44. On a Property of Two Triangles and Its Extension to Quadrilaterals

Author(s): F. S. Macaulay

Source: *The Mathematical Gazette*, Vol. 1, No. 12 (Oct., 1897), pp. 127-129

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3604534>

Accessed: 22-06-2016 23:14 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://about.jstor.org/terms>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*

MATHEMATICAL NOTES.

44. *On a property of two triangles and its extension to quadrilaterals.*

In the following note, when we say that two triangles are similar, such as for example the two triangles $BP'C$, QPR , we mean not only that the angles B , P' , C are equal to Q , P , R respectively, but also that the rotations denoted by $BP'C$, QPR are in the same sense both positive, or both negative. The following simple theorem is no doubt well known :

If ABC , PQR be any two triangles, and on the sides of ABC three other triangles $BP'C$, $CQ'A$, $AR'B$ be described similar to QPR , RQP , PRQ respectively, then the pairs of lines AP' , QR , and BQ' , RP , and CR' , PQ have one and the same product, and are equally inclined to one and the same line.

To prove this we may take a point O within the triangle ABC such that the angles BOC , COA , AOB are the supplements of P , Q , R respectively, and produce AO , BO , CO to meet the circles BOC , COA , AOB again in P' , Q' , R' .

Then the points P' , Q' , R' are clearly the same as those mentioned in the enunciation. The angle between AP' and BQ' is the angle $P'OQ'$ which is equal in magnitude and opposite in sense to the angle between QR and RP . Hence the pairs of lines AP' , QR , and BQ' , RP , and similarly the pair CR' , PQ , are equally inclined to one and the same line.

Also, since $OBP'C$ is a cyclic quadrilateral,

$$OP' \cdot BC = OB \cdot CP' + OC \cdot BP' ;$$

and since BC , CP' , BP' are proportional to QR , RP , PQ ,

therefore $OP' \cdot QR = OB \cdot RP + OC \cdot PQ$.

Adding $OA \cdot QR$ to each side, we have

$$\begin{aligned} AP' \cdot QR &= OA \cdot QR + OB \cdot RP + OC \cdot PQ \\ &= BQ' \cdot RP = CR' \cdot PQ. \end{aligned}$$

This establishes the theorem for two triangles. To extend it to quadrilaterals we require to prove the following :

If $ABCD$, $PQRS$ be any two quadrilaterals, and triangles $AP'B$, $BQ'C$, $CR'D$, $DS'A$ be described similar to QPS , RQP , SRQ , PSR respectively, then the pairs of lines $P'R'$, SQ , and $Q'S'$, PR have the same product, and are equally inclined to the same line.

Suppose that only two of the triangles, viz., $AP'B$ and $CR'D$, are described in the first instance. Let $P'R'$ cut the circle $P'AB$ in p , and the circle $R'CD$

in r ; let Bp and Cr meet in q , and Ap and Dr in s ; also let qs meet the circle qBC in Q' , and the circle sDA in S' . Then it can be easily proved that the points Q', S' are the same as those mentioned in the enunciation. For, if O is the point of intersection of pr and qs , and P, Q, R, S inverses of p, q, r, s with respect to O , it easily follows that $PQRS$ is similar to and in the same sense as the given quadrilateral $P'Q'R'S'$, and that it may be taken to be the given

quadrilateral for the purpose of proving the theorem. Hence it follows at once that the pairs of lines $P'R', SQ$, and $Q'S', PR$ are equally inclined to the same line.

To prove that the products $P'R' \cdot SQ$ and $Q'S' \cdot PR$ are equal, we have, since $pAP'B$ is a cyclic quadrilateral,

$$P'p \cdot AB = pA \cdot P'B + pB \cdot P'A;$$

therefore

$$P'p \cdot SQ = pA \cdot PS + pB \cdot PQ;$$

similarly

$$rR' \cdot SQ = rC \cdot RQ + rD \cdot RS.$$

Adding the two last equations, and subtracting $pr \cdot SQ$ from each side, we have

$$P'R' \cdot SQ = pA \cdot PS + pB \cdot PQ + rC \cdot RQ + rD \cdot RS - pr \cdot SQ.$$

Similarly

$$Q'S' \cdot PR = sA \cdot PS + qB \cdot PQ + qC \cdot RQ + sD \cdot RS + qs \cdot RP;$$

$$\therefore P'R' \cdot SQ - Q'S' \cdot PR = ps \cdot PS + pq \cdot PQ + rq \cdot RQ + rs \cdot RS - pr \cdot SQ - qs \cdot RP.$$

This last result may be shown to vanish from the fact that p, q, r, s are the inverses of P, Q, R, S with respect to the point O where pr, qs intersect. Hence

$$P'R' \cdot SQ = Q'S' \cdot PR.$$

In order to obtain the complete theorem for the quadrilateral we have to combine the two theorems enunciated above. Thus, applying the first of the two theorems to the triangles $DP'C, QRS$, we see that if $P'D'C$ be described similar to RQS , then DD', RS have the same product and are equally inclined to the same line as $P'R', SQ$. Thus, in all, we may construct six lines $AA', BB', CC', DD', P'R', Q'S'$ corresponding to SP, PQ, QR, RS, SQ, PR respectively. Any pair, such as AA', SP , have a fixed product, and are equally inclined to a fixed line. Thus the mutual ratios and inclinations of the six lines $AA', BB',$ etc. are independent of the shape of the quadrilateral $ABCD$ from which they are derived.

