On the Continuous Group that is defined by any given Group of Finite Order. By W. BURNSIDE. Received January 10th, 1898. Read January 13th, 1898.

It has been shown by Herr Study that, for every transitive linear homogeneous group in n variables with n independent parameters, the parameters may be so chosen that the finite equations of the group are linear and homogeneous in the parameters as well as in the variables (*Leipziger Berichte*, 1889, p. 201). If $x_1, x_2, \dots, x_n$ are the original variables, $x'_1, x'_2, \dots, x'_n$ the transformed variables, and $y_1, y_2, \dots, y_n$ the parameters thus chosen, the finite equations of the group are

$$x'_i = \sum_{j,k} a_{ijk} x_j y_k \quad (i, j, k = 1, 2, \dots, n).$$

The quantities a_{ijk} are absolute constants, which are connected by a number of relations necessary to ensure that the system of equations shall actually have the group property.

The object of the present paper is to call attention to a special determination of the constants a_{ijk} , which is obtained by means of a group of finite order n ; and to investigate some of the properties of the continuous group so defined. The determination in question may be stated thus:

If $S_1, S_2, \dots, S_n$ are the operations of a group of finite order n , then a_{ijk} is equal to unity or zero, according as $S_k S_i^{-1} S_j$ is or is not the identical operation.

The matrix of the n^2 coefficients in any transformation thus defined has formed the subject of investigation by Herr G. Frobenius ("Über die Primfactoren der Gruppendeterminante," *Berliner Sitzungsberichte*, 1896, pp. 1343-1382); and from a result given by him (*loc. cit.*, p. 1346) it follows at once that such a system of transformations have the group property. The main subject of Herr Frobenius's memoir is, however, entirely different from that of the present paper, and he makes no reference to the continuous group which is connected with the matrix.

In the simplest cases, $n = 2$ and $n = 3$, the sets of transformations

$$\text{are} \quad x'_1 = x_1 y_1 + x_2 y_2, \quad x'_2 = x_1 y_2 + x_2 y_1;$$

$$\text{and} \quad \begin{aligned} x'_1 &= x_1y_1 + x_2y_2 + x_3y_3, & x'_2 &= x_1y_3 + x_2y_1 + x_3y_2, \\ x'_3 &= x_1y_2 + x_2y_3 + x_3y_1. \end{aligned}$$

It is easily seen, on trial, that these sets of equations have the group property; and it was, in fact, a consideration of these simple cases that suggested to me the form for a system of transformations of n variables, which is given by any finite group of order n .

1. Let $S_1, S_2, \dots, S_n$ denote the operations of a group of finite order n , S_1 being the identical operation. The multiplication table of the group may be written in the form—

$$\left. \begin{array}{cccccc} S_1 & S_2 & \dots & S_k & \dots & S_n \\ S_2^{-1}S_1 & S_1 & \dots & S_2^{-1}S_k & \dots & S_2^{-1}S_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_l^{-1}S_1 & S_l^{-1}S_2 & \dots & S_l^{-1}S_k & \dots & S_l^{-1}S_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_n^{-1}S_1 & S_n^{-1}S_2 & \dots & S_n^{-1}S_k & \dots & S_1 \end{array} \right\}, \quad (1)$$

so that each place in the principal diagonal is occupied by the symbol of the identical operation. For each suffix l , the set of symbols

$$S_l^{-1}S_1, S_l^{-1}S_2, \dots, S_l^{-1}S_{l-1}, S_l^{-1}S_{l+1}, \dots, S_l^{-1}S_n$$

is identical with the set

$$S_2, S_3, \dots, S_n,$$

except as regards the sequence in which the symbols occur.

If we now regard $S_1, S_2, \dots, S_n$ as n continuously varying parameters capable of taking all real values, and if each compound symbol $S_l^{-1}S_k$ is replaced by the single symbol which is equivalent to it, when the above multiplication table (1) is explicitly written in terms of the original n symbols, then the set of homogeneous linear transformations—

$$\left. \begin{aligned} x'_1 &= S_1x_1 + S_2x_2 + \dots + S_kx_k + \dots + S_nx_n \\ x'_2 &= S_2^{-1}S_1x_1 + S_1x_2 + \dots + S_2^{-1}S_kx_k + \dots + S_2^{-1}S_nx_n \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ x'_l &= S_l^{-1}S_1x_1 + S_l^{-1}S_2x_2 + \dots + S_l^{-1}S_kx_k + \dots + S_l^{-1}S_nx_n \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ x'_n &= S_n^{-1}S_1x_1 + S_n^{-1}S_2x_2 + \dots + S_n^{-1}S_kx_k + \dots + S_1x_n \end{aligned} \right\} \quad (2)$$

defines a finite continuous group.

