be altered, and may under these circumstances contain a term depending on the magnetic force.

[February 3rd, 1892.

Since this paper was read, a paper has been communicated to the Society by Mr. Larmor, on "Gyrostatically Loaded Media," which suggests a theory which might possibly explain Kerr's experiments.]

On the Classification of Binodal Quartic Curves. By HENRY M. JEFFERY, F.R.S. Received June 18th, 1891. Read November 12th, 1891.*

1. The general form of their equation, if the nodes are real, is

$$u^3 (\xi x + \eta y - 1) = x^2 y^2 + u^3 (fx^2 + 2gxy + hy^2) \dots\dots (A).$$

The line $u (\equiv cx + dy - 1)$ connects the nodes, at which the pairs of tangents are $x^2 + hu^2$, $y^2 + fu^2$, and may be real, coincident, or imaginary. Six cases occur, according as h, f are negative, zero, or positive. It is proposed to examine all possible forms of (A), which can arise from variations of the parameters ξ, η . They are classified with reference (1) to their critical values, which yield a third double point, and (2) to those values of ξ, η , which correspond to points of undulation or folium-points.

2. Binodal quartics may be bipartite or unipartite, and have a maximum number of four inflexions.

A very instructive figure of a trinodal quartic is drawn by Plücker, and reproduced by Dr. Salmon with comments (*Higher Plane Curves*, p. 212). The two companion-figures are also drawn, the one unipartite and external, and the other quadripartite and internal, for small variations of a given parameter. Two real inflexions are seen in the trinodal, eight in each of the non-singular, quartics. When over two branches unite to form a node, two inflexions are lost, even if one remains at the node, as is the case when three inflexions pre-existed. Hence it follows that a binodal quartic has at most four inflexions.

* This paper, which has not been revised by the Author, is edited by the Secretaries.

Zeuthen based his conclusions, respecting the maximum number of eight inflexions, on Plücker's general form of a quartic curve,

$$xyz u + \kappa \phi^2 = 0,$$

in which (x, y, z, u) denote bitangents (of the first species), whose points of contact lie on the same conic (ϕ) . If the quartic be non-singular, each such bitangent touches a folium, or indentation in the curve circuit, characterized by two inflexions; moreover, two points of contact may unite, and become folium-points, or points of undulation.

The reader is also referred to a series of bicrunodal quartics, with their companion-curves, which illustrates the nodal forms, the four inflexions, and the conic complex of the quartic

$$(x^2 - a^2)^2 + (y^2 - b^2)^2 = c^2$$

(*Higher Plane Curves*, p. 43).

3. The critical values will be first considered.

For a third singularity,

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0,$$

if ϕ denote the quartic system (A).

$$\left. \begin{aligned} \xi &= \frac{2}{u^3} xy^2 - \frac{3}{u^4} cx^2y^2 + \frac{2}{u} (fx + gy) - \frac{c}{u^2} (fx^2 + 2gxy + hy^2) \\ \eta &= \frac{2}{u^3} x^2y - \frac{3}{u^4} dx^2y^2 + \frac{2}{u} (gx + hy) - \frac{d}{u^2} (fx^2 + 2gxy + hy^2) \end{aligned} \right\} \dots (B),$$

$$-1 = \frac{3}{u^4} x^2y^2 + \frac{1}{u^2} (fx^2 + 2gxy + hy^2) \dots\dots\dots (C).$$

By the aid of (C), the parameters may be written

$$\left. \begin{aligned} \xi &= \frac{2}{u^3} xy^2 + c + \frac{2}{u} (fx + gy) \\ \eta &= \frac{2}{u^3} x^2y + d + \frac{2}{u} (gx + hy) \end{aligned} \right\} \dots\dots\dots (B).$$

Every point in (C) is a singular point in some curve in the (A) system, whose parameters are these functions of its coordinates: this subsidiary curve (C) is the locus of nodes in (A).

The curve (B) may be considered as the envelop of the satellite-line (ξ, η) , when x, y are eliminated from (C) and (B). The curve

(B), when drawn, not only exhibits critical values, but *discriminates* other values of ξ, η in their vicinity, which indicate either separate branches in the companion-curves (A), or the same circuits with inflexions. See below, §§ 45, 47, for illustrations of its use in a bicuspidal system. The satellite-line of u , viz., $\xi x + \eta y - 1$, has geometrical significance in bicuspidal quartics (A), when $f = h = 0$.

4. A second subsidiary curve (D) is necessary to discriminate the points in (C) which denote crunodes, cusps, and acnodes in (ϕ), the (A) system. (D) is the function

$$\left(\frac{d^2\phi}{dx dy} \right)^2 - \frac{d^2\phi}{dx^2} \cdot \frac{d^2\phi}{dy^2}.$$

By differentiating twice,

$$\begin{aligned} \frac{d^3\phi}{dx^3} &= \frac{2y^3}{u^3} - \frac{12}{u^4} cxy^3 + \frac{12}{u^5} c^2x^2y^3 + \frac{2f}{u} - \frac{4c}{u^2} (fx + gy) \\ &\quad + \frac{2}{u^3} c^3 (fx^2 + 2gxy + hy^2), \\ \frac{d^3\phi}{dy^3} &= \frac{2x^3}{u^3} - \frac{12}{u^4} dx^2y + \frac{12}{u^5} d^2x^2y^2 + \frac{2h}{u} - \frac{4d}{u^2} (gx + hy) \\ &\quad + \frac{2}{u^3} d^3 (fx^2 + 2gxy + hy^2), \\ \frac{d^3\phi}{dx dy} &= \frac{4}{u^3} xy - \frac{6xy}{u^4} (cx + dy) + \frac{12}{u^5} cdx^2y^2 \\ &\quad + \frac{2g}{u} - \frac{2}{u^3} [d(fx + gy) + c(gx + hy)] + \frac{2}{u^3} cd(fx^2 + 2gxy + hy^2). \end{aligned}$$

When these values are substituted in (D), and reductions are made (most conveniently by separating the terms involving f, g, h), this function takes the simple form

$$(g^2 - fh) u^4 + 3x^2y^2 - u^2 (fx^2 - 4gxy + hy^2) \dots\dots\dots (D).$$

Critical values in (A) will indicate crunodes, cusps, or acnodes, according as the function (D) $>, =, <, 0$.

