ON SOME CASES OF TIDAL MOTION OF ROTATING SHEETS OF WATER

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1. The following investigations arose out of an attempt to obtain an expression for the free tidal motion of a rotating flat sheet of water in the form of a circular sector. This was suggested as a subject for research by Prof. Lamb. No progress, however, was made with the original problem except in the case of the semicircle, and in this case only when the angular velocity of rotation was small compared with the free periods of the relative motion of the water. Even in this restricted case the resulting expression was so complicated as to make interpretation very difficult, if not impossible, and consequently it is not given below.

After a statement of the general conditions (which were first investigated by Lord Kelvin *), and their application to the case of a nearly circular sheet of water, a discussion is given of the limiting forms of forced tides as the period of the disturbing force tends to become infinite. It appears that these limiting forms are, for rotating flat sheets of water of uniform depth, always different from the forms given by the equilibrium theory. The limiting form is calculated for a *rectangular* sheet, and the result may, perhaps, be of interest, as the complete determination of the tides on such a sheet has not yet been obtained.† Some partial solutions are then given, and, finally, the free symmetrical motion is obtained for a particular small circular sea on a sphere, as the next approximation after that which regards the sea as flat. This may, perhaps, give a little help

* "On Gravitational Oscillations of Rotating Water," *Proc. Roy. Soc. Edin.*, Vol. x, p. 92 (1879) [*Phil. Mag.* (5), Vol. x, p. 109 (1880); *Math. and Phys. Papers*, Vol. iv, p. 141].

† Cf. Lord Kelvin, *ibid.*, and Lord Rayleigh, "On the Vibrations of a Rectangular Sheet of Rotating Liquid," *Phil. Mag.* (6), Vol. v, p. 297 (1903).

in forming an estimate as to how far the results obtained for flat sheets of water may be applied to small seas on the earth's surface.

General Equations for Flat Rotating Sheets.

2. Suppose the sheet of water to be rotating with constant angular velocity ω about an axis perpendicular to its plane, and let the depth of the water (as rotating in free relative equilibrium) be uniform and equal to h . Suppose the disturbing forces to have a potential Ω , and let ξ denote the elevation of the free surface at any point. Then for a simple harmonic disturbance, in which the time factor is $e^{i\sigma t}$, we have the equation *

$$\nabla_1^2 \xi + \frac{\sigma^2 - 4\omega^2}{gh} \xi = \nabla_1^2 \bar{\xi}, \quad (1)$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in Cartesian coordinates, and $\bar{\xi} = -\Omega/g$, g being the acceleration due to gravity.

The relative velocity in any direction is given by

$$\frac{g}{\sigma^2 - 4\omega^2} \left(i\sigma \frac{\partial}{\partial s_1} + 2\omega \frac{\partial}{\partial s_2} \right) (\xi - \bar{\xi}), \quad (2)$$

where $\partial/\partial s_1$ denotes differentiation in that direction, and $\partial/\partial s_2$ in the direction which is a right angle in advance.

From this we see that the boundary condition is

$$\left(i\sigma \frac{\partial}{\partial n} + 2\omega \frac{\partial}{\partial s} \right) (\xi - \bar{\xi}) = 0, \quad (3)$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal, and $\partial/\partial s$ along the positive direction of the arc of the boundary.

The above equations do not necessarily hold when $\sigma = 0$, or $\sigma^2 = 4\omega^2$.

If we integrate (3) round any closed boundary line, we obtain

$$i\sigma \int \frac{\partial}{\partial n} (\xi - \bar{\xi}) ds = -2\omega \int \frac{\partial}{\partial s} (\xi - \bar{\xi}) ds. \quad (4)$$

Now ξ is essentially a one-valued function of position, and we assume $\bar{\xi}$ to be so also. It then follows that the integral on the right-hand side of (4)

* See Lamb, *Hydrodynamics*, 3rd ed., p. 303.

always vanishes, and consequently

$$\int \frac{\partial}{\partial n} (\xi - \bar{\xi}) ds = 0, \quad (5)$$

from (4) if $\sigma \neq 0$, the integral being taken round any closed boundary line. Now the "relative circulation" round such a boundary line is given by

$$\frac{g}{\sigma^2 - 4\omega^2} \left[i\sigma \int \frac{\partial}{\partial s} (\xi - \bar{\xi}) ds - 2\omega \int \frac{\partial}{\partial n} (\xi - \bar{\xi}) ds \right],$$

from (2), and by the above each integral in this is seen to vanish. A more general result, however, is the following: In any case of tidal motion under forces which have a single-valued potential, the circulation round any closed boundary line is constant. This easily follows from the theorem that the circulation in a closed chain of particles is constant.

We shall require equation (1) in polar coordinates. Let

$$\kappa^2 = \frac{\sigma^2 - 4\omega^2}{gh}. \quad (6)$$

Then (1) becomes in polars

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) (\xi - \bar{\xi}) + \kappa^2 \xi = 0. \quad (7)$$

For free motion we have $\bar{\xi} = 0$, and then normal solutions of (7) are of the type $(A \cos n\theta + B \sin n\theta) J_n(\kappa r)$, where A , B and n are any constants and $J_n(\kappa r)$ is the Bessel's function of the first kind of order n and argument κr .

For forced motion the forms of $\bar{\xi}$ which are of greatest interest are*

$$\bar{\xi} = C \left(\frac{r}{a} \right)^s e^{i(\sigma t + s\theta)}, \quad (8)$$

$$\text{and} \quad \bar{\xi} = C (\kappa r)^2 e^{i\sigma t}, \quad (9)$$

C being a constant.