To verify this statement let us consider the transformation TT' , where T represents the transformation written and T' is the corresponding transformation with accented coefficients. If in TT'

$$x'_q = \sum_{p=1}^{p=n} a_{q,p} x_p \quad (q = 1, 2, \dots, n),$$

then $a_{q,p} = S_q^{-1} S_p + S_q'^{-1} S_2' S_2^{-1} S_p + \dots + S_q'^{-1} S_n' S_n^{-1} S_p.$

Now, if $S_k^{-1} S_p = S_r,$

then $S'^{-1} S'_k = S'^{-1}_r.$

Hence $a_{p,p} = \sum_1^n S'^{-1}_r S_r,$

and therefore the coefficients in the leading diagonal of TT' are all the same.

Suppose now that $S_q^{-1} S_p = S_l^{-1} S_k.$

Then, the suffix y being determined uniquely, when l, q, x are given, by

$$S_q^{-1} S_x = S_l^{-1} S_y;$$

we have $S_p^{-1} S_x = S_k^{-1} S_y,$

or $S_x^{-1} S_p = S_y^{-1} S_k.$

Hence the relation $S_q^{-1} S_p = S_l^{-1} S_k$

implies that, corresponding to every term $S_q'^{-1} S'_x S_x^{-1} S_p$ in $a_{q,p}$, there is an equal term $S_l'^{-1} S'_y S_y^{-1} S_k$ in $a_{l,k}$, and conversely; or that

$$a_{q,p} = a_{l,k}.$$

In particular, for every value of the suffixes p and q , there is a single suffix r such that

$$S_q^{-1} S_p = S_r,$$

and corresponding to this we have the relation

$$a_{q,p} = a_{1,r}.$$

The coefficients in the transformation TT' thus follow the same law as those in T and T' , which is defined by the multiplication table of the finite group; and therefore the set of transformations form a group. Since its transformations contain a finite number n of continuously varying parameters, the group is a finite continuous group.

2. The infinitesimal transformations of the group are those for which all the parameters except S_1 are infinitesimal, while S_1 differs from unity by an infinitesimal quantity. Since the group in its finite form (2) contains n independent parameters, it will have n linearly independent infinitesimal transformations; and, in the phraseology which is now generally adopted, it may be described as a continuous group of order n . A set of n linearly independent infinitesimal operations is obtained from the finite form (2) of the group by taking, first,

$$S_2 = S_3 = \dots = S_n = 0;$$

and then

$$S_1 = 1,$$

$$S_2 = S_3 = \dots = S_{k-1} = S_{k+1} = \dots = S_n = 0 \\ (k = 2, 3, \dots, n).$$

If the corresponding differential operators are denoted by

$$X_k \quad (k = 1, 2, \dots, n),$$

then
$$X_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \dots + x_n \frac{\partial}{\partial x_n},$$

and
$$X_k = x_k \frac{\partial}{\partial x_1} + x_{k+1} \frac{\partial}{\partial x_2} + \dots + x_n \frac{\partial}{\partial x_n} \quad (k = 2, 3, \dots, n),$$

where x_{k_p} denotes the coefficient of S_k in the expression for x'_p given by (2). In fact, if

$$S_1 = 1, \quad S_2 = S_3 = \dots = S_{k-1} = S_{k+1} = \dots = S_n = 0,$$

then

$$x'_1 = x_1 + S_k x_k,$$

$$x'_2 = x_2 + S_k x_{k+1},$$

$$\dots \quad \dots \quad \dots$$

$$x'_n = x_n + S_k x_{k+n};$$

and, if S_k be regarded as infinitesimal, the result of replacing $x_1, x_2, \dots, x_n$ by $x'_1, x'_2, \dots, x'_n$ in any function $f(x_1, x_2, \dots, x_n)$ is

$$f(x_1 + S_k x_k, x_2 + S_k x_{k+1}, \dots, x_n + S_k x_{k+n}),$$

or

$$(1 + S_k X_k) f(x_1, x_2, \dots, x_n).$$

Since

$$x'_p = S_p^{-1} S_1 x_1 + S_p^{-1} S_2 x_2 + \dots + S_p^{-1} S_q x_q + \dots + S_p^{-1} S_n x_n,$$

it follows that

$$x_{k_p} = x_q,$$

if

$$S_p^{-1} S_q = S_k,$$

or if

$$S_p S_k = S_q.$$

This immediately suggests a more convenient form of notation than the suffix with a suffix that has been used provisionally. Thus we may write

$$X_k = x_{1k} \frac{\partial}{\partial x_1} + x_{2k} \frac{\partial}{\partial x_2} + \dots + x_{nk} \frac{\partial}{\partial x_n},$$

on the understanding that the suffixes

$$1, 2, 3, \dots, n$$

combine among themselves according to the multiplication table (1) of the original group of finite order, the suffix k corresponding to the operation S_k .

