

TABLE I.—AVERAGE DIFFERENCES IN A SINGLE MILE OF THE RESULTS OBTAINED BY TWO OBSERVERS ON GROUND OF DIFFERENT DEGREES OF INCLINATION IN GREAT BRITAIN, INDIA, AND SWITZERLAND.¹

	Great Britain.	India.	Switzerland.
	Foot.	Foot.	Foot.
(1) Average difference in a single mile on ground nearly level, and in very favourable circumstances of weather	·0230	·0142	·0125
(2) Average difference in a single mile on ground of a slightly undulating character, where the gradients do not exceed 1 in 100	·0238	·0168	·0148
(3) Average difference in a single mile on ground where the gradients lie entirely between 1 in 100 and 1 in 20	·0379	·0208	·0183
(4) Average difference in a single mile on ground where the gradients lie entirely between 1 in 20 and 1 in 10	·0566	·0350	·0308
(5) Average difference in a single mile on steep and rough ground where the gradients are frequently steeper than 1 in 10	..	..	·0416

¹ The quantities marked in black figures are estimated from the analogy afforded by the Swiss Levelling, as no direct data could be furnished.

To exhibit more clearly the nature of the law which regulates the magnitude of errors in levelling, the Author has prepared a diagram (Fig. 1), in which the successive changes of gradient from 'level' to upwards of 1 in 10 are represented by a curve of gradients, and the corresponding magnitudes of 'differences' by a curve plotted from the observed results of the Swiss Levelling. Such a diagram may be usefully applied to obtain the probable difference of results on ground of variable character. Thus, for

FIG. 1.

instance, if the gradients on a given line range from 1 in 100 to 1 in 10 (represented by the portion B C D E of the upper curve) in about equal proportions, the probable difference of results might be taken pretty correctly by inspection from the corresponding portion, *b c d e*, of the lower curve. But the inspection method will not be quite accurate, and it may be convenient to form accurately a table which would be applicable to most circumstances of levelling.

TABLE II.—AVERAGE DIFFERENCES in a SINGLE MILE DUE to GROUND of VARIABLE INCLINATION—COMPUTED from the RESULTS of TABLE A. by the FORMULA

Prob. Diff. = $\sqrt{\text{Mean of squares of diffs. due to the several slopes of ground.}}$

Character of the Ground.	Average Difference in a single Mile.		
	Great Britain.	India.	Switzerland.
(1) "Level ground" in very favourable circumstances of weather. Example.—A broad level valley or margin of a lake	Foot. ·0230	Foot. ·0142	Foot. ·0125
(2) "Gently undulating ground," where the gradients rarely, if at all, exceed 1 in 100. Example.—Most of the roads in India . . .	·0238	·0168	·0148
(3) "Steeply undulating ground," where the gradients rarely, if at all, exceed 1 in 20. Example.—Most of the roads in Great Britain	·0316	·0189	·0166
(4) "Mountain roads," where the gradients rarely, if at all, exceed 1 in 10. Example.—The mountain roads in Switzerland	·0416	·0254	·0224
(5) "Very steep and rough ground," where the gradients frequently exceed 1 in 10, and the levelling is carried over mere road tracks or paths	..	..	·0416

NOTE.—Quantities in black partly estimated as per Table I.

Having thus determined the 'average differences' from the results of standard levelling in different countries, it remains to form a table of the probable errors due to those differences. The proper method of doing this will be found in the "Theory of Errors of Observation," art. 60; and from the formulæ of that article it can be shown that if there be two independent measures of the same quantity the probable error of each measure is $0.477 \times$ difference of the measures, and the probable error of the mean of the two measures is $0.337 \times$ difference of the measures. With these co-efficients it is easy to frame a table of probable errors applicable to the cases of lines which have been levelled once or twice.

TABLE III.—PROBABLE ERRORS of the RESULTS of SINGLE or DUPLICATE LEVELLING ON GROUND of DIFFERENT CHARACTERS, FORMED from the RESULTS of the SWISS LEVELLING.

