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Source: *The Mathematical Gazette*, Vol. 3, No. 53 (Oct., 1905), pp. 229-231
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3602359
Accessed: 25-12-2015 03:49 UTC

elements does afford an intuition of a kind, though not analogous in the same immediate way as ordinary solid geometry is to the ordinary geometry of the plane. But with this reservation the passage quoted truly represents the proper way of regarding geometry of four or more dimensions. The power which this "artificial intuition" gives to those who cultivate it is really remarkable; an illustration has been recently given by Mr. Richmond, whose way of obtaining the Kummer (16, 6) configuration is summarised on pp. 129-30 of Hudson's book.

Naturally, Klein's applications of line geometry receive considerable attention. In many respects the fundamental properties of Kummer's Surface, and of its degenerate forms, are closely connected with a family of quadratic complexes; and this is one direction in which further study seems likely to be fruitful. Another is in connection with the application of theta-functions. It is proved by means of them (pp. 184-5; cf. pp. 138-40) that a surface can be found to touch Kummer's Surface all along any given algebraic curve lying thereon, and have no further intersection with the surface. Now this is a purely algebraic theorem, and has no intrinsic connection with theta-functions; it ought, therefore, to be possible to prove it without their aid. Hudson is possibly right in supposing that such a proof would be long and complicated (p. 185); but to provide it would be a real step in advance, and could hardly fail to bring out some characteristic properties of the surface.

No object would be gained by analysing the contents of this treatise in detail: it will be enough to say that, besides the general Kummer Surface, the degenerate forms receive due attention; that conscientious reference is made to the original sources; and that a photograph of a plaster model of the general surface, prefixed to the volume, will greatly help the reader to understand the account given of its geometrical properties. The duty of seeing the later part of the work through the press has been performed by Dr. H. F. Baker and Mr. H. Bateman; a prefatory note by the former contains a brief account of Hudson's career, and a sympathetic appreciation of the merits of this treatise.

G. B. MATHEWS.

Manual of Quaternions. By Professor C. J. JOLY. (Macmillan & Co., 1905.)

Tait has said somewhere that Hamilton first invented the quaternion and then discovered it. The invention consisted in the conception of the three imaginaries i, j, k, with their special laws of combination, and in the construction of an associative algebra with four fundamental units. The quaternion was then discovered to be a complex number representing the ratio of two vectors or directed lines in space. Hamilton was the first who clearly recognised the value of the associative law; and in spite of many imitations his system remains the only tridimensional system of vector analysis governed by this law.

The mathematical world has, however, been slow to recognise the essential merits of Hamilton's calculus. A vast deal of ingenuity and time has been spent by some in finding new notations of the quantities and operators peculiar to quaternions; some have even been beguiled by these notations into a rediscovery of theorems as old as Hamilton's

Lectures. The curious thing is that it does not appear that any really fresh mathematical truth has been brought to light by the inventors and users of vector notations intended to supersede Hamilton's original system. Here, of course, we refer only to tridimensional applications.

It is very refreshing then to open the pages of a book whose author, boldly accepting the form of the calculus as Hamilton developed it, proceeds to unfold its beauties and strength with all the skill of a practised hand. The book begins with a very brief chapter on the addition and subtraction of vectors, a part which necessarily occupied a considerable section of Hamilton's and Tait's treatises. The vector conception has now crept into our elementary books, and in due course will probably become a conspicuous feature even of our most elementary geometries. Graphical methods have within recent years transformed our teaching of algebra; and the vector as a geometrical entity is essentially graphical.