3. The alternants* $(X_k X_l) \quad (k, l = 1, 2, \dots, n),$

of the n differential operators may now be formed by direct calculation. Thus the coefficient of x_p in X_l is $\frac{\partial}{\partial x_q}$, if

$$p = ql,$$

or

$$q = pl^{-1};$$

and therefore the only term with a differential coefficient of the first order arising from

$$x_{pk} \frac{\partial}{\partial x_p} X_l$$

is

$$x_{pk} \frac{\partial}{\partial x_{pl^{-1}}}.$$

Hence

$$(X_k X_l) = \sum_{p=1}^{p=n} \left(x_{pk} \frac{\partial}{\partial x_{pl^{-1}}} - x_{pl} \frac{\partial}{\partial x_{pk^{-1}}} \right).$$

If

$$pl^{-1} = t,$$

and

$$pk = tx,$$

then

$$x = t^{-1}pk = lk.$$

Hence

$$\sum_1^n x_{pk} \frac{\partial}{\partial x_{pl^{-1}}} = \sum_{t=1}^{t=n} x_{lk} \frac{\partial}{\partial x_t} = X_{lk};$$

similarly,

$$\sum_1^n x_{pl} \frac{\partial}{\partial x_{pk^{-1}}} = X_{kl},$$

and therefore

$$(X_k X_l) = X_{lk} - X_{kl}.$$

* One of the referees pointed out to me that the use of the word "alternant," as an equivalent for the German "Klammerausdruck" and French "crochet," to denote the operation (XY) or $XY - YX$, is now generally established.

4. The continuous group of order n that has been co-ordinated to any given group of finite order n , whose operations are

$$S_1, S_2, \dots, S_n,$$

may therefore be defined as generated by n independent infinitesimal operations

$$X_1, X_2, \dots, X_n,$$

which satisfy the relations

$$(X_k X_l) = X_{lk} - X_{kl} \quad (k, l = 1, 2, \dots, n);$$

each double suffix lk in these relations being replaced by the single suffix m , which is given uniquely by

$$S_l S_k = S_m.$$

That the relations giving the alternants in terms of the n generating infinitesimal operations are self-consistent is a consequence of their having been derived from the finite form of the group. If, however, the definition just given is taken as the starting-point for the continuous group, it is necessary to show that the relations in question are actually self-consistent. The necessary conditions are that

$$(X_k X_l) + (X_l X_k) = 0$$

$$\text{and} \quad \{(X_k X_l) X_i\} + \{(X_l X_i) X_k\} + \{(X_i X_k) X_l\} = 0$$

hold for all values of k , l , and i .

The first condition is obviously satisfied. Moreover,

$$\begin{aligned} \{(X_k X_l) X_i\} + \{(X_l X_i) X_k\} + \{(X_i X_k) X_l\} &= X_{ilk} - X_{lki} - X_{kli} + X_{kli} \\ &\quad + X_{kil} - X_{ilk} - X_{kli} + X_{lik} \\ &\quad + X_{lki} - X_{kil} - X_{lik} + X_{lil} \\ &= 0, \end{aligned}$$

so that the second condition is also satisfied, whatever i , k , and l may be.

5. For a discussion of the properties of the continuous group, and especially for the determination of its composition, the definition of the group by means of its infinitesimal operations and the expressions for their alternants is more convenient than that which gives the operations of the group in finite form; it will therefore be used almost exclusively. The continuous group itself will be called G , the group of finite order g ; and the suffixes will be used as though they not only correspond to, but actually are, the operations of g .

A first step towards determining the composition of G is to find its self-conjugate infinitesimal operations—i.e., those of its infinitesimal operations which are permutable with every operation of the group. Let

$$Y = \sum_1^n e_s X_s,$$

be such an operation. Then

$$\begin{aligned} (X_t, Y) &= 0 = \sum_1^n e_s (X_t X_s) \\ &= \sum_1^n e_s (X_{st} - X_{ts}) \\ &= \sum_1^n (e_{pt^{-1}} - e_{t^{-1}p}) X_p. \end{aligned}$$

Hence, for every value of p and t ,

$$e_{pt^{-1}} - e_{t^{-1}p} = 0,$$

or

$$e_q - e_{t^{-1}qt} = 0,$$

for all values of q and t .

Now every operation of g which is conjugate to q , and no others, can be put in the form $t^{-1}qt$. Hence the necessary and sufficient conditions that Y may be a self-conjugate operation are that those coefficients e_s , whose suffixes form a set of conjugate operations of g , shall be all equal. If, then, g contains r distinct sets of conjugate operations, Y will contain r arbitrary parameters; and G contains r linearly independent self-conjugate infinitesimal operations. These generate a continuous group of order r , whose operations are all permutable with each other. This group is obviously a self-conjugate sub-group of G ; it will be called H .

6. Next, the derived group G_1 of G may be considered. This is the self-conjugate sub-group which is generated by all the alternants $(X_s X_t)$.

If q and $t^{-1}qt$ are any two conjugate operations of g , then $X_q - X_{t^{-1}qt}$ is the alternant of two of the generating operations of G ; for

$$\begin{aligned} X_q - X_{t^{-1}qt} &= X_{qt^{-1}} - X_{t^{-1}qt} \\ &= (X_{t^{-1}} X_q). \end{aligned}$$

Hence the generating operations of G_1 consist of such of the operations

$$X_q - X_{t^{-1}qt}$$

as are linearly independent.