5. Accordingly, in this investigation, the subsidiary quartics (C) and (D) are drawn so that, when they intersect each other, cusps are determined in (A), and acnodes and crunodes in points on (C) inside and outside the intersections respectively. Since (C) and (D) are also generally binodal—possessing generally the same nodes as

(A), but of opposite species—they intersect in eight points besides the common nodes, and consequently there may be eight cusps in the (A) system for the same constants f, g, h .

6. If the curves (C) and (D) touch each other, the four points of quadruple contact are indicated by parameters ξ, η , which resolve four binodal quartics (A) into nodal cubics and straight lines.

In such a complex the linear factor represents a tangent at the missing node, and touches the cubic in the given point of contact. Since this contact is imaginary when the (A) system is acnodal, write the subsidiary curves with the coefficients f, h negative.

$$u^4 + 3x^2y^2 - u^2 (fx^2 + 2gxy + hy^2) = 0 \dots\dots\dots (C),$$

$$(g^2 - fh) u^4 + 3x^2y^2 + u^2 (fx^2 - 4gxy + hy^2) = 0 \dots\dots\dots (D).$$

By addition, two conics are seen to pass through their eight intersections (apart from the eight intersections of the two nodes)

$$u^4 (1 + g^2 - fh) - 6gu^2xy + 6x^2y^2 = 0.$$

These coincide in pairs, if $g^2 = 2 - 2fh$, when $gu^2 = 2xy$.

By subtraction, $(1 + fh) u^2 - 2 (fx^2 + hy^2) = 0$,

or $(2fx - gy)(2hy - gx) = 0$,

which denote two radii vectores through the four points of contact. The curves (C) and (D) may now be written, so as to exhibit quadruple contact,

$$(gu^2 - 2xy)(2u^2 - 3gxy) + u^2 (2fx - gy)(2hy - gx) = 0 \dots\dots (C),$$

$$(gu^2 - 2xy) [(3g^2 - 2) u^2 - 3gxy] - u^2 (2fx - gy)(2hy - gx) = 0 \dots\dots (D).$$

For the first pair of points of contact $\left(\frac{x}{g} = \frac{y}{2f} = \frac{u}{2f^{\frac{1}{2}}} \right)$, the parameters (by § 3), are

$$\xi = c - 2gf^{\frac{1}{2}}, \quad \eta = d - \frac{1}{f^{\frac{1}{2}}} (1 + fh).$$

The corresponding quartic in (A) is the complex

$$u^2 \left[u^3 - \frac{1}{f^{\frac{1}{2}}} uy (1 + fh) + hy^3 \right] - 2gu^2x (f^{\frac{1}{2}}u - y) + x^2 (fu - y^2) = 0.$$

The line $(f^{\frac{1}{2}}u = y)$ meets the nodal cubic

$$u^2 \left(\frac{u}{f^{\frac{1}{2}}} - hy \right) - 2gu^2x + x^2 (f^{\frac{1}{2}}u + y) = 0$$

in the site of the node ($y = 0, u = 0$), and touches it in the above-named point of contact of (C) and (D).

For the second pair of points of contact of (C) and (D)

$$\left(\frac{x}{2h} = \frac{y}{g} = \frac{u}{2h^1}\right);$$

the corresponding complex is

$$(h^1u - x) \left[u^2 \left(\frac{u}{h^1} - fx \right) - 2gu^2y + y^2(h^1u + x) \right] = 0.$$

The line meets and touches this nodal cubic, similarly to the former. There are four such complexes found by giving both signs to f^1, h^1 .

COR.—If f, h are both or singly negative, the linear factors are imaginary, and no complexes occur.

7. If the subsidiary curves (C) and (D) have double contact of the third order, the radii vectores through their points of contact coincide.

When this occurs, $g^2 = \frac{c}{3} = 4fh$.

With these values, (C) and (D) may be written, to show double contact at the pair of points ($f^1x = h^1y = f^1h^1u$),

$$3(xy - f^1h^1u^2)^2 - u^2(f^1x - h^1y)^2 = 0 \dots\dots\dots(C),$$

$$3(xy - f^1h^1u^2)^2 + u^2(f^1x - h^1y)^2 = 0 \dots\dots\dots(D).$$

The complex (A) has the parameters

$$\xi = c - 2gf^1, \quad \eta = d - 2gh^1,$$

and consists of two lines, which were tangents at the nodes, and a conic, in which those lines intersect each other.

The quartic may take the two forms

$$u^4 - 2gu^3(f^1x + h^1y) + u^2(fx^2 + 2gxy + hy^2) - x^2y^2 = 0 \dots\dots(A),$$

$$(x - h^1u)(y - f^1u) \left[(x - h^1u)(y - f^1u) + 2f^1u(x - h^1u) + 2h^1u(x - f^1u) \right] = 0 \dots\dots\dots(A).$$

Two complexes are found, by giving both signs to f^1, h^1 .

COR.—Since the coordinates of every other point in (C), except the points of contact, render (D) positive, all other points in (C), (by § 4) are crunodes in the (A) system.