Tides on a Nearly Circular Sheet.†

3. Using polar coordinates, let the equation of the boundary be

$$r = a \{ 1 + f(\theta) \}, \quad (10)$$

* See Lamb, *l.c.*, chap. viii, App. ; (9) is taken as corresponding (as far as is necessary) for a plane, to equation (13) of Lamb's Appendix, κ being inserted for convenience. The value of C in (9), however, is of a higher order of small quantities than that in (8).

† The following method is similar to that used by Lord Rayleigh for the solution of an analogous problem, *Theory of Sound*, § 209.

where $f(\theta)$ is a differentiable function of θ , periodic in 2π , and so small that $|f(\theta)|^2$ and $|f'(\theta)|^2$ may be neglected.

Then for the boundary we have

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r} - f'(\theta) \frac{\partial}{a \partial \theta},$$

and
$$\frac{\partial}{\partial s} = \frac{\partial}{r \partial \theta} + f'(\theta) \frac{\partial}{\partial r} = \{1 - f(\theta)\} \frac{\partial}{a \partial \theta} + f'(\theta) \frac{\partial}{\partial r},$$

so that the boundary condition (3) becomes

$$\left[i\sigma \left\{ \frac{\partial}{\partial r} - f'(\theta) \frac{\partial}{a \partial \theta} \right\} + 2\omega \left\{ [1 - f(\theta)] \frac{\partial}{a \partial \theta} + f'(\theta) \frac{\partial}{\partial r} \right\} \right] (\xi - \bar{\xi}) = 0, \quad (11)$$

for
$$r = a \{1 + f(\theta)\},$$

and all values of θ .

If we take a to be the radius of the circle of area equal to that of the sheet under consideration, we shall have

$$\int_0^{2\pi} f(\theta) d\theta = 0. \quad (12)$$

We may suppose $f(\theta)$ expanded as a Fourier's series, and then the following method is applicable; but, owing to the complication of the results, we shall only consider the cases in which $f(\theta) = \epsilon \cos n\theta$, n being an integer and ϵ^2 negligible. Also we shall only consider the modes of vibration which are nearly symmetrical about the origin, but the others may be calculated with very little more trouble.

4. For the *free oscillations* in the modes which are *nearly symmetrical* about the origin we have to satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \kappa^2 \right) \xi = 0, \quad (13)$$

together with the condition (11).

We assume

$$\xi = J_0(\kappa r) + \epsilon (A \cos n\theta + B \sin n\theta) J_n(\kappa r), \quad (14)$$

where A and B are constants, and the time factor $e^{i\sigma t}$ has been omitted for shortness. This satisfies (13), and we shall see that A , B and κ can be chosen so as to make it satisfy (11).

Substituting from (14) in (11) we have, always neglecting ϵ^2 ,

$$i\sigma \{ \kappa J'_0(\kappa r) + \epsilon (A \cos n\theta + B \sin n\theta) \kappa J'_n(\kappa r) \} \\ + 2\omega \left\{ \frac{n\epsilon}{a} (-A \sin n\theta + B \cos n\theta) J_n(\kappa r) - \epsilon n \sin n\theta \kappa J'_0(\kappa r) \right\} = 0, \quad (15)$$

for $r = a(1 + \epsilon \cos n\theta)$,

and all values of θ .

Now on the boundary we may put

$$J'_0(\kappa r) = J'_0(\kappa a) + \kappa a \epsilon \cos n\theta J''_0(\kappa a),$$

and similarly for $J_n(\kappa r)$ and $J'_n(\kappa r)$, so that we see that in the terms in (15) involving ϵ we may replace r by a . Consequently we obtain, after a little rearrangement,

$$i\sigma(\kappa a) J'_0(\kappa a) + \epsilon \cos n\theta \{ i\sigma(\kappa a)^2 J''_0(\kappa a) + A i\sigma(\kappa a) J'_n(\kappa a) + B 2\omega n J_n(\kappa a) \} \\ + \epsilon \sin n\theta \{ B i\sigma(\kappa a) J'_n(\kappa a) - A 2\omega n J_n(\kappa a) - 2\omega n(\kappa a) J'_0(\kappa a) \} = 0, \quad (16)$$

for all values of θ .

To satisfy this we take

$$J'_0(\kappa a) = 0, \quad (17)$$

$$A i\sigma(\kappa a) J'_n(\kappa a) + B 2\omega n J_n(\kappa a) + i\sigma(\kappa a)^2 J''_0(\kappa a) = 0, \quad (18)$$

$$A 2\omega n J_n(\kappa a) - B i\sigma(\kappa a) J'_n(\kappa a) = 0. \quad (19)$$

Equation (17) gives the possible values of κ , and thence of σ from (6). We notice that the free periods are the same as for a circular sheet of water of the same area.* This remark also holds for the free modes which are not nearly symmetrical.

Solving (18) and (19) for A and B we obtain

$$A = \frac{\sigma^2(\kappa a)^3 J''_0(\kappa a) J'_n(\kappa a)}{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)}, \quad (20)$$

and

$$B = -i \frac{2\omega n \sigma(\kappa a)^2 J''_0(\kappa a) J_n(\kappa a)}{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)}. \quad (21)$$

Substituting these in (14) we obtain the desired result, viz.,

$$\xi = J_0(\kappa r) + \epsilon \frac{\sigma(\kappa a)^2 J''_0(\kappa a)}{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)} \\ \times \{ \sigma(\kappa a) J'_n(\kappa a) \cos n\theta - 2\omega n J_n(\kappa a) i \sin n\theta \} J_n(\kappa r), \quad (22)$$

subject to (17).

* See Lamb, *l.c.*, p. 306.

The above investigation will not be valid if κ and n are such as to make

$$4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)$$

small enough to be comparable with ϵ . The vanishing of this expression would mean that κ was also appropriate to that type of free oscillations of the sheet in which the principal term in ξ is $e^{in\theta} J_n(\kappa r)$ or $e^{-in\theta} J_n(\kappa r)$.