Character of the Ground. (For Definitions, see Table II.)	Average Difference in a single Mile of Two Observers (see Table B).	Probable Error in a single Mile of each Observer. Multiplier = .477.	Probable Error in a single Mile of the Mean of Two Observers. Multiplier = .337.
	Foot.	Foot.	Foot.
Level ground	.0125	.00596	.00421
Gently undulating ground	.0148	.00706	.00499
Steeply undulating ground	.0166	.00792	.00559
Mountain roads	.0224	.01068	.00755
Very steep and rough ground . . .	.0416	.01984	.01402

C. THE TREATMENT OF PROBABLE ERRORS IN THE CASE OF A COMPLEX SYSTEM OF LEVELLED LINES.

This is illustrated by working out in detail an example showing the application of the theory of probable errors in adjusting the levels of a number of points connected by lines of levelling on the 'polygon' system, the closing error of the chief polygon being 0.286 foot.

Let ABCDEFGA be the polygon system (Fig. 2), of which all the lines have been separately levelled.¹ The lines BC, CD, CF, and FG have been levelled twice each, and the remaining lines once. The length in miles, and the character of the ground, are written against each line.

FIG. 2.

The first thing to be done is to form a table of the probable errors of the several lines from Table III. If a line has been levelled once only, the 'probable error in a single mile' is to be taken from the third column of Table III.; but if twice, it is to be taken from the fourth column of the table, and applied to the mean of the observed results.

¹ The polygon system of the example, with the lengths of the lines and the observed levels of the several points, is taken from an actual case in the "Nivellement de la Suisse." The description of the ground is not given in the "Nivellement," and is to some extent hypothetical.

TABLE IV.—PROBABLE ERRORS OF the LINES A B, B C, C D, &c.

Section.	Observed Difference of Level of the two ends of the Section.	Character of the Ground.	Number of Times Levelled.	Length of Section.	Probable Error in a single Mile, from Table C.	Probable Error of the Section = Probable Error in a single Mile $\times \sqrt{\text{Length in Miles.}}$
A B	Feet. 193·458	Level . .	Once .	Miles. 48·1	Foot. ·00596	Foot. ·041
B C	15·860 ¹	Level . .	Twice	19·2	·00421	·018
C D	699·887 ¹	{ Steeply undulating . }	Twice	8·3	·00559	·016
D E	-1322·470	{ Steeply undulating . }	Once .	48·3	·00792	·055
E F	955·880	{ Gently undulating . }	Once .	96·5	·00706	·069
F G	155·125 ¹	{ Steeply undulating . }	Twice	19·5	·00559	·025
G A	-697·454	{ Gently undulating . }	Once .	47·2	·00706	·048
B G	503·961	{ Very rough and steep }	Once .	26·8	·01984	·103
C F	332·935 ¹	{ Mountain road . . }	Twice	24·9	·00755	·038

With these probable errors the equations for the determination of the levels of the points A, B, C, D, &c., can be formed, in the most probable manner, in accordance with the method of 'least squares.'² The details of the work are given in the Appendix (p. 187). The final results, together with the effect of the corrections, are shown in Table V. The heights are measured from a datum-level 500 feet below the point A.

TABLE V.—FINAL RESULTS and EFFECT of the CORRECTIONS.

		Height above Datum.		Difference.
		As obtained by levelling round from A through B, C, D, &c., to A.	As corrected by Theory.	
Level of point A		Feet. 500·000	Feet. 500·000	Foot. ·000
Do. do. B		693·458	693·464	+ ·006
Do. do. C		709·318	709·325	+ ·007
Do. do. D		1,409·205	1,409·202	- ·003
Do. do. E		86·735	86·618	- ·117
Do. do. F		1,042·615	1,042·319	- ·296
Do. do. G		1,197·740	1,197·445	- ·295
Do. do. A'		500·286	500·000	- ·286

¹ Mean results.² Vide "Theory of Errors of Observation," art. 95.