8. Two radii-vectores may unite, without necessary contact of the subsidiary curves (C) and (D).

In the equations of § 6, formed by addition and subtraction,

$$u^4(1+g^2-fh)-6gxyu^2+6x^2y^2=0,$$

$$u^2(1-g^2+fh)-2fx^2+2gxy-2hy^2=0,$$

write $u^2 = rxy$, so that

$$r^2(1+g^2-fh)-6rg+6=0.$$

The vectors will coincide, if

$$r^2(1+fh-g^2)^2+4rg(1+fh-g^2)+4(g^2-4fh)=0.$$

The elimination of r determines the condition of coincidence

$$\begin{aligned} & \{ (g^2+5fh)(g^2-fh)-8g^2+14fh+3 \}^2 \\ & = 12(3fh-1)g^2(g^2-fh-1)(g^2-fh-5). \end{aligned}$$

There are therefore eight cases to be examined for given values of f, h .

In general, cuspidal trinodal quartics (A) correspond; but if (C) and (D) touch each other quadruply, or doubly as in § 7, the quartics (A) are resolved into complexes.

9. This quartic in (g^2) , when expanded, bears the form

$$\begin{aligned} g^8-4(7fh+1)g^6+2(39f^2h^2+78fh-1)g^4-4(19f^3h^3+3f^2h^2+77fh-3)g^2 \\ + (fh-3)^2(5fh+1)^2=0 \end{aligned}$$

The invariants of this quartic are

$$I_4 = {}^n_3^4 (81f^3h^3-9fh+1),$$

$$I_6 = -256(81f^4h^4-45f^3h^3+9f^2h^2-fh+\frac{2}{3}).$$

The discriminant $(I_4^2-27I_6^2)$, after rejecting the factor 8^5 , is

$$(3fh-1)^5(4fh-1).$$

10. When $3fh=1$, $(g^2-\frac{1}{3})^2(g^2-\frac{1}{3})^2=0$; two sets of values of g^2 , and therefore of radii vectores, coincide.

Case (1), when $g^2 = \frac{1}{3}$, is considered in § 7.

Case (2), when $g^2 = \frac{1}{3}$: (C) and (D) may be written to show

double contact with each other, and with the conic ($u^2 = \frac{1}{3}\sqrt{3}xy$); (C) and (D) also intersect in four other points,

$$(u^2 - \frac{1}{3}\sqrt{3}xy)(u^2 - 3\sqrt{3}xy) - u^2(x\sqrt{f} - y\sqrt{h})^2 = 0 \dots (C),$$

$$(u^2 - \frac{1}{3}\sqrt{3}xy)(5u^2 - 3\sqrt{3}xy) + u^2(x\sqrt{f} - y\sqrt{h})^2 = 0 \dots (D).$$

The corresponding quartics are resolved into two conics

$$u^2 [\pm 2u + x\sqrt{f} + y\sqrt{h}]^2 - (xy - u^2\sqrt{3})^2 = 0 \dots (A).$$

11. When $4fh = 1$, the quartic of § 9 becomes

$$(g^2 - \frac{2}{3})^2 [g^2 - (\frac{1}{2} \pm \sqrt{3})^2] = 0.$$

Case (1), when $g^2 = \frac{2}{3}$: two values of g^2 coincide, as in § 10, and double contact takes place, from the forms of (C) and (D); the (A) quartics are resolved into conics,

$$(u^2 \mp xy)(u^2 \mp 3xy) - u^2(x\sqrt{f} \mp y\sqrt{h})^2 = 0 \dots (C),$$

$$(u^2 \mp xy)(2u^2 \mp 3xy) + u^2(x\sqrt{f} \mp y\sqrt{h})^2 = 0 \dots (D),$$

$$[u(x\sqrt{f} \pm y\sqrt{h}) - u^2\sqrt{2}]^2 - (xy \mp u^2)^2 = 0 \dots (A).$$

Case (2), when $g = \sqrt{3} - \frac{1}{2}$: (C) and (D) may take the forms

$$(u^2 - \sqrt{3}xy)^2 - u^2(x\sqrt{f} - y\sqrt{h})^2 = 0 \dots (C),$$

$$(u^2 - \sqrt{3}xy) [(3 - \sqrt{3})u^2 - \sqrt{3}xy] + u^2(x\sqrt{f} - y\sqrt{h})^2 = 0 \dots (D).$$

Two vectors coincide without contact, when (C) is resolved into two conics, and (D) passes through four double points, which are the mutual intersections of those conics (two of them being the given nodes), as well as through four other points in (C). There are in all six, not eight, intersections (other than the given nodes), which are cusps on the corresponding (A) trinodals. Thus, the point $[x\sqrt{f} = y\sqrt{h} = u(12)^{-1}]$ is a cusp on

$$u^4 - 4\sqrt{2}u^3(f^1x + h^1y)3^{-1} = x^2y^2 - u^2[(f^1x + h^1y)^2 + xy(2\sqrt{3} - 2)] \dots (A),$$

or, as it may be written, to show the cusp,

$$\left[u - \frac{\sqrt{2}}{2}3^{\frac{1}{2}}(f^1x + h^1y)\right]^2 \left[u - \frac{\sqrt{2}}{6}3^{\frac{1}{2}}(f^1x + h^1y)\right]$$

$$+ (f^1x - h^1y)^2 [u^2(1 - \sqrt{3}) + \frac{1}{4}(fx^2 + 6f^1h^1xy + hy^2)] = 0.$$

These examples will illustrate the three ways in which union of the vectors takes place.