Of course, to apply (22) we must restore the time factor $e^{i\sigma t}$ and then take the real part.

5. For the *forced oscillations* of *nearly symmetrical* type we have to satisfy, from (7) and (9),

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \kappa^2 \right) \xi = 4\kappa^2 C. \quad (23)$$

A particular integral of this is $\xi = 4C$, so that we assume

$$\xi = 4C + DJ_0(\kappa r) + \epsilon (A \cos n\theta + B \sin n\theta) J_n(\kappa r), \quad (24)$$

where A , B and D are constants, as this satisfies (23).

Substituting in (11) and proceeding as in the last section, we obtain

$$\begin{aligned} & D i \sigma(\kappa a) J_0'(\kappa a) - 2C i \sigma(\kappa a)^2 + \epsilon \cos n\theta \\ & \times \{ D i \sigma(\kappa a)^3 J_0''(\kappa a) + A i \sigma(\kappa a) J_n'(\kappa a) + B 2\omega n J_n(\kappa a) - 2C i \sigma(\kappa a)^2 \} \\ & + \epsilon \sin n\theta \{ B i \sigma(\kappa a) J_n'(\kappa a) - A 2\omega n J_n(\kappa a) - D 2\omega n(\kappa a) J_0'(\kappa a) \\ & + 4C \omega n(\kappa a)^2 \} = 0, \quad (25) \end{aligned}$$

for all values of θ .

To satisfy this we take

$$D J_0'(\kappa a) - 2C(\kappa a) = 0, \quad (26)$$

$$A i \sigma(\kappa a) J_n'(\kappa a) + B 2\omega n J_n(\kappa a) + D i \sigma(\kappa a)^2 J_0''(\kappa a) - 2C i \sigma(\kappa a)^2 = 0, \quad (27)$$

$$A 2\omega n J_n(\kappa a) - B i \sigma(\kappa a) J_n'(\kappa a) = 0. \quad (28)$$

Solving these for D , A and B , we obtain

$$D = \frac{2C(\kappa a)}{J_0'(\kappa a)}, \quad (29)$$

$$A = - \frac{2C \sigma^2(\kappa a)^3 J_n'(\kappa a) \{ J_0'(\kappa a) - (\kappa a) J_0''(\kappa a) \}}{J_0'(\kappa a) \{ 4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a) \}}, \quad (30)$$

and

$$B = i \frac{4C \omega n \sigma(\kappa a)^3 J_n(\kappa a) \{ J_0'(\kappa a) - (\kappa a) J_0''(\kappa a) \}}{J_0'(\kappa a) \{ 4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a) \}}, \quad (31)$$

so that on substituting in (24) we obtain the result

$$\xi = 4C + \frac{2C(\kappa a)}{J'_0(\kappa a)} \left[J_0(\kappa r) - \epsilon \frac{\sigma(\kappa a) \{ J'_0(\kappa a) - (\kappa a) J''_0(\kappa a) \}}{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)} \right. \\ \left. \times \{ \sigma(\kappa a) J'_n(\kappa a) \cos n\theta - 2\omega n J_n(\kappa a) i \sin n\theta \} J_n(\kappa r) \right]. \quad (32)$$

The above will not be valid if either or the product of both of $J'_0(\kappa a)$, $4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)$ is small enough to be comparable with ϵ . The vanishing of the first of these would mean that the period of the disturbing force was equal to one of the free periods investigated in the last section, while the significance of the vanishing of the second has already been given.

6. It is interesting to obtain the *limiting form* of (32) as $\sigma \rightarrow 0$, and to compare it with the value obtained by the equilibrium theory.

On making $\sigma \rightarrow 0$, since $n \neq 0$, (32) takes the form

$$\xi = 4C + \frac{2C(\kappa a)}{J'_0(\kappa a)} J_0(\kappa r), \quad (33)$$

where κ has its limiting value, which, however, is imaginary, since $\kappa^2 \rightarrow -4\omega^2/gk$. Let $\kappa \rightarrow ik$, so that $k^2 = 4\omega^2/gk$; then (33) becomes

$$\xi = 4C - \frac{2C(ka)}{I'_0(ka)} I_0(kr), \quad (34)$$

where $I_0(z) = J_0(iz)$. This, then, is the limiting form towards which the forced tide corresponding to

$$\bar{\xi} = -C(kr)^2 e^{i\sigma t} \quad (35)$$

tends, as the period tends to become infinite.

Now the "equilibrium tide," in which the elevation of the free surface is ξ_0 , say, is given by $\xi_0 = \bar{\xi} + A$, where A is a constant so chosen as to make $\iint \xi_0 dS = 0$, the integral being taken over the area of the sheet. For the area in question, viz, that bounded by $r = a(1 + \epsilon \cos n\theta)$, this gives

$$\xi_0 = \frac{1}{2} C(ka)^2 - C(kr)^2. \quad (36)$$

Thus, ξ and ξ_0 are different, or, in other words, the limiting form of the forced tide as the period tends to become infinite is not the same as that calculated on the equilibrium theory, when the sheet of water is rotating.

It is immediately verified, however, that $\bar{\xi}/\xi_0 \rightarrow 1$, as $k \rightarrow 0$, i.e., as the angular velocity tends to become zero, the case of infinite depth being excluded owing to the *tidal* nature of the motion.

The expressions (34) and (36) do not involve ϵ , so that both the limiting form and the "equilibrium value" of the forced symmetrical tides on a flat rotating sheet of water are not sensibly affected by a small departure from the circular form of boundary.

We shall return later to the subject of the present section in a more general manner.

Limiting Forms of Forced Tides of Long Period on Flat Rotating Sheets of Water.