Thus the effect of the corrections is to raise the points B and C a little, and to lower E, F, G, and A' considerably. This seems consistent with the nature of the ground, which from A to D is favourable, and the corrections are small. From D through E and F to G the distance is long and the ground rather steep, and it is accordingly on these lengths that the corrections are heavy. At the same time the cross lines BG and CF, though short, are steep, and would well allow of the heavy corrections required by the corrected levels of G and F. The final correction to A' is of course the closing error of the polygon, as shown by the observed levels.

D. REMARKS ON THE INSTRUMENTS AND METHODS OF WORKING USED IN ACCURATE LEVELLING.

For the standard levelling in Great Britain, India, and Switzerland, Y levels were used. This construction offers facilities for the quick and accurate determination of the collimation error and the level error. Its chief disadvantage is that, from want of rigidity in the mounting, it sometimes happens that the internal tube (which carries the object glass) does not slide truly in the external tube, so that the optical axis does not keep constant in position when focussing the telescope. This defect is remedied in part by Mr. Gravatt's construction, which, however, introduces inconvenience in respect of the collimation adjustment. In the instruments used for the Swiss levelling the spirit-level is movable, and is clipped upon the telescope tube—a construction which has no apparent advantage. In the English and in the Indian instruments the spirit-level is always attached to the telescope. In some levels recently constructed for foreign governments by Messrs. Troughton and Simms, the spirit-level is attached to the frame which carries the telescope, instead of to the telescope itself; by this arrangement the level is more firmly secured and less liable to accident.

Table VI. contains the essential information relating to the instruments and staves used in the initial levelling of different countries.

In the Swiss levelling the distances were always measured by taking the reading on the staff of two fixed wires in the diaphragm of the telescope, and corrections were afterwards applied with much labour in respect of all the instrumental errors due to the difference of distances. In Great Britain and in India the distances were always chained equal.

TABLE VI.—INSTRUMENTS and STAVES used in INITIAL LEVELLING.¹

		Great Britain.	India.	Switzerland.	France.
Spirit-level	{ Seconds of arc in tilt of telescope corresponding to $\frac{1}{10}$ th of an inch run of the bubble }	5"	3 $\frac{1}{2}$ "	3"	3" to 7"
Telescope	{ Focal length Diameter of object glass Magnifying power . . }	24 ins. 2 $\frac{1}{8}$ " 50	21 ins. 2 " 42	16 ins. 1 $\frac{1}{4}$ " 42	19 ins. 1 $\frac{1}{2}$ " 36
Staves . . .	{ Length Graduated to Read to }	10 ft. $\frac{1}{100}$ " $\frac{1}{1000}$ "	16 ft. $\frac{1}{100}$ " $\frac{1}{1000}$ "	9·84 ft. $\frac{3}{100}$ " $\frac{3}{1000}$ "	.. $\frac{6}{100}$ ft. $\frac{3}{1000}$ "
Maximum distance of staves from instrument	{	5 chs.	10 chs.	5 chs.	6 $\frac{1}{2}$ chs.

In India Colonel Walker discovered that the instruments had a tendency to settle on their axis between the reading of the back and forward staves. This would tend to produce a cumulative error, and he has therefore established the rule of alternating the order of observation at successive stations of the instrument, by reading the back staff first at one station and the forward staff first at the next. The special necessity for this appears to be mainly due to the practice of unshipping the instruments at every station for removal to the next station. To counteract the cumulative error originally noticed by Dr. Whewell, which is introduced by levelling a line continuously in the same direction, all the lines of the initial levelling in Great Britain were levelled both ways from end to end. In India the same end is gained by splitting up the long lines into short sections of 4 or 5 miles in length, and levelling alternate sections in opposite directions. But no precautions appear to have been taken in the Swiss and French levelling to guard against these accumulated errors.

Errors may easily be caused by an inaccurate method of placing benchmarks. Thus it has been usual in England to cut benchmarks deeply on the vertical faces of walls, and from the inconvenience of ascertaining the exact level of the centre of the mark, it is easy to make at once an error as large as would be generated

¹ The metrical values of the foreign measures have been reduced to English units for the sake of comparison.

in a considerable length of levelling. In the Swiss levelling the benchmarks were always defined by bronze bolts let into a horizontal face.

