On the Self-inductance of Single-layer Flat Coils

1919 Proc. Phys. Soc. London 32 31

(http://iopscience.iop.org/1478-7814/32/1/304)

IV. *On the Self-inductance of Single-layer Flat Coils.* *By* S. BUTTERWORTH, *M.Sc.*, *The National Physical Laboratory.*

RECEIVED OCTOBER 1, 1919.

1. The extended use of flat inductance coils in high-frequency electrical measurements renders desirable the development of simple formulæ for the predetermination of the inductance of such coils. A modification of Weinstein's formula* for coils of rectangular channel section has usually been employed for this purpose. This formula fails, however, when the inner radius is small.

It is the purpose of the present Paper to extend the modified Weinstein formula, to supply a formula suitable for coils of small inner radius, and to use the two formulæ to compute a table (Table I.) for the use of designers of flat inductance coils. Some applications of this table are also given.

2. The method of obtaining the formulæ is one of integration, starting with a suitable formula for the mutual inductance between two coaxial co-planar circles.

Maxwell† has given two elliptic integral formulæ for the mutual inductance between two coaxial circles. When the circles are also co-planar these reduce to

$$M=4\pi\sqrt{Aa}\,\{(2/k-k)K(k)-2/kE(k)\}, \quad . \quad . \quad . \quad . \quad (1)$$

or,
$$M=8\pi A\,\{K(a/A)-E(a/A)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which K and E are complete elliptic integrals of the first and second kind respectively ; a, A are the radii of the circles, and
$$k^2=4Aa/(A+a)^2.$$
In (2) $A>a$ always.

3. *Derivation of Extended Weinstein Formula.*

Let R be the mean radius of the flat coil, $2X$ be the coil depth, and n be the number of turns per unit of length. Also let $m(x, r)$ be the mutual inductance between two co-planar coaxial circles, whose mean radius is r and radius difference is $2x$. Then by summation of the mutual inductance between

* Weinstein, "Wied. Ann.," XXI., p. 329, 1884 ; Bull. Bureau of Standards, VIII., p. 137, 1912.

† "Electricity and Magnetism," Vol. II., 701.

the elementary filaments of the flat coil the self-inductance of the latter is

$$L=4n^2\int_0^X dx\int_{R-X+x}^{R+X-x} m(x,r)dr. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)^*$$

The integral (3) may be evaluated by expressing m in terms of x and r. The required expression may be derived from (1) by developing into a series involving only k' the complementary modulus of k, for

$$k'=(1-k^2)^{\frac{1}{2}}=(A-a)/(A+a)=x/r.$$

The suitable series† is

$$M=4\pi r\left(\Phi_0+\frac{1^2}{2^2}k'^2\Phi_1+\frac{1^2.1^2}{2^2.4^2}k'^4\Phi_2+\frac{1^2.1^2.3^2}{2^2.4^2.6^2}k'^6\Phi_3+\ldots\right) \qquad (4)$$

in which

$$\left.\begin{aligned}\Phi_0&=\log_e 4/k'-2\\ \Phi_1-\Phi_0&=\frac{1}{1}+\frac{2}{1}\\ \Phi_2-\Phi_1&=\frac{1}{2}-\frac{2}{1}\\ \Phi_3-\Phi_2&=\frac{1}{3}-\frac{2}{3}\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \Phi_n-\Phi_{n-1}&=\frac{1}{n}-\frac{2}{2n-3}\end{aligned}\right\}.$$

Applying (3) to the first four terms of (4)

$$L=16\pi n^2RX\left\{(\lambda-\tfrac{1}{2})+\tfrac{2}{3}z^2\left(\lambda+\frac{43}{12}\right)+\frac{44}{45}z^4\left(\lambda+\frac{96}{55}\right)\right.$$
$$\left.+\frac{412}{105}z^6\left(\lambda+\frac{98579}{86520}\right)+\ldots\right\} \qquad (A)$$

in which

$$\lambda=\log_e 4R/X,\ z=X/4R.$$

The first two terms of this series constitute the modified Weinstein formula.

The series converges for all possible values of X and R.

* For the method of building up this integral, *see* Butterworth, "Proc." Phys. Soc., XXXVII., p. 376, 1915.