7. It was mentioned in Section 2 that the equations there stated do not necessarily hold when $\sigma = 0$. In fact, when we start with $\sigma = 0$ we have the theory of steady motions * under the disturbing force (which is then constant as regards time). These are largely arbitrary, and include as a particular case, viz., that in which the relative motion vanishes, the ordinary equilibrium tides.

However, as we shall see, if we take the limiting forms of the equations in Section 2 as $\sigma \rightarrow 0$, we obtain conditions which give a determinate value for the tide in any particular instance. A special case was calculated in the last section.

Equations (1) and (3) become respectively

$$\nabla_1^2 \xi - \frac{4\omega^2}{gh} \xi = \nabla_1^2 \bar{\xi}, \quad (37)$$

$$\text{and} \quad \frac{\partial}{\partial s} (\xi - \bar{\xi}) = 0, \quad (38)$$

at the boundary. This last condition may be written in the form

$$\xi = \bar{\xi} + \text{constant}, \quad (39)$$

where the "constant" is the same for each distinct portion of the boundary, but may be different for different distinct portions if the sheet be not simply connected.

The solution of (37) and (39) is, however, not determinate; for we may add, without violating the equations, any solution of (37) with $\bar{\xi} = 0$, which takes constant values over the separate portions of the boundary. An additional necessary condition is taken to be given by (5) for each distinct portion of the boundary. This means that the relative circulation

* Lamb, *Hydrodynamics*, 2nd ed. (1895), p. 334; 3rd ed., p. 303.

round each closed boundary line vanishes, as it does so long as σ is not actually zero. The "limiting form" is defined to be such that this condition remains satisfied, so that the results may have approximate application to forced tides of very long, but not actually infinite, period. Of course, steady motions (*i.e.*, motions with $\sigma = 0$) for which (5) is not satisfied, or for which the relative circulation round each boundary line does not vanish, may exist at any time.

We can now shew that a unique solution of (37), (39), and (5) exists—at least, for any region, whatever the connectivity, for which the ordinary problem of Dirichlet can be proved to have a solution. However, to avoid a somewhat lengthy discussion, we will here only consider the case in which there is a single closed boundary.

Consider a solution ξ_1 of (37) which takes the value $\bar{\xi}$ on the boundary, and also a solution ξ_2 of (37) with $\bar{\xi} = 0$, which takes the value unity on the boundary. The question of the existence of ξ_1 and ξ_2 can be reduced to that of a solution of Dirichlet's problem.* We now show that

$$\int \frac{\partial \xi_2}{\partial n} ds \neq 0;$$

$$\begin{aligned} \text{for, } \int \frac{\partial \xi_2}{\partial n} ds &= \int \xi_2 \frac{\partial \xi_2}{\partial n} ds \\ &= \iint \xi_2 \nabla^2 \xi_2 dS + \iint \left\{ \left(\frac{\partial \xi_2}{\partial x} \right)^2 + \left(\frac{\partial \xi_2}{\partial y} \right)^2 \right\} dS \\ &= k^2 \iint \xi_2^2 dS + \iint \left\{ \left(\frac{\partial \xi_2}{\partial x} \right)^2 + \left(\frac{\partial \xi_2}{\partial y} \right)^2 \right\} dS, \end{aligned}$$

where we have used Green's formula. Since k is real, and ξ_2 not everywhere zero, we obtain the desired result.

We now add to ξ_1 such a constant multiple C of ξ_2 as will make $\xi = \xi_1 + C\xi_2$ satisfy the conditions (37), (39), and (5).

We can easily shew that the solution is unique. For, if there were two solutions, their difference ξ' would satisfy

$$(\nabla^2 - k^2)\xi' = 0,$$

$$\text{and } \int \frac{\partial \xi'}{\partial n} ds = 0,$$

at the boundary and be a constant there. By substituting ξ' for ξ_2 in the equations given above we see that ξ' must be everywhere zero.

* See, for instance, Heywood and Fréchet, *L'Equation de Fredholm*, chap. iii.

As far as the uniqueness of the solution in the general case is concerned this proof can be immediately extended.

Since in the present problem

$$\nabla_1^2 (\xi - \bar{\xi}) = k^2 \xi, \quad (40)$$

where

$$k^2 = 4\omega^2/gh,$$

$$\text{we have} \quad \iint \xi dS = \frac{1}{k^2} \iint \nabla_1^2 (\xi - \bar{\xi}) dS = \frac{1}{k^2} \Sigma \int \frac{\partial}{\partial n} (\xi - \bar{\xi}) ds, \quad (41)$$

where the surface integrals extend over the whole sheet, and the line integrals round all the portions of the boundary, and we see that the condition of constancy of total volume of water is satisfied by (5). For the case of a single boundary the condition

$$\iint \xi dS = 0, \quad (42)$$

may be taken instead of (5), to which it is then equivalent.

The reason why the necessary condition (42) is not included in (37) and (38), whereas it is included in the equations of which they are the limiting forms, can be seen from (4) and (41).

Since the "equilibrium tide" ξ_0 is given by $\xi_0 = \bar{\xi} + \text{constant}$, we immediately see from (40) that the equality $\xi = \xi_0$ requires $k^2 \xi = 0$, and therefore $k = 0$ or $\xi = 0$. Now when $k = 0$, $\omega = 0$, since the depth may not be infinite; but in this case the condition (38) would have to be replaced by $\partial(\xi - \bar{\xi})/\partial n = 0$ at the boundaries, and then $(\xi - \bar{\xi})$ would be a potential function with a zero normal derivative at the boundary, and, therefore, a constant. Hence we see that, when there is no rotation, the limiting form of the forced tide is the "equilibrium form"; but that when there is rotation the limiting form can *never* be exactly equal to the "equilibrium form," if it is other than null.*

8. We now proceed to the calculation, in some particular cases, of the limiting forms of the forced tides corresponding to the special value of $\bar{\xi}$ given by (35).