By due attention to all the ascertained causes of error a high degree of accuracy may be secured, and this Paper cannot better close than with a reference to the astonishing results obtained in India under the able management of Colonel Walker. From the Report on the Spirit Levelling Operations in India dated 1866 the following information is extracted :—

Section.	Length.	Maximum Divergence of two Observers.	Terminal Divergence.
	Miles.	Foot.	Foot.
Calcutta to Tilliagarhi	242	·20	·15
Tilliagarhi to Patka Gerouli. . .	346	·40	·38
Agra to Patka Gerouli	342½	·15	·06

APPENDIX.

DETAILS of WORKING-OUT of the EXAMPLE on page 182 by the METHOD of
'LEAST SQUARES.'

Let a, b, c, d , &c., be the corrected levels of the points A, B, C, D, &c. Then since the differences of level of the successive points have been determined approximately, as given below, the following equations are obtained:—

	Feet.		Foot.
$b - a =$	193·458 with probable error =		·041
$c - b =$	15·860 [†]	" "	= ·018
$d - c =$	699·887 [†]	" "	= ·016
$e - d =$	- 1322·470	" "	= ·055
$f - e =$	955·880	" "	= ·069
$g - f =$	155·125 [†]	" "	= ·025
$a - g =$	- 697·454	" "	= ·048
$g - b =$	503·961	" "	= ·103
$f - c =$	332·935 [†]	" "	= ·038

From the foregoing equations it follows that—

$\frac{b}{\cdot041} - \frac{a}{\cdot041} =$	$\frac{193\cdot458}{\cdot041}$	with probable error = 1
$\frac{c}{\cdot018} - \frac{b}{\cdot018} =$	$\frac{15\cdot860}{\cdot018}$	" " = 1
$\frac{d}{\cdot016} - \frac{c}{\cdot016} =$	$\frac{699\cdot887}{\cdot016}$	" " = 1
$\frac{e}{\cdot055} - \frac{d}{\cdot055} =$	$-\frac{1322\cdot470}{\cdot055}$	" " = 1
$\frac{f}{\cdot069} - \frac{e}{\cdot069} =$	$\frac{955\cdot880}{\cdot069}$	" " = 1
$\frac{g}{\cdot025} - \frac{f}{\cdot025} =$	$\frac{155\cdot125}{\cdot025}$	" " = 1
$\frac{a}{\cdot048} - \frac{g}{\cdot048} =$	$-\frac{697\cdot454}{\cdot048}$	" " = 1
$\frac{g}{\cdot103} - \frac{b}{\cdot103} =$	$\frac{503\cdot961}{\cdot103}$	" " = 1
$\frac{f}{\cdot038} - \frac{c}{\cdot038} =$	$\frac{332\cdot935}{\cdot038}$	" " = 1

From these equations are formed the seven equations which will give the most

[†] Quantities marked thus are the mean results of the duplicate levelling on those lines.

probable values of a , b , c , &c., according to the method of least squares, as follow :—

$$\begin{aligned} \text{For } a— & + \frac{a}{(\cdot 041)^2} - \frac{b}{(\cdot 041)^2} = - \frac{193 \cdot 458}{(\cdot 041)^2} \\ & + \frac{a}{(\cdot 048)^2} - \frac{g}{(\cdot 048)^2} = - \frac{697 \cdot 454}{(\cdot 048)^2} \\ & \hline a \left\{ \frac{1}{(\cdot 041)^2} + \frac{1}{(\cdot 048)^2} \right\} - \frac{b}{(\cdot 041)^2} - \frac{g}{(\cdot 048)^2} = - \frac{193 \cdot 458}{(\cdot 041)^2} - \frac{697 \cdot 454}{(\cdot 048)^2} \\ & a \times 1028 \cdot 9117 - b \times 594 \cdot 8838 - g \times 434 \cdot 0279 = - 417,799 \cdot 5 \end{aligned}$$