† Bromwich, "Quarterly Journal of Pure and Applied Mathematics," No. 176, p. 363, 1913; Butterworth, "Phil. Mag.," XXXI., p. 276, 1916.

In the worst case when the inner radius is zero $X/R=1$, and the value of the terms in $\{\ \}$ of (A) are then

0·886 0·207 0·012 0·002.

Thus, assuming the four terms evaluated approximate sufficiently closely to the true sum, the Weinstein formula gives a result about 1 per cent. low in the extreme case.

To verify this result a formula is now developed which is suitable for large values of X, that is, for a small inner radius of the flat coil.

4. Using Maxwell's second formula $\{(2)$ above$\}$ and putting G for $K-E$, the flux Φ through a circle of radius x in the plane of the coil is

$$8\pi n\left\{x\int_{r_1}^{x} G(a/x)da+\int_{x}^{r_2} aG(x/a)da\right\},$$

in which r_1, r_2 are the inner and outer radii, or by a change of variables

$$\Phi=8\pi nx^2\left(\int_{r_1/x}^{1} G(\mu)d\mu+\int_{x/r_2}^{1} G(\mu)d\mu/\mu^3\right). \quad . \quad . \quad . \quad (5)$$

The self-inductance of the flat coil is

$$L=n\int_{r_1}^{r_2} \Phi dx,$$

so that, integrating (5) by parts

$$L=\frac{16}{3}\pi n^2 r_2{}^3\int_{\alpha}^{1} (1-\alpha^3/\mu^3)G(\mu)d\mu, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which $\alpha=r_1/r_2$.

The evaluation of (6) involves integration of K and E with respect to the modulus. The required integrations have been discussed by the author in an earlier Paper.*

Using the notation and results obtained therein, (6) reduces to

$$L=\frac{8}{3}\pi n^2 r_2{}^3\{u_1-1-\alpha^3(1+v)-u+2\alpha E-\alpha(1-\alpha^2)K\}, \quad . \quad . \quad (7)$$

in which

$$u=\int_0^{\alpha} Kd\mu, \quad v=\int_{\alpha}^{1} Kd\mu/\mu,$$

$$u_1=\int_0^{1} Kd\mu=1\cdot831931248 \quad . \quad . \quad .$$

* "Phil. Mag.," XXIX., p. 584, 1915.

(7) may be expressed in series by making use of the series for K, E, u, v, giving

$$L=\frac{4}{3}\pi^2 n^2 r_2^3[2(u_1-1)/\pi+\alpha^3\{2(u_1-1)/\pi+\tfrac{1}{6}-\log 4/\alpha+\sigma\}] \quad (B)$$

in which

$$\sigma=6\sum_1^{\infty}\left(\frac{1\,.\,3\,.\,5\,\ldots\,2n-1}{2\,.\,4\,.\,6\,\ldots\ldots\,2n}\right)^2\frac{2n+1}{2n(2n+1)(2n+3)}\alpha 2n.$$

This formula is most convergent when α is small and is thus suitable for coils having a small inner radius.

5. For the case of zero inner radius formula (B) gives

$$L=6{\cdot}96957n^2r_2^3,$$

formula (A) gives

$$L=6{\cdot}956n^2r_2^3,$$

while the first two terms only give

$$L=6{\cdot}87n^2r_2^3.$$

Thus, in the most unfavourable case for formula (A) the error is of the order 0·2 per cent., while the Weinstein formula gives a result 1·5 per cent. below the true value. For the case $\alpha=0{\cdot}5$

$$L=4{\cdot}743502n^2r_2^3 \text{ by } (A),$$

$$L=4{\cdot}743500n^2r_2^3 \text{ by } (B).$$

The two formula thus check each other.

For general computation formula (A) usually the most rapid to work with, while formula (B) may be used as a check formula.

6. For practical calculations we may write

$$L=Qn^2r_2^3, \quad \ldots\ldots\ldots\ldots \quad (C)$$

and tabulate Q as a function of r_1/r_2. This is done in Table I.