Let us consider first the case of the *nearly circular* sheet defined by (10) with the condition (12). Of course, our results will be accurately true for a sheet which is exactly circular if we put $f(\theta) = 0$.

We have to satisfy

$$(\nabla_1^2 - k^2) \xi = -4Ck^2, \quad (43)$$

* Cf. Lamb, *l.c.*, p. 300.

with the conditions that

$$\xi = -C(kr)^2 + \text{constant}, \quad (44)$$

at the boundary $r = a \{1 + f(\theta)\}$,

and that
$$\int \frac{\partial}{\partial n} \{ \xi + C(kr)^2 \} ds = 0, \quad (45)$$

the integral being taken round the boundary.

We assume
$$\xi = 4C + AI_0(kr), \quad (46)$$

A being a constant. This satisfies (43), and we shall see that A can be chosen so as to make it satisfy both (44) and (45).

On the boundary we may put

$$I_0(kr) = I_0(ka) + (ka) f(\theta) I'_0(ka),$$

and
$$(kr)^2 = (ka)^2 \{1 + 2f(\theta)\},$$

so that condition (44) reduces to

$$AI'_0(ka) = -2C(ka). \quad (47)$$

In (45) we may replace $\partial/\partial n$ by $\partial/\partial r$, since the assumed value of ξ does not involve θ . This gives

$$\int \{ AkI'_0(kr) + 2Ck^2r \} ds = 0,$$

which, to our order, may be written

$$\int_0^{2\pi} \{ AI'_0(ka) + 2C(ka) + Bf(\theta) \} \{1 + f(\theta)\} d\theta = 0,$$

or
$$\int_0^{2\pi} \{ AI'_0(ka) + 2C(ka) + Df(\theta) \} d\theta = 0, \quad (48)$$

where B and D are certain constants. By virtue of (12), (48) reduces to the same as (47), and gives, on substituting in (46), the result

$$\xi = 4C - \frac{2C(ka)}{I'_0(ka)} I_0(kr), \quad (34)$$

which is independent of θ and is unaffected by the small departure from the circular boundary.

The expression (34) has already been obtained in Section 6 by other methods, but not with the same generality as at present.

The "equilibrium value" is again given by (36).

9. Let us now consider the case of the *rectangular* sheet of water. Using Cartesians we take the origin of coordinates at the centre. Let the length of the side parallel to the axis of x be $2a$, and of that parallel to the axis of y be $2b$.

The equations we have to satisfy are (43), (44), and (5); but we will take the last in the form (42), since the region is simply connected. In the first place, let us take the "constant" of (44) to be $4C + Ck^2(a^2 + b^2)$, and call the corresponding solution of (43) and (44) ξ' .

Then on the sides $x = \pm a$ we shall have

$$\xi' = 4C + Ck^2 \{ (a^2 + b^2) - (a^2 + y^2) \} = 4C + Ck^2(b^2 - y^2), \quad (49)$$

and similarly on the sides $y = \pm b$,

$$\xi' = 4C + Ck^2(a^2 - x^2). \quad (50)$$

As we shall then have the condition (42) unsatisfied, we next obtain a solution ξ'' of

$$(\nabla_1^2 - k^2) \xi = 0, \quad (51)$$

which takes a constant value over the boundary, and then add to ξ' such a multiple A of ξ'' as will make $\xi = \xi' + A\xi''$ satisfy the condition (42).

Since a particular solution of (43) is $\xi = 4C$, besides this we are only concerned with solutions of (51).

Now
$$\xi_s = \cosh m_s x \cos \frac{s\pi y}{2b},$$

satisfies (51), provided
$$m_s^2 = k^2 + \frac{s^2\pi^2}{4b^2},$$

and on $x = \pm a$, it takes the form

$$\xi_s = \cosh m_s a \cos \frac{s\pi y}{2b},$$

while on $y = \pm b$, it vanishes, provided s is an odd integer.

Now, by Fourier's theorem, or otherwise,

$$b^2 - y^2 = \frac{32b^2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} \cos \frac{(2n+1)\pi y}{2b},$$

for $-b \leq y \leq b$, so that

$$\sum \xi_s = \frac{32Ck^2b^2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b}, \quad (52)$$

where

$$\mu_n^2 = k^2 + \frac{(2n+1)^2\pi^2}{4b^2}, \quad (53)$$

which is uniformly convergent over the rectangle, satisfies (51) and is equal to $Ck^2(b^2 - y^2)$ on $x = \pm a$, while it vanishes on $y = \pm b$.

Similarly,

$$\frac{32Ck^2a^2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a}, \quad (54)$$

where
$$\nu_n^2 = k^2 + \frac{(2n+1)^2 \pi^2}{4a^2}, \quad (55)$$

which is also uniformly convergent over the rectangle, satisfies (51) and is equal to $Ck^2(a^2 - x^2)$ on $y = \pm b$, while it vanishes on $x = \pm a$.

Adding (52) and (54) to $4C$, we obtain an expression satisfying the conditions necessary for ξ' . Thus

$$\begin{aligned} \xi' = 4C + \frac{32Ck^2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} \\ \times \left\{ b^2 \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b} + a^2 \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a} \right\}. \end{aligned} \quad (56)$$

We now proceed to the determination of ξ'' , for which we make use of the known expansion

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \cos (2n+1)\theta,$$

which holds for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. From this and what has been said above we see that

$$\xi'' = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \left\{ \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b} + \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a} \right\}, \quad (57)$$

which is convergent over the rectangle, satisfies (51) and takes a constant value, viz., $\frac{1}{4}\pi$, over the sides (with the exception of the corners, which, however, need not trouble us).