OP_1P_2 , OQ_1Q_2 , the vectors through their four real intersections

$$2x^2 - (5 + \sqrt{3})xy + 2y^2 = 0,$$

$$(x - 3.037y)(y - 3.037x) = 0.$$

FIG. 2.

These four points P_1 , Q_1 ; P_2 , Q_2 are third (additional) cusps in the (A) system. All points on P_1 ac Q_1 , P_2 ac Q_2 are third acnodes in (A); all other points on (C) are third crunodes in (A).

This is proved by the aid of (D), without determining, as in § 2, the values of ξ , η for a given point in (C), and drawing the curve (A). Make $x = y$ in (C) :

$$(x-1)(3x-1)(x^2-4x+1) = 0.$$

Of the apses thus found, the coordinates $(2 \pm \sqrt{3}, 2 \pm \sqrt{3})$, when substituted in the function (D), render it negative,

$$3(2 \pm \sqrt{3})^2 - 2(3 \pm 2\sqrt{3})^2 \text{ or } -3(2 \pm \sqrt{3})^2.$$

These apses are therefore acnodes. Similarly, the other apses are crunodes. And, since no alteration occurs in a locus of acnodes or of crunodes, except at the points of junction which are cusps, P_1 ac Q_1 , P_2 ac Q_2 are loci of acnodes.

14. Quadruple contact occurs between (C) and (D), when $g = 0$

(§ 6), on the vectors $x = 0, y = 0$; the quartics (A) become complexes of lines and nodal cubics.

The four points of contact of

$$(u^3 - x^3)(u^3 - y^3) + 2x^3y^3 = 0 \dots\dots\dots (C),$$

$$(u^3 - x^3)(u^3 - y^3) - 2x^3y^3 = 0 \dots\dots\dots (D),$$

are $u \pm x = 0$, with $y = 0$; $u \pm y = 0$, with $x = 0$.

With the first pair (§ 2), $\xi = 1 \pm 2, \eta = 1$, (A) is resolved into the complex

$$(u \pm x) [u^3 (u \pm x) + y^3 (u \mp x)] = 0;$$

or, in full, $(x + y - 1)^3 (2x + y - 1) + y^3 (y - 1) = 0$,

$$(y - 1)(x + y - 1)^3 + y^3 (2x + y - 1) = 0.$$

With the second pair of values, (A) is resolved into the complex

$$(u \pm y) [u^3 (u \pm y) + x^3 (u \mp y)].$$

The detached lines were tangents at nodes, and touch the associated nodal cubics in the four assigned points.

15. Double contact of the third order cannot exist between (C) and (D) when $f = h = 1$, but takes place when $f = h = \frac{2}{3} = \frac{g}{4}$. See § 7.

16. Next consider the values of g which cause two radii vectores to unite when (C) and (D) intersect each other. (§ 8.)

When $f = h = 1$, g has eight values in the equation

$$(g^4 - 4g^3 + 12)^3 = 24g^3 (g^4 - 8g^3 + 12),$$

or $(g^3 - 6g + 6)(g^3 + 6g + 6)(g^3 - 2g - 2)(g^3 + 2g - 2) = 0$.

When $\pm g = 3 \pm \sqrt{3}$, the curves (C) and (D) touch each other, and the corresponding (A) quartics are resolved into conics; when $\pm g = \sqrt{3} \pm 1$, (C) is resolved into conics, and (D) intersects it in six points, which are cusps in the (A) system. See § 11.

17. When $g = 3 + \sqrt{3}$, and $(g - 1)(5 - g) = 1$. In this case

$$[u^3 (g - 1) - xy] [u^3 (5 - g) - 3xy] - u^3 (x - y)^3 = 0 \dots\dots\dots (C),$$

$$[u^3 (g - 1) - xy] [u^3 (g + 1) - 3xy] + u^3 (x - y)^3 = 0 \dots\dots\dots (D).$$

They are seen to have double contact with each other, and with the intermediate ellipse $[u^3 (g - 1) - xy]$, at the points

$$[x = y = \pm u \sqrt{g - 1}].$$

The biorunodal (C) consists of two circuits, which intersect in the common nodes; the biacnodal (D) of two elongated circuits, one of which is unifolium, and each touches a separate circuit of (C).

The parameters (§ 2) which correspond to these points of contact are

$$\xi = \eta = 1 \mp 4\sqrt{g-1}.$$

The corresponding (A) quartics are resolved into conics:

$$u^3 [u \mp 4(x+y)\sqrt{g-1}] = x^2y^2 - u^2(x^2 + 2gxy + y^2),$$

or
$$[xy - u^3(g-1)]^2 = u^3[x + y \mp 2u\sqrt{g-1}]^2.$$

The first consists of two ellipses, the second of an ellipse and hyperbola. All pass through the common nodes, and each pair touches each other in the points $[x = y = \pm u\sqrt{g-1}]$.

The curves (C) and (D) intersect each other in four real points, in which the conic ($u^3 = xy$) is met by the vectors

$$[x^2 + y^2 + (2g-4)xy = 0].$$

There are therefore acnodes, cusps, and crunodes in (A).

18. When $g = 3 - \sqrt{3}$, and $(g-1)(5-g) = 1$. The results of § 17 are equally true, with a modified interpretation.

The curve (C) consists of two intersecting circuits, each of which touches a separate circuit of (D); these last are short and long, without asymptotes. The two pairs of conic complexes (A) are an ellipse and hyperbola, which pass through the nodes and touch each other in pairs, as in § 17. But the vectors $[x^2 + y^2 + (2g-4)xy]$ are imaginary, so that (C) and (D) do not intersect; all points in (C) are therefore crunodes in (A), except the points $[x = y = \pm u\sqrt{g-1}]$.