TABLE I.

r_1/r_2.	Q.	r_1/r_2.	Q.	r_1/r_2.	Q.
0·00	6·970	0·35	5·996	0·70	2·528
0·05	6·964	0·40	5·632	0·75	1·946
0·10	6·930	0·45	5·213	0·80	1·397
0·15	6·845	0·50	4·743	0·85	0·8892
0·20	6·728	0·55	4·231	0·90	0·4574
0·25	6·544	0·60	3·682	0·95	0·1394
0·30	6·300	0·65	3·105	1·00	0·0000

Self-inductance in centimetres $=Qn^2r_2^3$.
r_1=inner radius (centimetres). n=turns per centimetre.
r_2=outer radius (centimetres).

7. *Flat Coil of Best Time Constant.*

The (direct current) resistance of a flat coil wound closely with circular wire is

$$4\rho n^3(r_2^2 - r_1^2) = 4\rho n^3 r_2^2(1-\alpha^2),$$

ρ being the resistivity of the wire, so that the time constant is proportional to $Qr_2/n(1-\alpha^2)$.

Also for a given length and section of wire $r_2^2(1-\alpha^2)$ and n are fixed, and therefore (by eliminating r_2) the time constant is proportional to $Q/(1-\alpha^2)^{3/2}$. By calculating this quantity for various radius ratios we obtain the variation of time-constant. It is found (*see* Table II.) that the maximum time constant occurs when the inner radius is about 2/5 the outer radius. The maximum is, however, very flat, the coil having 90 per cent. of the maximum efficiency if $r_1/r_2 < 0{\cdot}7$.

TABLE II.

$\alpha=0{\cdot}0$	0·1	0·2	0·3	0·4	0·5
$Q/(1-\alpha^2)^{3/2}=6{\cdot}97$	7·03	7·15	7·25	7·32	7·30
$\alpha=0{\cdot}6$	0·7	0·8	0·9	1·0	
$Q/(1-\alpha^2)^{3/2}=7{\cdot}18$	6·94	6·46	5·50	0·00	

The best time constant for a single layer *cylindrical* coil is obtained when the length (b) is 4/5 of the radius (a), and in that case $L=14{\cdot}90n^2a^3$, n again being the number of times per centimetre. For a flat coil of the best time constant

$$L' = 5{\cdot}632n^2r_2^3.$$

To compare the two cases we require the radius a of the cylindrical coil which can be wound with the same length of wire as the flat coil. This is $a=0{\cdot}725r_2$, with $r_1=0{\cdot}4r_2$.

Using this $$L=5{\cdot}666n^2r_2^3.$$

The cylindrical coil is thus slightly better than the corresponding flat coil.

8. *Mutual Inductance between Coaxial Coplanar Flat Coils.*

The mutual inductance between two coaxial flat coils in the same plane can be obtained from Table I. as follows :—

Let the inner and outer radii of the $\left\{\dfrac{\text{inner}}{\text{outer}}\right\}$coils be

$$\left\{\frac{r_1,\ r_2}{r_3,\ r_4}\right\}.$$

Denote the self-inductances of the two coils by L_α, L_γ, and that of the coil which would fill the interspace by L_β.

Denote the mutual inductances between these three coils by $M_{\alpha\beta}$, $M_{\beta\gamma}$, $M_{\gamma\alpha}$. Further, let L_α, L_β in series have self-inductance L_A; L_β, L_γ in series have inductance L_B; L_α, L_β, L_γ in series have inductance L_C.

Then $L_A=L_\alpha+L_\beta+2M_{\alpha\beta}$, $L_B=L_\beta+L_\gamma+2M_{\beta\gamma}$, $L_C=L_\alpha+L_\beta+L_\gamma+2M_{\alpha\beta}+2M_{\beta\gamma}+2M_{\gamma\alpha}$, from which the three M's can be found since the L's are known from Table I.

In particular

$$M_{\alpha\gamma}=\tfrac{1}{2}(L_C-L_A-L_B-L_\beta),$$

or, in terms of Q

$$M_{\alpha\gamma}=\tfrac{1}{2}n^2\{r_4{}^3(Q_{r_1/r_4}-Q_{r_2/r_4})-r_2{}^3(Q_{r_1/r_3}+Q_{r_2/r_3})\}.$$

A simple case is where there is no interspace, and the inner coil has zero inner radius. Then

$$M=\tfrac{1}{2}n^2R^3\{Q_0(1-\alpha^3)-Q_\alpha\},$$

in which $\alpha=r/R$, R is the outer radius, r the dividing radius.