Suppose that $a_1, a_2, \dots, a_s$ is a set of conjugate operations of g . The $s-1$ operations

$$X_{a_1} - X_{a_2}, X_{a_1} - X_{a_3}, \dots, X_{a_1} - X_{a_s}$$

are linearly independent, and by a linear combination of them every operation

$$X_{a_p} - X_{a_q} \quad (p, q = 1, 2, \dots, s)$$

can be represented.

Hence each set of conjugate operations of g gives a set, one less in number, of linearly independent operations which must appear among the generating operations of G_1 . If then r still denotes the number of distinct sets of conjugate operations in g , G_1 is a continuous group of order $n-r$. Moreover, since the operation

$$X_{a_1} + X_{a_2} + \dots + X_{a_s}$$

cannot be represented as a linear combination of

$$X_{a_1} - X_{a_2}, X_{a_1} - X_{a_3}, \dots, X_{a_1} - X_{a_s},$$

the derived group G_1 has no operation except identity in common with the group H of the self-conjugate operations. The self-conjugate sub-groups G_1 and H , which have no common operation except identity, are therefore such that every operation of one is permutable with every operation of the other, while every infinitesimal operation of G can be formed by a linear combination of the infinitesimal operations of G_1 and H . Adopting a phrase from the theory of groups of finite order, G may be spoken of as the *direct product* of G_1 and H .

7. This idea of the direct product of two continuous groups may at once be used to verify almost intuitively a property of G_1 , the direct proof of which is not so obvious. From the definition of the derived group it follows that the derived group of the direct product of two continuous groups is the direct product of the derived groups of the two continuous groups considered separately. Hence the derived group of G is the direct product of the derived groups of G_1 and H . Since the operations of H are all permutable, the derived group of H consists of the identical operation only; and therefore the derived group of G is the same as the derived group of G_1 ; or, in other words, G_1 is its own derived group.

The derived group G_1 can only reduce to the identical operation if r is equal to n ; that is, if every operation of g is self-conjugate, and

g therefore Abelian. Now the necessary and sufficient conditions that a continuous group H may be integrable is that the series of groups $H, H_1, H_2, \dots$, each of which is the derived group of the preceding, shall terminate with the identical operation. If this is the case for the group G , then G_1 must reduce to the identical operation. Hence G is integrable if, and only if, g is Abelian.

8. When g is not Abelian, it may be shown that G_1 , the derived group of G , is semi-simple; i.e., that it contains no self-conjugate integrable sub-group.

In a memoir with the title *Sur la Structure des Groupes de Transformations finis et continus* (Paris, 1894), M. Cartan has given (p. 109) a method by which the greatest self-conjugate integrable sub-group of any continuous group can be directly determined. The method may be stated thus. Let the characteristic equation of the group H , namely,

$$\begin{vmatrix} \sum e_i c_{i11} + \omega, & \sum e_i c_{i21}, & \dots, & \sum e_i c_{in1} \\ \sum e_i c_{i12}, & \sum e_i c_{i22} + \omega, & \dots, & \sum e_i c_{in2} \\ \dots & \dots & \dots & \dots \\ \sum e_i c_{i1n}, & \sum e_i c_{i2n}, & \dots, & \sum e_i c_{inn} + \omega \end{vmatrix} = 0,$$

be written at length in the form

$$\omega^n + \psi_1(e) \omega^{n-1} + \psi_2(e) \omega^{n-2} + \dots = 0.$$

Let $X_1, X_2, \dots, X_n$ be the n infinitesimal operations of the group, and suppose that all the infinitesimal transformations of the derived group H_1 can be expressed linearly in terms of the s operations

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n \quad (i = 1, 2, \dots, s).$$

Then, if

$$e_1X_1 + e_2X_2 + \dots + e_nX_n$$

is an infinitesimal operation of the greatest self-conjugate integrable sub-group I of H , the relations

$$a_{i1} \frac{\partial \psi_2}{\partial e_1} + a_{i2} \frac{\partial \psi_2}{\partial e_2} + \dots + a_{in} \frac{\partial \psi_2}{\partial e_n} = 0 \quad (i = 1, 2, \dots, s)$$

are satisfied; and, conversely, every infinitesimal operation for which these relations are satisfied belongs to I .

The coefficients $\psi_1, \psi_2, \dots$ in the characteristic equation may be calculated by direct expansion of the determinant. In particular*

$$\psi_1 = \sum_{i,k}^{1,2,\dots,n} e_i c_{ikk},$$

$$\psi_1^2 - 2\psi_2 = \sum_{i,j,k,l}^{1,2,\dots,n} e_i e_j c_{ikl} c_{jlk}.$$

9. For the group G $c_{ikk} = 0$ ($i, k = 1, 2, \dots, n$);
and therefore $\psi_1 = 0$.

For given values of i, j, k , the product $c_{ikl} c_{jlk}$ is different from zero only when one of the relations

$$l = ki \quad \text{or} \quad l = ik,$$

and one of the relations

$$l = kj^{-1} \quad \text{or} \quad l = j^{-1}k$$

are satisfied. Hence either

$$j^{-1} = i,$$

or

$$j^{-1} = kik^{-1},$$

or

$$j^{-1} = k^{-1}ik.$$

The term $e_i e_j$ can therefore only appear in ψ_2 , if j^{-1} is either conjugate to or identical with i .

