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PROBLEMS IN THE EXPERIMENTAL PEDAGOGY OF GEOMETRY.

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Geometry has always been considered a subject of high educational value, and therefore it has held its place in the curriculum without much question. Of late years it has, however, in common with other mathematical subjects, had to face the challenge to prove its right to the prominent place it has heretofore held. This challenge has for the most part come from the psychologists, who demand experimental proof for its claims. Another challenge, as to whether the traditional textbook method of teaching the subject is satisfactory, has recently come, more strongly, perhaps, than ever before, from some of the mathematics teachers themselves.

The different methods in vogue today of teaching any subject have not been built on speculative theory only, but are rather the cumulative results of the experiences of teachers in the past. These results have to a large extent come down to us by tradition, and therefore have been subject to the modifications of the changing conditions in the history of the world. Experiments have undoubtedly been performed here and there, and the results have been embodied in the formation of those methods, but what psychologists and others are now asking is that experiments and observations be made under careful conditions and in a systematic and scientific manner. The results can then be carefully studied, properly interpreted and recorded for future use.

The results which we should aim to obtain in the teaching of geometry are, for the ordinary student, as follows:

1. A knowledge of that part of the content which is fundamental and of most importance.
2. The ability to think logically, and a knowledge of the methods and principles of geometric proof and of the nature of proof in general.

These involve: (1) The ability to tell whether a given argument is logically sound and the habit of logically sound thinking. The student must learn to distinguish between necessary and contingent conclusions, and when the judgment has come to a decision as to the most probable hypothesis on which a contingent conclusion is based he should understand that the reasoning in reaching the conclusion is the same as that in reaching a necessary conclusion. (2) The ability to construct a chain of reasoning from given material. In geometry the student learns that he has certain tools by means of which he is to prove a proposition. He also learns how to examine these tools to see what ones are applicable, and then learns the general method of applying them. He learns to analyze, arrange and classify his material. He should then acquire the general methods which underlie all proofs so that when he comes to proofs in other subjects he has a clear knowledge of what constitutes a proof and of its general technique, and can apply it to whatever he may have to do. (3) The ability to think for himself. He must learn to look at a thing from all sides, to size it up in its true relations (which is good judgment). He must force every argument to its conclusion, and acquire the habit of thinking through and through a subject.

3. Ability to express himself clearly. The student must be taught the value and acquire the habit of clear expression.

4. Concentration. The ability to shut out everything else and fix the mind on the subject at hand is a prerequisite to clear thinking, and must be insisted upon in geometry.

5. Constructive imagination. The ability to construct new images out of old elements. To be able to visualize and form the habit of doing so is of great importance.

6. Rational memory. The habit of associating things so as to retain them.

7. Liberal-mindedness. The habit of always looking for new truth and leaving the mind open for its entrance. The student of geometry should be a discoverer, and get into the habit of looking for new light, as well as of looking at things from a broad standpoint.

8. Esthetic ability. Geometry possesses not only beauty of form, but of idea and of method. It should cultivate a habit of neatness and accuracy.

9. A spirit of honesty. The study of geometry soon shows the futility of dishonest thinking. It shows the reward of honest effort.

10. Will power. A well-governed will is essential to usefulness, and the study of geometry makes the student exercise it.

These results are not always obtained, but the aim should be to develop them as fully as possible. It is somewhat of a question as to how they are considered by teachers. Are all but content the incidentals—the mere by-products in the study of the subject, or are they the main object? It is to be feared that they are too often considered incidentals, and that altogether too little attention is given to them. In my estimation they are the chief aim, and the content the incidental, or the occasion for acquiring them. The mere content of most subjects is of little value as compared with the other mental and moral qualities, and the teacher of geometry should see to it that they are obtained from its study.

Not only should these powers be sought for, but they should also be sought for in a general way, and not as related to the subject at hand only. Concentration, for instance, should be insisted upon in connection with geometry, but the teacher should at the same time not limit its value to geometry,¹ but carefully point out that it is an ability of great value in everything. So all these powers, if they are insisted upon in connection with every school subject and taught from this general standpoint, will be applied by the student in connection with everything he undertakes in life.