$$\begin{aligned} \text{For } b— & + \frac{b}{(\cdot 041)^2} - \frac{a}{(\cdot 041)^2} = \frac{193 \cdot 458}{(\cdot 041)^2} \\ & + \frac{b}{(\cdot 018)^2} - \frac{c}{(\cdot 018)^2} = - \frac{15 \cdot 860}{(\cdot 018)^2} \\ & + \frac{b}{(\cdot 103)^2} - \frac{g}{(\cdot 103)^2} = - \frac{503 \cdot 961}{(\cdot 103)^2} \\ & \hline b \left\{ \frac{1}{(\cdot 041)^2} + \frac{1}{(\cdot 018)^2} + \frac{1}{(\cdot 103)^2} \right\} - \frac{a}{(\cdot 041)^2} - \frac{c}{(\cdot 018)^2} - \frac{g}{(\cdot 103)^2} \\ & = \frac{193 \cdot 458}{(\cdot 041)^2} - \frac{15 \cdot 860}{(\cdot 018)^2} - \frac{503 \cdot 961}{(\cdot 103)^2} \\ & b \times 3775 \cdot 5634 - a \times 594 \cdot 8838 - c \times 3086 \cdot 420 - g \times 94 \cdot 2596 = 18,631 \cdot 20 \end{aligned}$$

$$\begin{aligned} \text{For } c— & + \frac{c}{(\cdot 018)^2} - \frac{b}{(\cdot 018)^2} = \frac{15 \cdot 860}{(\cdot 018)^2} \\ & + \frac{c}{(\cdot 016)^2} - \frac{d}{(\cdot 016)^2} = - \frac{699 \cdot 887}{(\cdot 016)^2} \\ & + \frac{c}{(\cdot 038)^2} - \frac{f}{(\cdot 038)^2} = - \frac{332 \cdot 935}{(\cdot 038)^2} \\ & \hline c \left\{ \frac{1}{(\cdot 018)^2} + \frac{1}{(\cdot 016)^2} + \frac{1}{(\cdot 038)^2} \right\} - \frac{b}{(\cdot 018)^2} - \frac{d}{(\cdot 016)^2} - \frac{f}{(\cdot 038)^2} \\ & = \frac{15 \cdot 860}{(\cdot 018)^2} - \frac{699 \cdot 887}{(\cdot 016)^2} - \frac{332 \cdot 935}{(\cdot 038)^2} \\ & c \times 7685 \cdot 1908 - b \times 3086 \cdot 420 - d \times 3906 \cdot 250 - f \times 692 \cdot 5208 \\ & = - 2,915,546 \cdot 78 \end{aligned}$$

$$\begin{aligned} \text{For } d— & + \frac{d}{(\cdot 016)^2} - \frac{c}{(\cdot 016)^2} = \frac{699 \cdot 887}{(\cdot 016)^2} \\ & + \frac{d}{(\cdot 055)^2} - \frac{e}{(\cdot 055)^2} = \frac{1322 \cdot 470}{(\cdot 055)^2} \\ & \hline \end{aligned}$$

$$d \left\{ \frac{1}{(\cdot 016)^2} + \frac{1}{(\cdot 055)^2} \right\} - \frac{c}{(\cdot 016)^2} - \frac{e}{(\cdot 055)^2} = \frac{699 \cdot 887}{(\cdot 016)^2} + \frac{1322 \cdot 470}{(\cdot 055)^2}$$

$$d \times 4236 \cdot 8285 - c \times 3906 \cdot 250 - e \times 330 \cdot 5785 = 3,171,113 \cdot 1$$

$$\text{For } e - \quad + \frac{e}{(\cdot 055)^2} - \frac{d}{(\cdot 055)^2} = - \frac{1,322 \cdot 470}{(\cdot 055)^2}$$

$$+ \frac{e}{(\cdot 069)^2} - \frac{f}{(\cdot 069)^2} = - \frac{955 \cdot 880}{(\cdot 069)^2}$$