In order to give some idea of the magnitude of the various inductances, the following table has been calculated from the above formula :—

TABLE III.—*Mutual-inductances Between Flat Coils.*

α.	L_1/n^2R^3.	L_2/n^2R^3.	M/n^2R^3.	k.
0·1	0·00697	6·93	0·0162	0·074
0·2	0·0557	6·73	0·0930	0·152
0·3	0·1880	6·30	0·240	0·220
0·4	0·446	5·63	0·446	0·280
0·5	0·871	4·74	0·678	0·333
0·6	1·503	3·68	0·892	0·379
0·7	2·39	2·53	1·025	0·421
0·8	3·56	1·397	1·005	0·451
0·9	5·08	0·457	0·715	0·470
0·92	5·23	0·317	0·611	0·474
0·94	5·79	0·1910	0·495	0·470
0·96	6·17	0·0942	0·356	0·465
0·98	6·56	0·0272	0·192	0·454
1·00	6·97	0 0000	0·000	0·000

L_1=Self-inductance of inner coil.
L_2=Self-inductance of outer coil.
M=Mutual inductance between coils.
k=Coefficient of coupling=$M/\sqrt{L_1L_2}$.
r=Dividing radius.
$\alpha=r/R$. n=Turns per centimetre.

ABSTRACT.

Two formulæ are established for the computation of the self-inductance of single layer flat coils, one for the case when the inner and outer radii are not very different and the other for the case of small inner radius. The two formulæ are shown to be consistent and capable

of including all possible cases. From the formula a table is calculated which enables the inductance to be expressed in the form $L=Qn^2r^3$, in which n is the number of turns per cm., r the outer radius and Q is a tabulated function of the ratio of the inner and outer radii. Some applications of the table are given.

DISCUSSION.

Dr. Eccles recalled the very useful Paper published by the author some years ago, giving formulæ for thick coils. He had found these extremely useful, and felt sure that the results of the present Paper would prove equally valuable. He felt that people hardly realised their indebtedness to those who undertook these very laborious calculations for the help of the users of coils.

Dr. E. H. Rayner pointed out how useful such formulæ were in the construction of inductances. Almost anyone could make up a resistance to have any desired value; but it was another matter with regard to inductances. With formulæ such as these, however, it was possible with the simplest appliances to construct an inductance to within 1 per cent. of the required value.

Prof. G. W. O. Howe communicated the following: In the "Archiv für Elektroteknik," Vol. 3, 1915, page 187, is a Paper by J. Spielrein, entitled "The Self-Induction of Air-Core Spirally-wound Coils" (*see* "Science Abstracts," Elec. Eng., 1915, No. 660). Spielrein's final formula is $L=T^2r_2f(r_1/r_2)$, where T is the total number of turns, whereas Mr. Butterworth's formula is $L=n^2r^3Q$, where n is the turns per cm. Since $T=n(r_2-r_1)$ Spielrein's formula can be written $L=n^2(r_2-r_1)^2r_2f(r_1/r_2)$. Hence $Q=(r_2-r_1/r_2)^2f(r_1/r_2)=(1-r_1/r_2)^2f(r_1/r_2)$.

Now Spielrein gives a table of $f(r_1/r_2)$ to seven places for all values of r_1/r_2 from 0 to 1 by steps of 0·01. Comparing Spielrein's $f(r_1/r_2)$ with Mr. Butterworth's Q we have:—

r_1/r_0.	$f(r_1/r_2)$.	$(1-r_1/r_2)^2f(r_1/r_2)$.	Q.
0·0	6·969573	6·969573	6·970
0·5	18·97400	4·74350	4·743
0·9	45·74241	0·4574241	0·4574
1·0	∞	$0\times\infty$	0

It seems obvious that Mr. Butterworth was unaware of this Paper.

The Author, in reply, said he had not known that Spielrein had tackled the problem. He was afraid it rendered his Paper somewhat unnecessary.

Dr. Eccles pointed out that it was of great importance that two independent investigators obtained results in such striking agreement.