Now $\iint \xi' dx dy$, when taken over the area of the rectangle, has the value

$$16Cab + \frac{256Ck^2}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \left\{ \frac{b^3}{\mu_n} \tanh \mu_n a + \frac{a^3}{\nu_n} \tanh \nu_n b \right\},$$

and similarly $\iint \xi'' dx dy$ has the value

$$\frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left\{ \frac{b}{\mu_n} \tanh \mu_n a + \frac{a}{\nu_n} \tanh \nu_n b \right\},$$

since term-by-term integration can be justified in each case, so that the required value of ξ is given by

$$\begin{aligned} \xi = & 4C + \frac{32Ck^2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} \\ & \times \left\{ b^3 \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b} + a^2 \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a} \right\} \\ & - AC \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \\ & \times \left\{ \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b} + \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a} \right\}, \quad (58) \end{aligned}$$

where the constant A is given by

$$A = \frac{2\pi^4 ab + 32k^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \left\{ \frac{b^3}{\mu_n} \tanh \mu_n a + \frac{a^3}{\nu_n} \tanh \nu_n b \right\}}{\pi^3 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left\{ \frac{b}{\mu_n} \tanh \mu_n a + \frac{a}{\nu_n} \tanh \nu_n b \right\}}, \quad (59)$$

and μ_n, ν_n are given by (53) and (55).

The "equilibrium value" is given by

$$\xi_0 = \frac{4}{3} Ck^2 ab (a^2 + b^2) - Ck^2 (x^2 + y^2). \quad (60)$$

10. Let us now discuss the case of a sheet of water which has the shape of a *circular sector*, whose sides are given by $\theta = \pm \alpha$, in polar coordinates.

We know that, unless $\cos s\alpha = 0$, the function

$$\xi_s = \frac{\cos s\theta}{\cos s\alpha} I_s(kr), \quad (61)$$

satisfies (51) and becomes equal to $I_s(kr)$ on the sides. When $\cos s\alpha = 0$, we may replace this by

$$\frac{\frac{\partial}{\partial s} \{ \cos s\theta I_s(kr) \}}{\frac{\partial}{\partial s} \cos s\alpha} = \frac{\theta \sin s\theta I_s(kr) - \cos s\theta \frac{\partial}{\partial s} I_s(kr)}{a \sin s\alpha}, \quad (62)$$

which then satisfies the same conditions.

By means of the known expansions

$$1 = I_0(kr) + 2 \sum_{n=1}^{\infty} (-1)^n I_{2n}(kr),$$

and
$$(kr)^2 = 8 \sum_{n=1}^{\infty} (-1)^{n-1} n^2 I_{2n}(kr),$$

together with (61) and (62), we can form expressions ξ' and ξ'' , each of which satisfies (51), whilst at the sides $\theta = \pm \alpha$,

$$\xi' = -C(kr)^2, \quad \xi'' = 1. \quad (63)$$

Further, by means of Fourier's theorem and the assumption

$$\xi'_n = \cos \frac{(2n+1)\pi\theta}{2\alpha} I_{[(2n+1)\pi]/2\alpha}(kr), \quad (64)$$

which satisfies (51) and vanishes at the sides, we can satisfy the conditions (63) on the enclosed arc $r = a$.

Then taking $\xi = \xi' + A\xi''$, and determining A by means of the condition (42), we obtain the required solution; but it would, perhaps, be difficult to interpret. For the case of the semicircle the solutions (62) would not be required.

Some Partial Solutions.

11. Leaving now the question of limiting forms of forced tides, we proceed to investigate the solution of the general equations of free motion given in Section 2 for regions, however, which are not bounded in all directions. The results will only be applicable over limited portions of the regions considered, since the fundamental equations are really only valid for limited regions.

Let us first consider the case of the *semi-infinite plane* given in polar coordinates by $0 \leq \theta \leq \pi$. We have to satisfy (13), while condition (3) becomes

$$\left(i\sigma \frac{\partial}{r\partial\theta} - 2\omega \frac{\partial}{\partial r} \right) \xi = 0, \quad (65)$$

for $\theta = 0$ and $\theta = \pi$.

We assume
$$\xi = \sum_n (A_n \cos n\theta + B_n \sin n\theta) J_n(kr), \quad (66)$$

where the values of n are as yet unrestricted. This satisfies (13), and on substituting in (65) we obtain

$$\sum_n \left\{ \frac{i\sigma n}{r} (-A_n \sin n\theta + B_n \cos n\theta) J_n(kr) - 2\omega (A_n \cos n\theta + B_n \sin n\theta) \kappa J'_n(kr) \right\} = 0, \quad (67)$$

for $\theta = 0$ and $\theta = \pi$.

Now
$$J'_n(\kappa r) = \frac{n}{(\kappa r)} J_n(\kappa r) - J_{n+1}(\kappa r),$$

and
$$\frac{n}{(\kappa r)} J_n(\kappa r) = \frac{1}{2} \{J_{n+1}(\kappa r) + J_{n-1}(\kappa r)\},$$

so that, on dividing through by $\frac{1}{2}\kappa$, (67) may be written in the form

$$\sum_n \{ [i\sigma(-A_n \sin n\theta + B_n \cos n\theta) - 2\omega(A_n \cos n\theta + B_n \sin n\theta)] J_{n-1}(\kappa r) \\ + [i\sigma(-A_n \sin n\theta + B_n \cos n\theta) + 2\omega(A_n \cos n\theta + B_n \sin n\theta)] J_{n+1}(\kappa r) \} = 0, \quad (68)$$

for $\theta = 0$ and $\theta = \pi$.