19. When $-g = 3 + \sqrt{3}$, the forms of the equations to the subsidiary curves may be derived from § 17, by changing the sign of xy , and leaving (g) unaltered:

$$[u^3(g-1) + xy][u^3(5-g) + 3xy] - u^2(x+y)^2 = 0 \dots\dots\dots(C),$$

$$[u^3(g-1) + xy][u^3(g+1) + 3xy] + u^2(x+y)^2 = 0 \dots\dots\dots(D).$$

They have double contact at the points $[x = -y = \pm u\sqrt{g-1}]$.

The corresponding quartics (A) may be resolved into pairs of conics

$$[u^3(g-1) + xy]^2 = u^3[2u\sqrt{g-1} \mp (x-y)]^2.$$

Cor.—When $-g = 3 - \sqrt{3}$, the results of § 19 are applicable, with the altered value of (g).

20. When $g = \sqrt{3} + 1$, the subsidiary curves only meet in the given nodes; all points in (C) are (third) crunodes in the (A) system. In this case

$$(u^2 - \sqrt{3}xy)^2 - u^2(x+y)^2 = 0 \dots\dots\dots (C),$$

$$\sqrt{3}(u^2 - \sqrt{3}xy)[(2 + \sqrt{3})u^2 - xy] + u^2(x+y)^2 = 0 \dots\dots (D).$$

The contact is imaginary, as well as the intersections.

Cor.—These results are applicable when $-g = \sqrt{3} + 1$, by changing the sign of xy , and leaving g unaltered.

21. When $g = \sqrt{3} - 1$, (C) and (D) meet in six points, besides the given nodes:

$$(u^2 - \sqrt{3}xy)^2 - u^2(x-y)^2 = 0 \dots\dots\dots (C),$$

$$\sqrt{3}(u^2 - \sqrt{3}xy)[(2 - \sqrt{3})u^2 + xy] - u^2(x-y)^2 = 0 \dots\dots (D).$$

(D) consists of a single circuit, and each hyperbolic branch meets (C) in three points, of which one is double, without contact at any, as in § 11 (2). There are six points of intersection on (C), which are cusps on (A); and internal points on (C) between two cusps are acnodes on (A); the rest are crunodes. Thus, for the pair of points $[x = y = \pm 3^{-\frac{1}{2}}u]$,

$$\xi = \eta = 1 \mp 4(3^{-\frac{1}{2}})$$

in (B). The corresponding quartics (A), have (third) cusps

$$u^4 \mp 4(x+y)u^2 3^{-\frac{1}{2}} = x^2y^2 - u^2[(x+y)^2 + 2xy(\sqrt{3} - 2)];$$

or, as it may be written,

$$[u \mp \frac{1}{2}3^{\frac{1}{2}}(x+y)]^3 [u \pm \frac{1}{2}3^{\frac{1}{2}}(x+y)]$$

$$+ (x-y)^2 [u^3 (1 - \frac{1}{2}\sqrt{3}) + \frac{1}{16}(x^3 + 6xy + y^3)] = 0.$$

These trinodal quartics have cusps on one of the two intersecting circuits, of which they are composed, situated on the coincident radii vectores.

22. Let $-g = \sqrt{3} - 1$. *Mutatis mutandis*, as in § 19,

$$(u^2 + \sqrt{3}xy)^2 = u^2(x+y)^2 \dots\dots\dots (C),$$

$$\sqrt{3}(u^2 + \sqrt{3}xy)[u^2(2 - \sqrt{3}) - xy] = u^2(x+y)^2 \dots\dots\dots (D).$$

The curve (D) has a single bifolium circuit, which intersects (C) in four single points, and two double points, besides the acnodes. There

are, therefore, six cusps in the (A) binodal system; of which two

$$\text{are } \left[u \mp \frac{1}{2} 3^{\frac{1}{2}} (x-y) \right]^3 \left[u \pm \frac{1}{6} 3^{\frac{1}{2}} (x+y) \right] \\ + (x+y)^2 \left[u^2 \left(1 - \frac{1}{2} \sqrt{3} \right) + \frac{1}{16} (x^2 + 6xy + y^2) \right] = 0.$$

23. More generally, the number of cusps can be determined for an assigned value of (g).

The equations (C) and (D) may be combined into the forms

$$gu^2 = \sqrt{3} (\sqrt{3} \mp 1) xy, \\ 2g(x^2 + y^2) + (1 \mp \sqrt{3})(g^2 \pm 2\sqrt{3})xy = 0.$$

It is thus found that, if numerically $\pm g > 3 + \sqrt{3}$, or $< \sqrt{3} - 1$, (C) and (D) intersect in eight points; if $\pm g < 3 + \sqrt{3}$ and $> \sqrt{3} - 1$, they meet in four, unless $\pm g = \sqrt{3} + 1$, when they meet in no point but the nodes. In these several cases, there are eight, four, and no (third) cusp in the (A) binodal system.

II. *Binodal Quartics.*

24. In this system f, h are negative, and assumed to be unity in the main and subsidiary equations; $u \equiv x + y - 1$.

$$u^3 (\xi x + \eta y - 1) = x^2 y^2 + u^2 (x^2 + 2gxy + y^2) \dots\dots\dots (A),$$

$$u^4 + 3x^2 y^2 + u^2 (x^2 + 2gxy + y^2) = 0 \dots\dots\dots (C),$$

$$(g^2 - 1) u^4 + 3x^2 y^2 - u^2 (x^2 - 4gxy + y^2) = 0 \dots\dots\dots (D).$$

It is seen that any value of $\pm g$, numerically $< \sqrt{3} + 1$, reduces (C) to the acnodes (1, 0), (0, 1), when it is written

$$(u^2 + \sqrt{3} xy)^2 + u^2 [(x+y)^2 + 2(g - \sqrt{3} - 1)xy] = 0,$$

Consequently there are no trinodal quartics in the (A) system.