$$e \left\{ \frac{1}{(\cdot 055)^2} + \frac{1}{(\cdot 069)^2} \right\} - \frac{d}{(\cdot 055)^2} - \frac{f}{(\cdot 069)^2} = - \frac{1322 \cdot 470}{(\cdot 055)^2} - \frac{955 \cdot 880}{(\cdot 069)^2}$$

$$e \times 540 \cdot 6184 - d \times 330 \cdot 5785 - f \times 210 \cdot 0399 = - 637,953 \cdot 1$$

$$\text{For } f - \quad + \frac{f}{(\cdot 069)^2} - \frac{e}{(\cdot 069)^2} = \frac{955 \cdot 880}{(\cdot 069)^2}$$

$$+ \frac{f}{(\cdot 025)^2} - \frac{g}{(\cdot 025)^2} = - \frac{155 \cdot 125}{(\cdot 025)^2}$$

$$+ \frac{f}{(\cdot 038)^2} - \frac{c}{(\cdot 038)^2} = \frac{332 \cdot 935}{(\cdot 038)^2}$$

$$f \left\{ \frac{1}{(\cdot 069)^2} + \frac{1}{(\cdot 025)^2} + \frac{1}{(\cdot 038)^2} \right\} - \frac{e}{(\cdot 069)^2} - \frac{g}{(\cdot 025)^2} - \frac{c}{(\cdot 038)^2} \\ = \frac{955 \cdot 880}{(\cdot 069)^2} - \frac{155 \cdot 125}{(\cdot 025)^2} + \frac{332 \cdot 935}{(\cdot 038)^2}$$

$$f \times 2502 \cdot 5307 - e \times 210 \cdot 0399 - g \times 1600 \cdot 0 - c \times 692 \cdot 5208 = 183,137 \cdot 4$$

$$\text{For } g - \quad + \frac{g}{(\cdot 025)^2} - \frac{f}{(\cdot 025)^2} = \frac{155 \cdot 125}{(\cdot 025)^2}$$

$$+ \frac{g}{(\cdot 048)^2} - \frac{a}{(\cdot 048)^2} = \frac{697 \cdot 454}{(\cdot 048)^2}$$

$$+ \frac{g}{(\cdot 103)^2} - \frac{b}{(\cdot 103)^2} = \frac{503 \cdot 961}{(\cdot 103)^2}$$

$$g \left\{ \frac{1}{(\cdot 025)^2} + \frac{1}{(\cdot 048)^2} + \frac{1}{(\cdot 103)^2} \right\} - \frac{f}{(\cdot 025)^2} - \frac{a}{(\cdot 048)^2} - \frac{b}{(\cdot 103)^2} \\ = \frac{155 \cdot 125}{(\cdot 048)^2} + \frac{697 \cdot 454}{(\cdot 048)^2} + \frac{503 \cdot 961}{(\cdot 103)^2}$$

$$g \times 2128 \cdot 2875 - f \times 1600 \cdot 0 - a \times 434 \cdot 0279 - b \times 94 \cdot 2596 = 598,417 \cdot 68$$

Collecting the equations we have for the determination of a, b, c, d, e, f, g :—

$$\begin{aligned}
 (1) \quad & a \times 1028 \cdot 9117 - b \times 594 \cdot 8838 - g \times 434 \cdot 0279 = -417,799 \cdot 5 \\
 (2) \quad & b \times 3775 \cdot 5634 - a \times 594 \cdot 8838 - c \times 3086 \cdot 420 - g \times 94 \cdot 2596 = 18,631 \cdot 20 \\
 (3) \quad & c \times 7685 \cdot 1908 - b \times 3086 \cdot 420 - d \times 3906 \cdot 250 - f \times 692 \cdot 5208 = -2,915,546 \cdot 78 \\
 (4) \quad & d \times 4236 \cdot 8285 - c \times 3906 \cdot 250 - e \times 330 \cdot 5785 = 3,171,113 \cdot 1 \\
 (5) \quad & e \times 540 \cdot 6184 - d \times 330 \cdot 5785 - f \times 210 \cdot 0399 = -637,953 \cdot 1 \\
 (6) \quad & f \times 2502 \cdot 5607 - e \times 210 \cdot 0399 - g \times 1600 \cdot 00 - c \times 692 \cdot 5208 = 183,137 \cdot 4 \\
 (7) \quad & g \times 2128 \cdot 2875 - f \times 1600 \cdot 000 - a \times 434 \cdot 0279 - b \times 94 \cdot 2596 = 598,417 \cdot 68
 \end{aligned}$$