To satisfy this we take n in the form $s + 2m$, where s is a constant and m takes positive integral values, and proceed to make the coefficients of the different Bessel's functions in equation (68) vanish for $\theta = 0$ and $\theta = \pi$. The coefficient of $J_{s-1}(\kappa r)$ gives, except for $s = 0$,

$$i\sigma B_s - 2\omega A_s = 0,$$

and
$$i\sigma(-A_s \sin s\pi + B_s \cos s\pi) - 2\omega(A_s \cos s\pi + B_s \sin s\pi) = 0,$$

from which we deduce

$$\frac{A_s}{B_s} = \frac{i\sigma}{2\omega} = \frac{A_s \cos s\pi + B_s \sin s\pi}{B_s \cos s\pi - A_s \sin s\pi}. \quad (69)$$

From this we see that we must have $\sin s\pi = 0$, so that s must be an integer. The case $s = 0$ is excluded from (69), since $J_1(\kappa r)$ and $J_{-1}(\kappa r)$ are not independent; but, on looking back at (67) we see that zero is a possible value of s .

With this determination of s we observe that the two conditions involved in (68) now reduce to the same and become in fact

$$\sum_{m=0}^{\infty} \{ (i\sigma B_{s+2m} - 2\omega A_{s+2m}) J_{s+2m-1}(\kappa r) + (i\sigma B_{s+2m} + 2\omega A_{s+2m}) J_{s+2m+1}(\kappa r) \} = 0, \quad (70)$$

so that the determination of the constants is partly arbitrary. We have the equation

$$(i\sigma B_{s+2m} - 2\omega A_{s+2m}) + (i\sigma B_{s+2(m-1)} + 2\omega A_{s+2(m-1)}) = 0, \quad (71)$$

for $m = 1, 2, 3, \dots$, when $s \neq 0$, and $m = 2, 3, 4, \dots$, when $s = 0$.

We can make $A_{s+2m} = B_{s+2m} = 0$, for $m > 1$ if we take

$$i\sigma B_{s+2} + 2\omega A_{s+2} = 0.$$

A_s and B_s have their ratio determined by (69), except for $s = 0$, when B_0 does not exist, so that we obtain, when $s \neq 0$,

$$A_{s+2} = A_s = 1$$

say. When $s = 0$, (70) gives, from the coefficient of $J_1(\kappa r) [= -J_{-1}(\kappa r)]$,

$$(i\sigma B_2 - 2\omega A_2) + 4\omega A_0 = 0,$$

so that again we have $A_2 = A_0 = 1$.

Therefore solutions of our problem are given by

$$\begin{aligned} \xi = & \left\{ \cos s\theta - \frac{2\omega}{\sigma} i \sin s\theta \right\} J_s(\kappa r) \\ & + \left\{ \cos (s+2)\theta + \frac{2\omega}{\sigma} i \sin (s+2)\theta \right\} J_{s+2}(\kappa r), \quad (72) \end{aligned}$$

where s is any integer, including zero.

It was from expressions of this type that the solution mentioned in Section 1, for a slowly rotating semicircle of radius a , was obtained.

The solution given by Lord Kelvin † for a rotating canal, in which the relative motion of the water is everywhere parallel to the banks, can be constructed from the expressions (72) or obtained directly by a certain choice of the constants in (71).

12. Let us now consider the case of a *channel* bounded by $x = 0$ and $x = a$, using Cartesian coordinates.

We have to satisfy

$$(\nabla_1^2 + \kappa^2) \xi = 0, \quad (73)$$

for $0 < x < a$, together with the condition

$$\left(i\sigma \frac{\partial}{\partial x} + 2\omega \frac{\partial}{\partial y} \right) \xi = 0, \quad (74)$$

at $x = 0$ and $x = a$.

Let us consider

$$\xi = (A_n e^{inx} + B_n e^{-inx}) e^{im y}, \quad (75)$$

which satisfies (73), provided

$$m^2 + n^2 = \kappa^2.$$

* *L.c.*; see Lamb, *l.c.*, p. 304.

Now
$$i\sigma \frac{\partial \xi}{\partial x} + 2\omega \frac{\partial \xi}{\partial y} = -n\sigma (A_n e^{inx} - B_n e^{-inx}) e^{imy} \\ + 2i\omega m (A_n e^{inx} + B_n e^{-inx}) e^{imy}. \quad (76)$$

For this to vanish at $x = 0$ and $x = a$ we must have either $n = s\pi/a$, where s is any integer, and

$$-A_n(n\sigma - 2i\omega m) + B_n(n\sigma + 2i\omega m) = 0,$$

or else $B_n = 0$ and $n\sigma = 2i\omega m$.

The second of these alternatives makes (74) to be satisfied for all values of x , and leads to Lord Kelvin's solution for the canal which was mentioned in the last section.

The first gives

$$\xi = \frac{1}{2\sigma} \left[\left(\frac{s\pi\sigma}{a} + 2i\omega m_s \right) e^{is\pi x/a} + \left(\frac{s\pi\sigma}{a} - 2i\omega m_s \right) e^{-is\pi x/a} \right] e^{im_s y},$$

or
$$\xi = \left(\frac{s\pi}{a} \cos \frac{s\pi x}{a} - \frac{2\omega}{\sigma} m_s \sin \frac{s\pi x}{a} \right) e^{im_s y}, \quad (77)$$

where
$$m_s^2 = \kappa^2 - \frac{s^2 \pi^2}{a^2},$$

A_n having been taken equal to $(s\pi\sigma/a + 2i\omega m_s)/2\sigma$.

However, if m_s is imaginary (77) is not applicable, so that the only admissible values of the integer s are those which make m_s^2 positive. There need not be any if κ is small enough, as $s = 0$ does not lead to a solution.

Symmetrical Tides of a Small Circular Sea on a Sphere.

13. Finally, we consider the free tidal motion of a sea which is situated on a rotating sphere and bounded by a small circle with its centre at one of the poles.