There seem to be three fairly-well defined lines of experimental work which might be undertaken in connection with this subject. As to the foregoing results, three questions might be asked:

¹It would be well at the very start to impress upon students of geometry the general value of the subject.

1. Are they obtained under any given method?
2. What is the best method of obtaining them?
3. If obtained, is there a transfer from geometry to other subjects?

Experiments in these directions, to be of value, must be performed with the utmost care and under the most certain conditions. So many elements of uncertainty enter that any one person may easily overlook some important factor and thereby spoil the value of the experiment. If the three lines of investigation indicated were carefully carried out and the questions involved settled, there would be no doubt as to the correct aims and proper methods in teaching the subject.

Considering these three questions in order, I would offer under the first, as guides for testing for three of the results and as suggested tests, the following:

A. KNOWLEDGE OF FUNDAMENTALS; SUGGESTIONS FOR TESTS.

1. Give a series of geometrical figures and have the pupils write the names opposite in a limited time.
2. Give a list of geometrical terms and have the pupils in a limited time—
 - (a) Define them informally.
 - (b) Illustrate such as admit of illustration by a drawing.
 - (c) Give additional properties (not included in the definition) of such as admit of them. These additional properties should not be made to include their relation to external things.
3. State the hypotheses of the fundamental propositions and have the student give the conclusions; time limited.
4. State the conclusions of such propositions and have the students give as many sufficient hypotheses as possible; time limited.

Suggested Test.

Name the following, using a single word or phrase. The marks indicate in several instances what is meant:

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

(10)

2. Define informally, illustrate by a drawing such as admit of illustration, and give additional properties (not included in the definition) of such as admit of them :

- (1) Polygon.
- (2) Parallelogram.
- (3) Hypotenuse.
- (4) Circle.
- (5) Regular polygon.
- (6) Apothem.
- (7) Similar polygons.
- (8) Adjacent angles.
- (9) Inscribed angle.
- (10) Perpendicular lines.
- (11) Supplementary adjacent angles.
- (12) Exterior angle of a triangle.
- (13) Equilateral triangle.
- (14) Projection of a line segment on another line.
- (15) Locus.
- (16) Altitude of a trapezoid.
- (17) Base of a trapezoid.
- (18) Area of a surface.
- (19) Circumscribed polygon.
- (20) Converse.
- (21) Secant.

- (22) Chord.
- (23) Diagonal of a pentagon.
- (24) A line segment divided externally.

3. Give conclusions for the following:

- (1) If two parallel lines are cut by a transversal.....
(Three conclusions.)
- (2) If two angles of a triangle are unequal... ..
- (3) The sum of the interior angles of a polygon equals
.....
- (4) If a line is the perpendicular bisector of a sect.....
- (5) If a straight line is perpendicular to a radius where
the radius meets the circumference.....
- (6) In a triangle if a line is drawn cutting the sides and
parallel to the base.....
- (7) An angle formed by two chords intersecting within
the circumference is measured by.....
- (8) In any triangle the square on the side opposite an
acute angle is equal to.....
- (9) Two circumferences have the same ratio as.
- (10) The areas of similar polygons are as.....
- (11) The area of a regular polygon is equal to.....
- (12) If two triangles have an angle of one equal to an
angle of the other, their areas... ..
- (13) The area of a circle is equal to.....
- (14) The locus of the vertex of a right angle opposite a
given fixed hypotenuse is.....
- (15) If a perpendicular is drawn from the vertex to the
hypotenuse of a right triangle.....

4. State the sufficient hypotheses in each of the following:

- (1) Two angles are equal if.....
- (2) Two straight lines are parallel if.....
- (3) Two straight lines are perpendicular if.....
- (4) Two sects (segments of straight lines) are unequal
if.....
- (5) Two triangles are congruent if.....
- (6) A quadrilateral is a parallelogram if.....
- (7) A sect is a mean proportional between two other
sects if.....

- (8) Two triangles are similar if.....
- (9) A polygon is regular if.....
- (10) Two parallelograms are congruent if.....

B. REASONING, GEOMETRICAL AND GENERAL; SUGGESTIONS FOR TESTS.

1. Tests in geometry.

- (a) Give tests on originals (with time limited) at various stages of the study of the subject.
- (b) Have the students give a full description of the method used in analyzing the questions and the steps leading to their conclusions.