Suppose first that $j = i$. Then the term e_i^2 will occur in ψ_2 only if i is an operation of g of order 2, or if i is conjugate to its own inverse. Let i be one of a set of x conjugate operations of order 2 in g . For a given k , $c_{ikl} c_{ilk}$ is different from zero only when either

$$l = ki \quad \text{or} \quad l = ik,$$

and either

$$k = il \quad \text{or} \quad k = li,$$

are satisfied; i and k not being permutable. Moreover, when these conditions are satisfied, c_{ikl} and c_{ilk} are either both +1 or both -1.

Now g contains $\frac{n(x-1)}{x}$ operations which are not permutable with i .

Hence the coefficient of e_i^2 in $-2\psi_2$ is $\frac{2n(x-1)}{x}$.

Suppose next that i is conjugate to its own inverse, but that it is

* Cartan, *loc. cit.*, p. 25.

not an operation of order 2. For a given k , $c_{ik}c_{uk}$ is only different from zero if either

$$l = ki \quad \text{or} \quad l = ik,$$

and either $l = ki^{-1}$ or $l = i^{-1}k$,

are satisfied. Hence k must be an operation which transforms i into its inverse; and each such k gives two admissible values of l . For each of these c_{ik} and c_{uk} have opposite signs. Moreover the number of operations which transform i into its inverse is equal to the number of operations which are permutable with i . Hence, in this case, if i is one of y conjugate operations in g , the coefficient of e_i^2 in $-2\psi_2$ is $-\frac{2n}{y}$.

By precisely similar reasoning it may be shown that, if $j = i^{-1}$, i not being an operation of order 2, then the coefficient of $e_i e_{i^{-1}}$ in $-2\psi_2$ is $\frac{4n(z-1)}{z}$, where z is the number of operations in the conjugate set to which i belongs; and that, if j is conjugate to, but not identical with i^{-1} , then the coefficient of $e_i e_j$ in $-2\psi_2$ is $-\frac{4n}{z}$.

It follows from this calculation that the value of $\frac{1}{2n} \frac{\partial \psi_2}{\partial e_i}$ is in general

$$-\frac{z-1}{z} e_{i^{-1}} + \frac{1}{z} \sum e_j,$$

where the summation is extended to all values of j which are conjugate to i^{-1} , except i^{-1} itself.

This may be written in the form

$$-e_{i^{-1}} + \frac{1}{z} \sum e_j,$$

where the summation is now extended to all values of j which are conjugate to i^{-1} .

If, however, i is an operation of g which is conjugate to its own inverse, and at the same time is not of order 2, then

$$\frac{1}{2n} \frac{\partial \psi_2}{\partial e_i} = -e_{i^{-1}} + \frac{1}{z} \sum e_j + \frac{1}{z} e_i;$$

and, if i is an operation of order 2, then

$$\frac{1}{2n} \frac{\partial \psi_2}{\partial e_i} = \frac{1}{z} \sum e_j - e_i.$$

Now the derived group of G is generated by all infinitesimal operations of the form

$$X_i - X_{i'},$$

where i and i' are conjugate operations of g . Hence, for the infinitesimal operations of the greatest self-conjugate integrable sub-group, the e 's must satisfy the relations

$$\frac{\partial \psi_2}{\partial e_i} - \frac{\partial \psi_2}{\partial e_{i'}} = 0.$$

Unless i is an operation which is not of order 2 and is conjugate to its own inverse, this gives

$$e_{i'-1} - e_{i-1} = 0.$$

In the exceptional case

$$e_{i'-1} - e_{i-1} = \frac{1}{z} (e_{i'} - e_i);$$

but corresponding to this there will be the relation

$$e_{i'} - e_i = \frac{1}{z} (e_{i'-1} - e_{i-1}),$$

derived from

$$\frac{\partial \psi_2}{\partial e_{i-1}} - \frac{\partial \psi_2}{\partial e_{i'-1}} = 0;$$

and this gives

$$e_{i'-1} - e_{i-1} = e_{i'} - e_i = 0.$$

Hence, finally, the necessary and sufficient condition that

$$e_1 X_1 + e_2 X_2 + \dots + e_n X_n$$

should belong to the greatest self-conjugate integrable sub-group of G is that

$$e_i = e_{i'},$$

if i and i' are conjugate operations of g . The greatest self-conjugate integrable sub-group of G is therefore the sub-group H generated by the self-conjugate operations. Hence G_1 contains no integrable self-conjugate sub-group of G . Now G is the direct product of G_1 and H , while every operation of H is permutable with every operation of G . If, then, g_1 is a self-conjugate sub-group of G_1 , it is also a self-conjugate sub-group of the group generated by G_1 and H , that is, of G . It follows therefore that G_1 contains no integrable self-conjugate sub-group.