Since, however, it is seen that the last equation would result from the addition of the first six equations, it is not an independent equation, and must be excluded. There are then six equations among seven unknown quantities, and the values of a, b, c , &c., are indeterminate; this ought evidently to be the case, since in the formation of the equations no account was taken of absolute levels, but only of differences of level. Adopting, however, the level of the point A as the datum-level, and putting $a = 0$. Then the other six quantities, b, c, d, e, f, g , may be determined as follows :—

$$\begin{aligned}
 \text{From (1)} \quad & g = \frac{417,799 \cdot 5}{434 \cdot 0279} - b \times \frac{594 \cdot 8838}{434 \cdot 0279}; \\
 \therefore g \times 94 \cdot 2596 &= 90,735 \cdot 22 - b \times 129 \cdot 1934 \\
 g \times 1600 \cdot 00 &= 1,540,176 \cdot 0 - b \times 2192 \cdot 980
 \end{aligned}$$

Therefore equations 2, 3, 4, 5, 6, become

$$\begin{aligned}
 (a) \quad & b \times 3904 \cdot 7568 - c \times 3086 \cdot 420 = 109,366 \cdot 42 \\
 (b) \quad & c \times 7685 \cdot 1908 - b \times 3086 \cdot 420 - d \times 3906 \cdot 250 - f \times 692 \cdot 5208 = -2,915,546 \cdot 78 \\
 (c) \quad & d \times 4236 \cdot 8285 - c \times 3906 \cdot 250 - e \times 330 \cdot 5785 = 3,171,113 \cdot 1 \\
 (d) \quad & e \times 540 \cdot 6184 - d \times 330 \cdot 5785 - f \times 210 \cdot 0399 = -637,953 \cdot 1 \\
 (e) \quad & f \times 2502 \cdot 5607 - e \times 210 \cdot 0399 + b \times 2192 \cdot 980 - c \times 692 \cdot 5208 = 1,723,313 \cdot 4
 \end{aligned}$$

$$\text{From (a)} \quad c = -\frac{109,366 \cdot 42}{3086 \cdot 420} + b \times \frac{3904 \cdot 7568}{3086 \cdot 420}$$

$$\begin{aligned}
 \therefore c \times 7685 \cdot 1908 &= -272,322 \cdot 6 + b \times 9722 \cdot 852 \\
 \text{and } c \times 3906 \cdot 250 &= -138,416 \cdot 8 + b \times 4941 \cdot 958
 \end{aligned}$$

Also it is noticed that by adding together all the equations $\alpha, \beta, \gamma, \delta, \epsilon$, the co-efficients of c, d and e vanish, and the resulting equation is as follows :—

$$b \times 3011 \cdot 3168 + f \times 1600 \cdot 00 = 1,450,293 \cdot 04$$

$$\text{This gives} \quad f = \frac{1,450,293 \cdot 04}{1600} - b \times \frac{3,011 \cdot 3168}{1600}$$

$$\begin{aligned}
 \therefore f \times 692 \cdot 5208 &= 627,723 \cdot 6 - b \times 1303 \cdot 374 \\
 f \times 210 \cdot 0399 &= 190,387 \cdot 1 - b \times 395 \cdot 3104
 \end{aligned}$$