Observe from (a) the degree of ability to reason correctly, and from (b) the students' knowledge of the methods of logical attack and the ability to analyze their own thought processes. As the study of geometry proceeds there should be improvement if the pupil is grasping the subject.

2. Tests in other subjects.

- (a) Give tests in some subject, other than geometry (with time limited), involving reasoning of like logical difficulty and concerning data of equal familiarity as in geometry, but such tests should not be to reproduce proofs they may have already had.
- (b) As in 1 (b).

Observations similar to those made in 1 should be recorded. The test consists in observing whether the improvement in 2 (a) and (b) bears any ratio to the improvement in 1 (a) and (b).

Directions.—Test 2 should be given just before beginning the study of geometry, and both 1 and 2 at regular intervals throughout the course, and, if possible, other tests in 2 should be given at similar intervals after finishing the study of geometry. Test 2 should also be given, when possible, to pupils in the same year of high school who are not taking geometry, and the results compared with the results of those who are taking it. The differences should be carefully noted.

Suggested Tests.

1. Test in geometry.

- (a) Draw and letter the figures and state the hypothesis and conclusion in terms of your notation in each case:
- (1) The median from the vertex to the base of an isosceles triangle bisects the vertical angle.
 - (2) If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.
 - (3) The bisector of one of two vertical angles bisects the other.
 - (4) If the perpendiculars from two vertices upon the opposite sides in a triangle are equal, the triangle is isosceles.
 - (5) The line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the bases.
- (b) Demonstrate the following; formulate the general statement without the use of reference letters:

Given $\angle 2 = \angle 3$, $\angle 4 = \angle 5$, and $KA \perp CH$, prove $AB \parallel CD$.

(2)

Given AE perpendicular to BC; given BD perpendicular to AC, and angle BAC equal to angle ABC, show that angle BAE is equal to angle ABD.

(3)

Given AB equal to BC equal to CA, shew that AD is less than CD.

(4)

Given the angles 1, 2, 3 equal, and AD , BE , CF equal, show that DE is equal to DF .

Given E , F , G , H , the midpoints of the sides of $ABCD$, show that FE is parallel to GH .

Given $ABCD$, a square, and AN , AE , BF , CG , DH , EI , FK , GL , each equal to one-half of AD , prove $EFGHIJKL$ is a regular octagon.

(7)

ABCD is a parallelogram and E and F are the midpoints of AB and CD. Prove that $CG = GH = HB$.

(8)

DE and DF bisect the angles HBA and KCA. Prove that $\angle EDF = \frac{1}{2} (\angle ABC + \angle BCA)$.

NOTE.—A good test would be to give figures and ask the student to prove all the propositions they can connected with it. For instance, the figure

would involve several propositions.

(c) Demonstrate:

- (1) If two circles are tangent externally at A, the tangents to them from any point on the common tangent at their point of contact are equal.
- (2) If two circles intersect at P and Q, and PA and PB are diameters, then QA and QB are in the same straight line.
- (3) If the bisector of the angle A of an inscribed triangle ABC meets BC in D and the circumference in P, then $AB \times AC = AD \times AP$.
- (4) To draw a circle with a given radius that will pass through a given point and be tangent to a given line.
- (5) A chord 24 inches long is 9 inches from the center of a circle; find the length of the tangents drawn from the extremities of the chord and produced till they meet.

(d) Demonstrate:

- (1) If the diagonals of a trapezoid are equal, the trapezoid is isosceles.
- (2) If perpendiculars be drawn from any point within an equilateral triangle to the three sides, their sum is equal to the altitude of the triangle.
- (3) The area of the ring included between two concentric circles is equal to that of a circle whose radius is one-half of a chord of the outer circle drawn tangent to the inner.
- (4) Prove that if one leg of a right triangle is the diameter of a circle, the tangent at the point where the circumference cuts the hypotenuse bisects the other leg.

- (5) What is the locus of the midpoint of one leg of a right triangle whose hypotenuse is fixed? Prove the correctness of your answer.
2. Tests in other subjects.
- (a) Starting with each of the following propositions in turn as true, state which of the others must then be true, which may then be true, and which cannot then be true.