Professor Joly wisely assumes that for students who have made some progress in mathematics the law of vector addition calls for little elaboration. He devotes five pages to it and then plunges into quaternions proper. He takes what he believes to be the "shortest and simplest route." The student he says "cannot be expected to undertake the study of quaternions in the hope of being rewarded by the beauty of the ideas and by the elegance of the analysis. And for his sake, though with reluctance I must confess, I have abandoned Hamilton's methods of establishing the laws of quaternions." Perhaps so, but it is a poor student who despises logical development, beauty of idea, elegance of method, and is content with a "working knowledge of the calculus." We confess to an uneasy feeling that principle has here been sacrificed to expediency. There is a suggestion of "tumbling over the wall" and not coming "in at the Gate which standeth at the beginning of the way." We have often wondered how many students study Clifford's Dynamic for the sake of learning dynamics. Possibly none; and probably as few have *learned* quaternions from that book. Although there is some initial similarity between Professor Joly's method and the tentative nibbling at quaternions which characterises Clifford's book, there is almost immediately a vast divergence. The true quaternion is introduced on page 9, and it dominates the whole treatise. This is as it ought to be. It is possible, as Heaviside has shown, to use effectively much of the notation of quaternions without explicit use of the quaternion itself; but sooner or later the lack of it will be felt. The student should never lose sight of the fact that $S\alpha\beta$ and $V\alpha\beta$ together form by addition a quantity which, however it may operate on or be taken in conjunction with a like quantity, gives rise to a quantity of the same analytical nature. This is the central doctrine of the quaternion calculus.

To give any complete idea of all that Professor Joly's volume contains would be practically to reproduce his table of contents. Beginning with the simpler applications to trigonometry and to the geometry of plane and sphere, he quickly passes into the peculiarities of quaternion differentiation and into the exquisite theory of the linear vector function (or matrix), after which he is ready for all kinds of applications in geometry of curves and surfaces, and in kinematics and dynamics. The reader cannot fail to be impressed with the directness of the method in all these applications, especially if he is familiar with the ordinary modes of attack. In virtue of this directness of attack and the extraordinary conciseness of notation more detail can be packed into one quaternion page than into three or four pages of ordinary analysis. By what other method, for example, could systematic discussions of line, surface and volume integrals, of spherical harmonics, heterogeneous strain, elastic vibrations, and electro-magnetic theory be given in less than fifty pages? In the variety of the mathematical and physical subjects taken up there are only two other books which can compare with Professor Joly's *Manual*, and these are Hamilton's *Elements* and Tait's *Treatise*.

The greater part of the book is necessarily a development of much that is to be found in the pages of Hamilton, Tait, and M'Aulay; but Professor Joly has a characteristic style of his own, more nearly akin to Hamilton's than to Tait's. In the last two chapters especially are the author's additions more in evidence. These are on Projective Geometry and Hyperspace. The former is based upon a new interpretation of the quaternion; and in the latter Professor Joly gives a sketch of the properties of associative algebras applicable to n-dimensional space.

Professor Joly has certainly succeeded in his aim of providing the student with a *working* book. He takes excursions into many fields of mathematics pure and applied, and the treatment is not superficial. Important applications are worked out in detail; and numerous examples are given by which the student may test his progress. Let the reader accept on trust the initial assumptions and developments, and work earnestly through the succeeding chapters. He will come out in the end a practised quaternionist. C. G. KNOTT.

Leçons sur les fonctions de variables réelles, par E. BOREL; **Leçons sur les fonctions discontinues**, par RENÉ BAIRE; **Le calcul des residus et ses applications à la théorie des fonctions**, par E. LINDELÖF. (Paris, Gauthier-Villars, 1905, 3 f. 50 c. each.)

M. Borel's book is the sixth of his series of monographs on the theory of functions, of which the first appeared as recently as 1898. M. Borel is only human, and by now he has decided to leave to his pupils the work of preparing his lectures for publication. It cannot be said that this method has proved in every case an unqualified success. The first few volumes, prepared by M. Borel himself, and particularly the admirable *Leçons sur les fonctions entières*, were remarkable alike for their originality, for the judgment shewn in the selection of material, and for the lucidity and proportion of the exposition. This high standard has not been maintained in all of the later volumes, some of which have been rather scrappy, and have given the impression of hasty, and at times perfunctory composition. In these respects, however, the present volume is an improvement upon its immediate predecessors, M. Maurice Fréchet having performed his task unusually well. But I cannot help thinking that M. Borel would be fortunate if he could find the time to write his books himself.

The principal problem with which M. Borel deals in this volume is