Therefore, substituting the values of c and f in equation β ,

$$b \times 7939 \cdot 806 - d \times 3906 \cdot 250 = -2,015,500 \cdot 58$$

This gives

$$d = \frac{2,015,500 \cdot 58}{3906 \cdot 250} + b \times \frac{7939 \cdot 806}{3906 \cdot 250}$$

$$\begin{aligned} \therefore d \times 4236.8285 &= 2,186,069.0 + b \times 8611.738 \\ d \times 330.5785 &= 170,568.0 + b \times 671.9308 \end{aligned}$$

Therefore, substituting in equation γ the values of c and d already obtained,

$$b \times 3669.780 - e \times 330.5785 = 846.627.3$$

This gives

$$e = -\frac{846,627.3}{330.5785} + b \times \frac{3669.780}{330.5785}$$

$$\therefore e \times 540.6184 = -1,384,549.0 + b \times 6001.450$$

Substituting the values so found for e , d and f in equation 8,

$$b \times 5724.8296 = 1,107,551.0$$

$$h = 193.4644$$

and $b - a = 193.4644$ I.

And since

$$c = -\frac{109,366.42}{3086.420} + b \times \frac{3904.7568}{3086.420}$$

$$\therefore c = 209.3251$$

and $c - b = 15.8607$ II.

And since

$$d = \frac{2,015,500 \cdot 58}{3906 \cdot 250} + b \times \frac{7939 \cdot 806}{3906 \cdot 250}$$

$$\therefore d = 909.2024$$

and $d - c = 699.8773$ III.

And since

$$e = -\frac{846,627 \cdot 3}{330 \cdot 5785} + b \times \frac{3669 \cdot 780}{330 \cdot 5785}$$

$$\therefore e = -413.382$$

and $e - d = -1322.5844$ IV.

And since

$$f = \frac{1,450,293.04}{1600} - b \times \frac{3011.3168}{1600}$$

$$\therefore f = 542.319$$

and $f - e = 955.701$ V.

And since

$$g = \frac{417,799.5}{434.0279} - b \times \frac{594.8838}{434.0279}$$

$$\therefore g = 697.445$$

and $g - f = 155.126$ VI.

also $\alpha - q = -697.445$ VII.

$$q - b = 503 \cdot 981 \quad . \quad . \quad . \quad . \quad . \quad . \quad , \text{ VIII.}$$

$$f - c = 332.994 \quad . \quad . \quad . \quad . \quad . \quad . \quad IX.$$

THE FOLLOWING PUBLISHED WORKS RELATING to the SUBJECT of the PAPER
HAVE BEEN CONSULTED.

- (1) Ordnance Survey. "Abstracts of the Principal Lines of Spirit Levelling in England and Wales." By Col. Sir HENRY JAMES, R.E. 1861.
- (2) Ordnance Survey. "Abstracts of the Principal Lines of Spirit Levelling in Scotland." By Col. Sir HENRY JAMES, R.E. 1861.
- (3) "Tables of Heights in Sind, the Punjaub, North-West Provinces, and Central India." Calcutta, 1863.
- (4) "Tables of Heights in North-West Provinces and Bengal." Roorkee, 1866.
- (5) "On the Methods of determining Heights in the Trigonometrical Survey of India." By J. T. WALKER, Lieut.-Col. R.E.—Royal Astronomical Society. Vol. 33.
- (6) "Nivellement de précision de la Suisse," under the direction of Messrs. Hirsch and Plantamour. Five Nos., 1867 to 1874.
- (7) "Nivellement Général de la France." Published by the Minister of Agriculture, Commerce, and Public Works.
- (8) "Account of a Level Line measured from the Bristol Channel to the English Channel during the year 1837-8 by Mr. Bunt." Drawn up by the Rev. W. WHEWELL. Eighth Report of the British Association for the Advancement of Science. 1839.
- (9) "A Treatise on the Principal Mathematical Instruments employed in Surveying, Levelling, and Astronomy." By FREDERICK W. SIMMS. 1836.
- (10) "On the Algebraical and Numerical Theory of Errors of Observations." By Sir G. B. AIRY. Second Edition, 1875.

[Sir G. B. AIRY