25. When $g = \sqrt{3} + 1$, (C) consists of the given acnodes and two others $[x = -y = \pm 3^{-\frac{1}{2}} u]$. For this last pair the corresponding (A) quartics

$$\left[u \mp \frac{1}{2} 3^{\frac{1}{2}} (x-y) \right]^3 \left[u \pm \frac{1}{6} 3^{\frac{1}{2}} (x-y) \right] \\ + (x+y)^2 \left[-u^3 \left(1 + \frac{1}{2} \sqrt{3} \right) + \frac{1}{16} (x^2 + 6xy + y^2) \right] = 0$$

exhibit (third) cusps in the second and fourth quadrants.

The values of the parameters ξ, η in these (A) quartics are, by § 3,

$$\xi = 1 \mp 4(3^{-\frac{1}{2}}), \quad \eta = 1 \pm 4(3^{-\frac{1}{2}}).$$

26. When $-g = \sqrt{3} + 1$, the cusped trinodal (A) quartics are

$$\left[u \mp \frac{1}{2} 3^{\frac{1}{2}} (x+y) \right]^3 \left[u \pm \frac{1}{6} 3^{\frac{1}{2}} (x+y) \right] \\ + (x-y)^2 \left[-u^2 \left(1 + \frac{1}{2} \sqrt{3} \right) + \frac{1}{18} (x^2 + 6xy + y^2) \right] = 0.$$

These four are the only trinodal quartics when $\pm g = \sqrt{3} + 1$.

27. The biacnodal quartics (A) can never degenerate into complexes, since the subsidiary curves (C) and (D) cannot touch each other.

In §§ 6, 7 it was shown that (C) and (D) cannot have quadruple contact, or double contact of the third order. They can only have double contact when $\pm g = \sqrt{3} + 3$, and it will be proved that this contact is imaginary.

28. Let $g = 3 + \sqrt{3}$, $(g-1)(5-g) = 1$. As in § 17, (C) and (D) may be written, to exhibit imaginary contact,

$$\left[u^2 (g-1) + xy \right] \left[u^2 (5-g) + 3xy \right] + u^2 (x-y)^2 = 0 \dots\dots (C),$$

$$\left[u^2 (g-1) + xy \right] \left[u^2 (g+1) + 3xy \right] - u^2 (x-y)^2 = 0 \dots\dots (D).$$

They intersect in four real points, in which the vectors

$$x^2 + (2g-4)xy + y^2 = 0$$

intersect the conic

$$u^2 + xy = 0.$$

When $-g = 3 + \sqrt{3}$, like results are obtained by writing $-xy$ for xy , and leaving (g) unaltered.

29. For all values of $(\pm g)$ numerically $> \sqrt{3} + 1$, (C) and (D) intersect in four real points, which are cusps in the corresponding (A) quartics. (C) is therefore a locus of crunodes, cusps, and acnodes in (A).

Let $g = 3$. (Fig. 3.) The locus of nodes (C) consists of two unifolium circuits besides the acnodes A, B ; and each circuit is divided into two portions by branches of the bicrunodal quartic (D). P ac Q is the locus of acnodes, P cr Q of crunodes, while P, Q are cusps in the corresponding (A) quartics.

Ex.—The point $(1, -1)$ lies in P ac Q ; the corresponding parameters (§ 3) are 3, -1 ; the corresponding (A) quartic has $(1, -1)$ for a third acnode

$$u^3 (3x-y+1) = x^2 y^2 + u^2 (x^2 + 6xy + y^2) \dots\dots\dots (A).$$

Ex.—The point $(\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3})$ lies on P or Q , and is a crunode on

$$u^4 + 10u^3(x-y)3^{-\frac{1}{2}} = x^2y^3 + u^2(x^2 + 6xy + y^2) \dots\dots\dots (A).$$

FIG. 3.

III. Quartics with one crunode and one acnode.

30. In this system f is positive, h negative, and assumed to be unity.

$$u^3(\xi x + \eta y - 1) = x^2y^2 + u^2(x^2 + 2gxy - y^2) \dots\dots\dots (A),$$

$$u^4 + 3x^2y^2 + u^2(x^2 + 2gxy - y^2) = 0 \dots\dots\dots (C),$$

$$(g^2 + 1)u^4 + 3x^2y^2 - u^2(x^2 - 4gxy - y^2) = 0 \dots\dots\dots (D).$$

$$u \equiv x + y - 1,$$

as before.

The subsidiary curves (C) and (D) cannot have double contact of the third order (§ 7), but they have quadruple contact (§ 6), when $g^2 = 4$. The points of contact are thus defined :

$$2u^2 + gxy = 0, \quad x^2 - y^2 = 0.$$

31. Let $g = 2$. The bisector $(x + y = 0)$ passes through P , Q , $(-1, 1)$, $(1, -1)$, in which (C) and (D) touch each other ; the other

points of contact are imaginary. The curve (C) consists of a crunodal circuit with an acnode at B. (Fig. 4.) The curve (D) has its crunode at B, and acnode at A.

FIG. 4.

To the point $P(-1, 1)$ the paramoters are $\xi = 1$, $\eta = 5$; and the (A) quartic

$$u^3(x+5y-1) = x^2y^2 + u^2(x^2+4xy-y^2)$$

is resolved into an acnodal cubic, and a line ($y = 1$) which touches it at P .

$$(u^2+y^2)(u^2-x^2) + 4u^2y(u-x) = 0.$$

For $Q(1, -1)$, the corresponding complex is

$$(u^2+y^2)(u^2-x^2) - 4u^2y(u+x) = 0.$$

32. Let $g = -2$. The internal bisector ($x = y$) passes through the points of real contact of (C) and (D), $(1, 1)$, $(\frac{1}{3}, \frac{1}{3})$.