The continuous group G therefore, which has been coordinated to every group g of finite order n , is the direct product of a group of permutable operations of order r , and a semi-simple group of order $n-r$; r being the number of distinct sets of conjugate operations in g .

10. In illustration of this result, let g be a dihedral group of order $2p$, where p is an odd prime. Such a group is defined by

$$a^2 = 1, \quad s^p = 1, \quad asa = s^{-1}.$$

There are $\frac{1}{2}(p+3)$ conjugate sets, viz.,

$$1; s, s^{-1}; s^2, s^{-2}; \dots; s^{\frac{1}{2}(p-1)}, s^{-\frac{1}{2}(p-1)};$$

$$a, as, as^2, \dots, as^{p-1}.$$

The semi-simple group G_1 is therefore a group of order $\frac{3}{2}(p-1)$, generated by the infinitesimal operations

$$X_s - X_{s^{-a}} \quad [a = 1, 2, \dots, \frac{1}{2}(p-1)],$$

$$X_a - X_{as^\beta} \quad (\beta = 1, 2, \dots, p-1).$$

To avoid the use of suffixes, these may be represented symbolically by

$$s^a - s^{-a} \quad \text{and} \quad a - as^\beta.$$

If G_1 has a self-conjugate sub-group I_1 , it is easy to see that it must contain infinitesimal operations in the form

$$\sum e_a (s^a - s^{-a}) \quad [a = 1, 2, \dots, \frac{1}{2}(p-1)].$$

Now

$$(a, s^a - s^{-a}) = 2(as^{-a} - as^a),$$

and

$$(as^\beta, as^{-a} - as^a) = s^{a+\beta} - s^{-a-\beta} + s^{a-\beta} - s^{-a+\beta}.$$

Hence, since I_1 is also a self-conjugate sub-group of G , it must contain

$$\sum e_a (s^{a+\beta} - s^{-a-\beta} + s^{a-\beta} - s^{-a+\beta}).$$

If β is 1, this is

$$e_2(s - s^{-1}) + \sum_r (e_{r-1} + e_{r+1})(s^r - s^{-r}) + (e_{\frac{1}{2}(p-3)} - e_{\frac{1}{2}(p-1)})(s^{\frac{1}{2}(p-1)} - s^{-\frac{1}{2}(p-1)})$$

$$[r = 2, 3, \dots, \frac{1}{2}(p-3)].$$

Suppose now that I_1 contains only one independent infinitesimal operation of the form assumed. Then

$$\frac{e_2}{e_1} = \frac{e_1 + e_2}{e_2} = \dots = \frac{e_{\frac{1}{2}(p-5)} + e_{\frac{1}{2}(p-1)}}{e_{\frac{1}{2}(p-3)}} = \frac{e_{\frac{1}{2}(p-3)} - 1}{e_{\frac{1}{2}(p-1)}} = \lambda;$$

so that λ is determined by

$$-1 = \lambda - \frac{1}{\lambda - \lambda - \dots \text{ to } \frac{1}{2}(p-1) \text{ } \lambda\text{'s}},$$

and the e 's are then given by

$$e_1 = 1, \quad e_2 = \lambda, \quad e_{r+1} = \lambda e_r - e_{r-1}.$$

Conversely, if the e 's are thus chosen, the self-conjugate sub-group I_1 contains only one infinitesimal operation of the form assumed.

Now the roots of the equation for λ are all different, for they are

$$\omega^r + \omega^{-r} \quad [r = 1, 2, \dots, \tfrac{1}{2}(p-1)],$$

where ω is a p^{th} root of unity.

There are therefore $\tfrac{1}{2}(p-1)$ infinitesimal operations of the form assumed, which are clearly independent. Each of these occurs in a self-conjugate sub-group of G_1 which contains none of the others. Suppose now that

$$I_1, I_2, \dots, I_{\frac{1}{2}(p-1)}$$

are the smallest self-conjugate sub-groups of G_1 , each of which contains one of the above $\tfrac{1}{2}(p-1)$ linearly independent infinitesimal operations. No two of these can have a common sub-group; for, if they had, it would be a self-conjugate sub-group of G_1 with no operation of the form

$$\Sigma f_a(s^a - s^{-a}),$$

and a consideration of the alternants makes it clear that no such sub-group can exist. Hence the order of none of them can exceed 3; and, since every continuous group of order 2 is integrable, the order of none of them can be less than 3.

Finally, therefore, the semi-simple group G_1 is in this case composed of $\tfrac{1}{2}(p-1)$ simple groups of order 3, no two of which contain a common operation, and each two of which are permutable.