An interesting particular case of the theorem is that if P', R' coincide with the same point X , then Q', S' coincide with the same point Y , and also A, A' coincide, etc. If we put $SP = a, PQ = b, QR = c, RS = d, SQ = x, PR = y$, we may enunciate this case in the following way: *If there is a point X such that the triangles XAB, XCD are similar to x, a, b and x, c, d respectively, then there is also a point Y such that the triangles YBC, YDA are similar to y, b, c*

and y, d, a respectively. The general theorem can be simply deduced from this particular case.

For an extension of the theorem to polygons with any number of sides see the *Proceedings of the London Mathematical Society*, vol. xxviii., p. 442.

F. S. MACAULAY.

45. To simplify expressions of the form

$$\frac{a^n(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^n(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^n(x-a)(x-b)}{(c-a)(c-b)}.$$

Denote the given expression by $f(x)$.

Then $f(x) - x^n$ vanishes for the three values a, b, c of x . It must also be kept in mind that $f(x)$ is a quadratic function of x .

(i.) Let $n=0, 1$ or 2 , then $f(x) - x^n$ must vanish identically, and therefore $f(x) = x^n$.

(ii.) Let $n=3$, then $f(x) - x^3 = -(x-a)(x-b)(x-c)$, and therefore $f(x) = (a+b+c)x^2 - (bc+ca+ab)x + abc$.

(iii.) Let $n=4$, then $f(x) - x^4 = -(x+a+b+c)(x-a)(x-b)(x-c)$ (since the coefficient of $x^3=0$), therefore $f(x) = (a^2+b^2+c^2+bc+ca+ab)x^2 - \{(a+b+c)(bc+ca+ab) - abc\}x + (a+b+c)abc$.

(iv.) Let $n > 4$, then $f(x) = p_{n-2}x^2 - (s_2p_{n-3} - s_3p_{n-4})x + s_3p_{n-5}$, where $s_1 = a+b+c$; $s_2 = bc+ca+ab$; $s_3 = abc$, and p_r = sum of homogeneous products of r dimensions of a, b, c .

For $f(x) - x^n = -(x^{n-3} + p_1x^{n-4} + p_2x^{n-5} \dots p_{n-3})(x^3 - s_1x^2 + s_2x - s_3)$, where $p_1, p_2, p_3 \dots p_{n-3}$ are determined by the $n-3$ equations,

$$\begin{aligned} p_1 - s_1 &= 0, \\ p_2 - p_1s_1 + s_2 &= 0, \\ p_3 - p_2s_1 + p_1s_2 - s_3 &= 0, \\ &\dots\dots\dots \\ p_{n-3} - p_{n-4}s_1 + p_{n-5}s_2 - p_{n-6}s_3 &= 0. \end{aligned}$$

Hence $(1 + p_1x + p_2x^2 \dots)(1 - s_1x + s_2x^2 - s_3x^3) = 1$,

and therefore $1 + p_1x + p_2x^2 \dots = \frac{1}{(1-ax)(1-bx)(1-cx)}$

and therefore p_r = sum of homogeneous products of r dimensions of a, b, c .

$$\begin{aligned} \text{Hence } f(x) &= (s_3p_{n-5} - s_2p_{n-4} + s_1p_{n-3})x^2 + (s_2p_{n-4} - s_3p_{n-3})x + s_3p_{n-5} \\ &= p_{n-2}x^2 + (p_{n-1} - s_1p_{n-2})x + s_3p_{n-5}. \end{aligned}$$

Similarly, if $\phi(x)$ denote

$$\begin{aligned} \frac{a^n(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + \frac{b^n(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)} + \frac{c^n(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)} \\ + \frac{d^n(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}; \end{aligned}$$

(i.) if $n=0, 1, 2$, or 3 , $\phi(x) = x^n$;

(ii.) if $n=4$, $\phi(x) = (a+b+c+d)x^3 - (ab+ac+ad+bc+bd+cd)x^2 + (bcd+cda+dab+abc)x - abcd$
 $= s_1x^3 - s_2x^2 + s_3x - s_4$;

(iii.) if $n=5$, $\phi(x) = p_2x^3 - (s_1s_2 - s_3)x^2 + (s_1s_3 - s_4)x - s_1s_4$;

(iv.) if $n > 5$, $\phi(x) = p_{n-3}x^3 + (p_{n-2} - s_1p_{n-3})x^2 + (s_3p_{n-4} - s_4p_{n-5})x - s_4p_{n-5}$, where p_r = sum of homogeneous products of r dimensions of a, b, c, d .

E. M. LANGLEY.