The corresponding (A) complexes are

$$(u-x) [(u+x)(u^2+y^2) - 4u^2y] = 0,$$

$$(u+x) [(u-x)(u^2+y^2) + 4u^2y] = 0.$$

Each acnodal cubic is touched by a linear factor, which represents a tangent at a node, in one of the specified points.

33. Every point on (C), besides P and Q , is a second crunode in the

(A) system. For, if the coordinates of two assigned points on (C), $(x^2 = y^2 = \frac{1}{3})$, be substituted in (D), that function is positive (§ 4).

34. If $g^2 < 4$, the curves (C) and (D) do not meet. Every point on (C) is therefore a second crunode on (A), for the reasoning of § 33 is applicable to the modified forms.

35. If $g^2 > 4$, (C) and (D) intersect in four real points. Let $g = 3$. Through the real intersections of the curves

$$u^4 + 3x^2y^2 + u^2(x^2 + 6xy - y^2) = 0 \dots\dots\dots(C),$$

$$10u^4 + 3x^2y^2 - u^2(x^2 - 12xy - y^2) = 0 \dots\dots\dots(D),$$

the two vectors $(x + 2.645y)(x + .6465y) = 0$ pass.

These vectors intercept in (C) (see Fig. 4, modified) on AP , that part of the loop which is remote from O , and on Q , two series of points which are acnodes in (A); all other points are crunodes, except the four points of intersection, which are cusps in (A).

36. In this system, two radii vectores cannot unite (§ 8), unless (C) and (D) touch each other.

For the values of g^2 necessary for such union, when $f = -h = 1$, are (by § 9) given by the quartic

$$g^8 + 24g^6 - 80g^4 + 384g^2 + 256 = 0;$$

but this has no positive value.

Moreover, no lower value of $(-fh)$ will give positive values of g^2 in the corresponding quartic.

This result is confirmed by the discriminant of the quartic in g^2 (§ 9), which only yields two positive limiting values of fh .

IV. Quartics with a crunode and a cusp.

37. In this system one of the constants is zero, and the other negative, and here assumed to be unity. The quartics (A), (C), (D) are

$$u^3(\xi x + \eta y - 1) = x^2y^2 - u^2(x^2 + 2gxy) \dots\dots\dots(A),$$

$$u^4 + 3x^2y^2 - u^2(x^2 + 2gxy) = 0 \dots\dots\dots(C),$$

$$g^2u^4 + 3x^2y^2 + u^2(x^2 - 4gxy) = 0 \dots\dots\dots(D),$$

$$u \equiv x + y - 1.$$

The curves (C) and (D) touch at their common tacnode $(0, 1)$, which is always real in (D), but imaginary in (C), if $g^2 < 3$.

By combining (C) and (D) in the forms

$$6x^2y^2 - 6gxyu^2 + (g^2 + 1)u^4 = 0,$$

$$(g^2 - 1)u^2 + 2x^2 - 2gxy = 0,$$

we see that (C) and (D) intersect in four points, if $g^2 > 2$, and touch in four points at the tacnode, if $g^2 = 2$ (see § 6). If $g^2 = 3$, two of the four points are merged in the tacnode (Fig. 7).

38. Let $g^2 = 2$; then $u^2 = gxy$, and $gx^2 = xy$. (Figs. 5 and 6.)

FIG. 5.

FIG. 6.

The curves (C) and (D) have double contact at the points P, Q ($gx = y = \pm u$).

The four corresponding (A) quartics

$$u^3 [(1 - 2g)x - 1] = x^2y^2 - u^2(x^2 + 2gxy),$$

$$u^3 [(1 + 2g)x + 2y - 1] = x^2y^2 - u^2(x^2 + 2gxy),$$

are resolved into complexes of lines and cubics

$$(x-1) [u^3 - 2gxu^2 + x^3(u+y)] = 0,$$

$$(u+y) [u^3 + 2gxu^2 + x^3(x-1)] = 0.$$

The cubics are cusped at A , and touched by the lines in P, Q .

39. When $g^2 = 3$, (C) and (D) intersect in two points only, besides the common nodes A, B . (Fig. 7.) Two radii vectores do not unite,

FIG. 7.

as it might be inferred from the formula of § 9,

$$(g^2 + 1)^3 (g^2 - 3)^2 = 0.$$

That formula is inapplicable, since it was obtained on the hypothesis that (h) was constant.

In this case $(u^2 - gxy)^2 - u^2x^2 = 0$ (C),

$(u^2 - gxy)(3u^3 - gxy) + u^2x^2 = 0$ (D).

By combining, $(u^2 - gxy)(2u^3 - gxy) = 0,$

$$u^2(u^3 - gxy + x^2) = 0,$$

and the vectors are $x^2 = 0, \quad x^3 - 2gxy = 0. \quad [? 2x^2 - gxy = 0.]$

The conics have a contact of five points with (D) at the tacnode, and meet (D) in the points $(gy = 2x = \pm 2u)$.