11. The sub-group H , formed of the self-conjugate operations of G , being generated by r independent infinitesimal operations, is a linear homogeneous group of order r in n variables. Its parameter group is therefore a group of permutable operations in r variables which is completely defined by g . It may be easily verified that the sub-group H is given in finite form by the transformations

$$x'_s = a_{s-1_1}x_1 + a_{s-1_2}x_2 + \dots + a_{s-1_k}x_k + \dots + a_{s-1_n}x_n$$

$$(s = 1, 2, \dots, n),$$

where the parameters are such that

$$a_i = a_j,$$

if i and j are conjugate operations in g .

If a'_s ($s = 1, 2, \dots, n$) are the parameters of the operation of G resulting from successive transformations with parameters a_s and a'_s ; then, § 1,

$$\begin{aligned} a''_s &= a'_1 a_s + a'_2 a_{s-1} + \dots + a'_k a_{k-1} + \dots + a'_n a_{n-1}, \\ &= a'_s a_1 + a'_{s2-1} a_2 + \dots + a'_{sk-1} a_k + \dots + a'_{sn-1} a_n; \\ &\quad (s = 1, 2, \dots, n). \end{aligned}$$

If here a_s be regarded as the original variables, a''_s the transformed variables, and a'_s the parameters, the equations define the so-called *parameter group* of G .

If in these equations it be assumed that

$$a_i = a_j,$$

and

$$a'_i = a'_j,$$

where i and j are conjugate operations in g , then will

$$a''_i = a''_j,$$

and the equations will define the parameter group of H .

Suppose, in fact, that 2, 3, ..., t is a set of conjugate operations in g ; and that s, σ are any two conjugate operations of g . Then a''_s contains the set of terms

$$(a'_{s2-1} + a'_{s3-1} + \dots + a'_{st-1}) a_2,$$

and a''_σ the set $(a'_{\sigma2-1} + a'_{\sigma3-1} + \dots + a'_{\sigma t-1}) a_2$.

But, since 2, 3, ..., t , and therefore also $2^{-1}, 3^{-1}, \dots, t^{-1}$, is a set of conjugate operations, it follows that, if

$$r^{-1} s r = \sigma,$$

then the set

$$r^{-1} (s2^{-1}, s3^{-1}, \dots, st^{-1}) r$$

is identical except as regards sequence with the set

$$\sigma 2^{-1}, \sigma 3^{-1}, \dots, \sigma t^{-1}.$$

Hence the written sets of terms in a''_s and a''_σ are identical, and therefore

$$a''_s = a''_\sigma.$$

12. Since only r of the n parameters are independent, the form above written for the parameter group of H is redundant. It may be reduced as follows.

Let $k_1, k_2, \dots, k_m$ be the k^{th} of the r conjugate sets, and write

$$a_{k_1} = a_{k_2} = \dots = a_{k_m} = a_k.$$

Then $a_s'' = a_s' a_1 + \dots + (a_{sk_1-1} + a_{sk_2-1} + \dots + a_{sk_m-1}) a_k + \dots$

Let the set of operations

$$sk_1^{-1}, sk_2^{-1}, \dots, sk_m^{-1}$$

contain $\gamma_{s, k^{-1}, t}$ belonging to the t^{th} conjugate set. If s is transformed by r into σ , then each of the above set is transformed into one of the set

$$\sigma k_1^{-1}, \sigma k_2^{-1}, \dots, \sigma k_m^{-1}$$

by r ; and therefore $\gamma_{s, k^{-1}, t}$ does not depend on the particular operation s , but only on the conjugate set to which s belongs.* It is an integer, which is completely defined for each value of s, k , and t from 1 to r by the group g . In particular, if t is unity, γ must be zero or unity. In terms of this symbol the coefficient of a_k in a_s'' may be written

$$\sum_{t=1}^{t=r} \gamma_{s, k^{-1}, t} a_t',$$

and the parameter group is given by

$$a_s'' = \sum_{t,k}^{1,2,\dots,r} \gamma_{s, k^{-1}, t} a_t' a_k \quad (s = 1, 2, \dots, r).$$

This parameter group is isomorphic with the group H ; it is therefore a group of permutable operations.

13. A simple example in illustration of the last paragraph is given by the symmetric group of three symbols,

$$A^3 = 1, \quad B^3 = 1, \quad BAB = A^2.$$

There are three conjugate sets of operations, viz.,

$$1; \quad A, A^2; \quad B, BA, BA^2.$$

* The symbol $\gamma_{s, k^{-1}, t}$ differs only by a factor from the symbol $h_{s, \beta, \gamma}$ used by Herr Frobenius in a memoir "Über Gruppencharaktere" (*Berliner Sitzungsberichte*, 1896, p. 987). Herr Frobenius defines the symbol $h_{s, \beta, \gamma}$ as the number of solutions of the equation $ABC = 1$, when for A, B, C respectively are put each of the operations belonging to the $\alpha^{\text{th}}, \beta^{\text{th}}$, and γ^{th} conjugate sets in a group of finite order. If n_s is the number of operations in the s^{th} set, and if the inverses of the operations of the k^{th} and t^{th} sets form the k^{th} and t^{th} sets, then