Let R be the radius of the sphere, h the (uniform) depth of the sea, and ω the angular velocity of the sphere which is supposed to be rotating about the axis through the pole P of the small circle. Let r be the distance (measured along a great circle) of any point on the sphere from P , and let $r = a$ be the equation of the small circle.

We suppose that we may neglect a^3/R^3 . Now the equation of free

motion (symmetrical about the axis) of the water in such a sea is *

$$\frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left\{ \frac{\sin \theta}{\frac{\sigma^2}{4\omega^2} - \cos^2 \theta} \frac{\partial \xi}{\partial \theta} \right\} + \frac{4\omega^2}{gh} \xi = 0, \quad (78)$$

where ξ and σ have their previous meanings and $\theta = r/R$.

The condition to be satisfied at the boundary $r = a$ is that $\partial \xi / \partial \theta = 0$ there.

$$\text{We now put} \quad \sin \theta = \frac{r}{R}, \quad \cos \theta = 1 - \frac{1}{2} \frac{r^2}{R^2},$$

and the equation (78) becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{r}{\frac{\sigma^2}{4\omega^2} - 1 + \frac{r^2}{R^2}} \frac{\partial \xi}{\partial r} \right\} + \frac{4\omega^2}{gh} \xi = 0. \quad (79)$$

Now to our order of small quantities

$$\left(\frac{\sigma^2}{4\omega^2} - 1 + \frac{r^2}{R^2} \right)^{-1} = \left(\frac{\sigma^2}{4\omega^2} - 1 \right)^{-1} \left(1 - \frac{r^2}{b^2} \right),$$

where

$$b^2 = \left(\frac{\sigma^2}{4\omega^2} - 1 \right) R^2, \quad (80)$$

$$\text{and so (79) becomes} \quad \frac{1}{r} \frac{\partial}{\partial r} \left\{ \left(r - \frac{r^3}{b^2} \right) \frac{\partial \xi}{\partial r} \right\} + \kappa^2 \xi = 0, \quad (81)$$

$$\text{where} \quad \kappa^2 = \left(\frac{\sigma^2}{4\omega^2} - 1 \right) \frac{4\omega^2}{gh} = \frac{\sigma^2 - 4\omega^2}{gh}. \quad (82)$$

The equation to be satisfied at $r = a$ is $\partial \xi / \partial r = 0$. To solve (81) we assume

$$\xi = J_0(\kappa r) + \frac{1}{b^2} F(\kappa r). \quad (83)$$

Then to our order we have

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} F(\kappa r) \right\} + \kappa^2 F(\kappa r) = \frac{1}{r} \frac{d}{dr} \left\{ r^3 \frac{d}{dr} J_0(\kappa r) \right\}. \quad (84)$$

It is easily verified that a particular solution of this equation is given by

$$F(\kappa r) = \frac{1}{6\kappa^2} [2(\kappa r)^2 J_0(\kappa r) + \{4(\kappa r) + (\kappa r)^3\} J_0'(\kappa r)]; \quad (85)$$

* See Lamb, *l.c.*, p. 316. The mutual attraction of the particles of water is here neglected.

and, as any other admissible solution can be obtained from this by adding a constant multiple of $J_0(\kappa r)$, this is the one we require.

The condition to be satisfied at the boundary $r = a$ gives

$$J'_0(\kappa a) + \frac{1}{b^3} F'(\kappa a) = 0. \quad (86)$$

Let us assume for a moment that a solution of (86) differs from one of $J'_0(\kappa a) = 0$ by a quantity of the order a^2/b^3 . Suppose $J'_0(a) = 0$; then to our order (86) may be written

$$(\kappa a - a) J''_0(a) + \frac{1}{b^3} F'(a) = 0, \quad (87)$$

in which we may put

$$b^3 = \frac{\kappa^2 g h R^2}{4\omega^2} = \frac{a^2 g h R^2}{4\omega^2 a^2}, \quad (88)$$

so that

$$\kappa a = a - \frac{4\omega^2 a^2}{a^2 g h R^2} \frac{F'(a)}{J''_0(a)}. \quad (89)$$

The above assumption will be justified if the last term in (89) is of the order a^2/R^2 .

Since $J'_0(a) = 0$, we may put in (89) $J'_0(a) = -J_0(a)$, and we find that $F'(a) = -\frac{1}{6} a^2 a J_0(a)$, so that κa is given by

$$\kappa a = a - \frac{2\omega^2 a^4}{3a^2 g h R^2}. \quad (90)$$

The elevation of the water ξ is then given by

$$\xi = J_0(\kappa r) + \frac{2\omega^2 a^4}{3a^2 g h R^2} [2(\kappa r)^2 J_0(\kappa r) + \{4(\kappa r) + (\kappa r)^3\} J'_0(\kappa r)], \quad (91)$$

where the time factor $e^{i\omega t}$ is omitted, σ being given by (82).

The first three values of a are 3.8317, 7.0156, 10.1735 approximately.*

It is interesting to notice that, to our order of approximation, when $\omega = 0$, the results are the same as for a plane.

At the boundary the ratio of the second term in (91) to the first is

$$\frac{4\omega^2 a^4}{3a^2 g h R^2},$$

* See Gray and Mathews, *Treatise on Bessel Functions*, p. 280.

which is positive, so that the effect of the curvature is to make the rise and fall of the water near the boundary slightly more marked, as compared with that at the pole.

Again, since the second term on the right-hand side of (90) is small compared with the first and of opposite sign, we see that the value of κ^2 and therefore of σ^2 , is diminished by the curvature. This means that the periods are slightly increased.

The *forced* symmetrical tides on such a sea may be easily calculated to the same order of approximation; the results are a little more complicated.