(NOTE.—It is assumed that there exist things which are solids, things which are not solids, things which are crystals, and things which are not crystals.)

- (1) All crystals are solids.
- (2) Some solids are not crystals.
- (3) Some not crystals are not solids.
- (4) No crystals are not solids.
- (5) Some solids are crystals.
- (6) Some not solids are not crystals.
- (7) All solids are crystals.

For instance:

If 7 is true, 3, 5, 6 must be true; 1, 4 may be true, and 2 cannot be true.

- (b) What conclusions (if any) can be drawn from each pair of the following statements?

- (1) None but residents are members of the club.
- (2) Some members of the club are not officers.
- (3) All members of the club are invited to compete.
- (4) All officers are invited to compete.

- (c) Point out any defects in the reasoning in the following cases:

- (1) It is impossible to maintain that the virtuous alone are happy, and at the same time that selfishness is compatible with happiness, but incompatible with virtue.
- (2) Bullies are always cowards, but not always liars; liars, therefore, are not always cowards.

(NOTE.—Test (a) can be repeated at frequent intervals by changing the order of the sentences and substituting other words for "crystal" and "solids.")

- (3) Most M is P; most S is M; therefore, some S is P.

C. CONCENTRATION; SUGGESTIONS FOR TESTS.

1. Tests with unusual distractions.

- (a) Give (under distractions of some definite kind, such as the ticking of a metronome, the reading aloud of an interesting book, or the conducting of a recitation in the same room) tests (with time limited) to work out some originals that require concentration. Give these from time to time, and in so far as possible use originals of about the same difficulty and familiarity of principle involved, and see if the time required diminishes with the length of time geometry has been studied.
- (b) Same as in (a), using other than geometrical exercises.

2. Tests without unusual distractions.

- (a) Same as in 1, (a).
- (b) Same as in 1, (b).

Directions.—If possible, test 2, (b), should also be given to pupils in the same grade of high school who are not studying geometry, and the results compared with the results of those who are taking it. The differences should be carefully noted.

Suggested Tests.

The foregoing tests would serve very well for tests in concentration. Series (a), (b), (c) could be given at different times.

Under the second question as to what is the best method of obtaining the desired results it may be said that, besides many special devices, there has come to be three fairly well recognized general methods of teaching the subject. They are: the textbook method, the suggestive method, and the syllabus method. To determine which of these is best, or the relative merits of special devices, one should by well-selected experiments test the results. As different schools use different methods, the results might be compared by giving the same test to different schools.

To test for transfer is not by any means in all cases an easy

problem, and in some of its phases it would seem almost, if not entirely, impossible. Tests for some phases, viz., reasoning, concentration, etc., have been already suggested, and tests for other phases could be devised.

As has been pointed out, to obtain the results from the study of geometry enumerated above as general, and not peculiar to the subject only, it must be taught with them in view. They must be made objects themselves worthy of acquisition. In view of this, experiments to test the transfer of accuracy from numerical calculation to reasoning in arithmetic do not signify much unless we know from what standpoint and with what object accuracy in connection with numerical work has been taught. It is not sufficient to leave a power or habit to purely accidental acquirement. It must be taught as something in itself worthy of possession. We would hardly expect a boy to be honest if honesty was never mentioned to him except to have it impressed upon him that he should be honest with his teacher. He must be taught the importance of honesty with everyone and everywhere.

Possibly some will say that teaching abilities and habits such as accuracy in this general way in connection with numerical computation one is not teaching it here only, but in connection with everything. While there may be some truth in this, it is also true that if accuracy is taught in this way in connection with every subject in the curriculum, it will go with the pupil all through life and be transferred by him to everything he undertakes. That which is impressed upon one from a few standpoints, or taught as of importance in connection with but one or two subjects, is not likely to be considered of much importance in others, and will not become a habit in all.

Perhaps no subject teaches anything in itself, but simply furnishes the means for bringing out the lesson. As some occasions furnish better opportunities than others for instilling lessons, so some subjects are perhaps better suited than others for getting the mind in the best psychological condition to be impressed by and to possess some of these powers.