$$n_s \gamma_{s, k^{-1}, t} = h_{s, k, t}.$$

Taking suffixes 1, 2, 3 to denote these in the order written, the sets 2^{-1} and 3^{-1} coincide with 2 and 3. Then by the multiplication table,

$$\begin{aligned}
 \gamma_{1,1,1} &= 1, & \gamma_{2,1,1} &= 0, & \gamma_{3,1,1} &= 0, \\
 \gamma_{1,2,1} &= 0, & \gamma_{2,2,1} &= 1, & \gamma_{3,2,1} &= 0, \\
 \gamma_{1,3,1} &= 0, & \gamma_{2,3,1} &= 0, & \gamma_{3,3,1} &= 1, \\
 \gamma_{1,1,2} &= 0, & \gamma_{2,1,2} &= 1, & \gamma_{3,1,2} &= 0, \\
 \gamma_{1,2,2} &= 2, & \gamma_{2,2,2} &= 1, & \gamma_{3,2,2} &= 0, \\
 \gamma_{1,3,2} &= 0, & \gamma_{2,3,2} &= 0, & \gamma_{3,3,2} &= 2, \\
 \gamma_{1,1,3} &= 0, & \gamma_{2,1,3} &= 0, & \gamma_{3,1,3} &= 1, \\
 \gamma_{1,2,3} &= 0, & \gamma_{2,2,3} &= 0, & \gamma_{3,2,3} &= 2, \\
 \gamma_{1,3,3} &= 3, & \gamma_{2,3,3} &= 3, & \gamma_{3,3,3} &= 0.
 \end{aligned}$$

Hence the parameter group of the sub-group of self-conjugate operations is expressible in the form

$$\begin{aligned}
 \alpha'' &= \alpha' + 2\beta'\beta + 3\gamma'\gamma, \\
 \beta'' &= \beta'\alpha + (\alpha' + \beta')\beta + 3\gamma'\gamma, \\
 \gamma'' &= \gamma'\alpha + 2\gamma'\beta + (\alpha' + 2\beta')\gamma.
 \end{aligned}$$

It is easy to verify that all the operations of this group are, as they should be, permutable.

[*Note (February 3rd, 1898).*—It was suggested by one of the referees that it might be desirable to add the proof of the theorem due to M. Cartan quoted in § 8. This would, however, involve too long a quotation from M. Cartan's memoir. The theorem appears to me of so great importance in the discussion of the composition of a group that I have allowed §§ 8, 9 to remain unaltered. The result to which it is applied may however be arrived at by means of another theorem given by M. Cartan in the same memoir (p. 52) which admits of ready quotation.

The theorem is that, if H , a continuous group of order n , has a self-conjugate integrable sub-group, then the discriminant of $\psi_1(e)$ (§ 8) must be zero; and therefore that, if the discriminant of $\psi_1(e)$ is not zero, H must be semi-simple.

M. Cartan's proof is as follows. Let h be an integrable self-conjugate sub-group of H ; and form the series of derived groups

of h . Each of these is a self-conjugate sub-group of H , and the last but one is certainly a group all of whose operations are permutable. Let

$$X_{m+1}, X_{m+2}, \dots, X_n$$

be the independent infinitesimal operations of this self-conjugate sub-group of permutable operations, and take these for the last $n-m$ of the independent infinitesimal operations of H . When the characteristic equation of H is formed for this set of independent infinitesimal operations, it may be at once verified that it is independent of

$$e_{m+1}, e_{m+2}, \dots, e_n.$$

Hence $\psi_2(e)$ is a function of $e_1, e_2, \dots, e_m$ only, and therefore, regarded as a function of n symbols, its discriminant vanishes. If, therefore, the discriminant of $\psi_2(e)$ does not vanish, there can be no such self-conjugate sub-group as h .

From this result it may be shown at once that G_1 (§ 8) is a semi-simple group. From the result of § 9 the value of ψ_2 for G may be written down. If in this the number of e 's is reduced to $n-r$, by taking

$$e_i + e_{i_1} + \dots + e_{i_k} = 0$$

for each set of suffixes $i_1, i_2, \dots, i_k$ which form a conjugate set in g , the resulting value of ψ_2 is the coefficient of ω^{n-r-2} in the characteristic equation of G_1 . If i_1 is an operation of g which is not conjugate to its inverse, the terms in ψ_2 which contain suffixes conjugate to i_1 are

$$-\frac{1}{n}(e_{i_1}e_{i_1^{-1}} + \dots + e_{i_{k-1}}e_{i_{k-1}^{-1}}) - \frac{1}{n}(e_{i_1} + \dots + e_{i_{k-1}})(e_{i_1^{-1}} + \dots + e_{i_{k-1}^{-1}}),$$

and these can clearly be expressed as the sum of $2(k-1)$ linearly independent squares. A similar result is easily shown to hold for the set of terms in ψ_2 which arise from a suffix which is conjugate to its inverse; and ψ_2 is thus expressed as the sum of $(n-r)$ linearly independent squares.—W. B.]