These points are cusps in the corresponding (A) trinodal quartics

$$u^4 \mp \frac{1}{3}u^3(10x + 2gy) = x^2y^2 - u^2(x^2 + 2gxy) \quad \text{..... (A).}$$

These may be written to show both cusps as well as the crunode, and the bitangent, which touches a folium at points situated on the axes

$$(u-x)^3(3u-x-2gy) = x^2(3u-x-gy)^2,$$

$$(u+x)^3(3u+x+2gy) = x^2(3u+x+gy)^2.$$

There are four bicuspidal and singly crunodal quartics, since (g) has two signs. Only one figure of (C) and (D) is drawn in illustration. The portion of the ellipse, intercepted by the upper lobe of (D), and the portion of the hyperbola, intercepted by the lower lobe of (D), yield acnodes on (A).

40. For all values of $g^2 > 2$, (C) is a locus of acnodes, cusps, and crunodes on the corresponding (A) trinodal quartics.

V. *Quartics with an acnode and a cusp.*

41. In this system f is positive, and here taken to be unity;

$$u \equiv x + y - 1,$$

$$u^3(\xi x + \eta y - 1) = x^2 y^3 + u^2(x^2 + 2gxy) \dots\dots\dots (A),$$

$$u^4 + 3x^2 y^2 + u^2(x^2 + 2gxy) = 0 \dots\dots\dots (C),$$

$$g^2 u^4 + 3x^2 y^2 - u^2(x^2 - 4gxy) = 0 \dots\dots\dots (D).$$

If $g^2 =$ or < 3 , (C) is reduced to the node and tacnode. There are consequently no critical values of the parameters ξ, η (§ 3), and no trinodal quartics in the system (A). The curves (C) and (D) touch at their common real tacnode, when $g^2 =$ or > 3 .

42. If $g^2 > 3$, (C) and (D) intersect each other in four points, two of which are imaginary. All points on (C) which lie between the (y) axis and these points of intersection, are acnodes on the corresponding (A) quartics, and are thus separated by cusps from the other points, which are crunodes.

Ex. $g = 3$. The curves (C), (D) meet in two of the points in which the conics

$$(u^2 + .4417xy)(u^2 + 1.358xy) = 0$$

are intersected by the conics

$$4u^2 - x^2 + 3xy = 0,$$

or by the vectors

$$x(x - 1.233y) = 0, \quad x(x + 2.433y) = 0.$$

The last pair only gives real intersections, which discriminate the nodes on (C). Both (C) and (D) have real tacnodes; the acnodal curve (C) has two asymptotes, the crunodal (D) has four.

VI. Bicuspidal Quartics.

43. The general system is

$$u^3 (\xi x + \eta y - 1) = x^3 y^3 + 2gxyu^3 \dots\dots\dots (A).$$

$$\text{The locus of nodes} \quad u^4 + 2u^2 gxy + 3x^2 y^2 = 0 \dots\dots\dots (C).$$

$$\text{The subsidiary curve} \quad (gu^2 + xy)(gu^3 + 3xy) = 0 \dots\dots\dots (D).$$

The parameters ξ , η , and the discriminating curve (B), which is their envelop, are thus determined.

$$\text{By § 3, } \frac{1}{y} (\xi - c) = \frac{1}{x} (\eta - d) = \frac{2}{u} \left(\frac{xy}{u^2} + g \right) = \frac{2}{u} (G + g),$$

since, by (C), $xy = \frac{1}{3}u^3 [-g \pm \sqrt{(g^2 - 3)}] = Gu^2$, suppose.

Eliminate x, y from these equations.

$$(\xi - c)(\eta - d) = 4G(G + g)^2 \dots\dots\dots (B).$$

44. The line $(\xi x + \eta y = 1)$ intersects each curve (A) in the coordinate axes, and may be regarded as a satellite of the line of nodes $(u \equiv cx + dy - 1)$, of which (B) is the envelop. All conjoint values of ξ, η , which satisfy (B), determine a third node in the quartics (A); all slightly varying values of ξ, η , which denote lines external to (B), or intersecting it, denote quartics in (A), which are companion-curves to the preceding, but not trinodal. See below, § 47, with Figs. 9, 10.

When $g^2 < 3$, the locus (C) is resolved into two imaginary conics; there are therefore no trinodal values in the (A) system.

45. When $g^2 = 3$, (C) becomes two coincident conics

$$(u^2 + gxy)^2 = 0;$$

(D) two distinct conics

$$(u^2 + gxy)(gu^2 + xy) = 0.$$

Every point in (C) is also on (D), and is therefore a cusp in a corresponding curve in (A). For a point in the locus $(U^2 + gXY = 0)$, the parameters ξ, η are (if $c = d = 1$)

$$\xi - 1 : \eta - 1 : 4 :: Y : X : gU.$$

The corresponding quartic (A) is tricuspidal,

$$u^4 + \frac{4}{gU} u^3 (xY + yX) - 2gu^2xy - x^2y^2 = 0,$$

and may be written, to exhibit the third cusp (X, Y, U),

$$\left[uU + \frac{g}{2} (xY + yX) \right]^3 \left[uU - \frac{g}{6} (xY + yX) \right] + \frac{3}{16} (xY - yX)^2 (x^2Y^2 + 6xyXY + y^2X^2 - 8u^2U^2) = 0.$$

Ex.—Let $g = -\sqrt{3}$, and let the points in (C) be the apses

$$\pm u = x\sqrt[4]{3} = y\sqrt[4]{3} = \sqrt[4]{3} (2 \mp \sqrt[4]{3})^{-1}.$$

The corresponding coordinates in the discriminating curve (B) are

$$\xi - 1 = \eta - 1 = \mp 1.755.$$

With these parameters, the corresponding tricuspidal quartics (A) are

$$u^4 \mp 1.755u^3 (x+y) = x^2y^2 - 2\sqrt{3} u^2xy.$$

For convenient calculation these may be written

$$(xy - \sqrt{3}u^2)^2 = 2.245u^4 - 1.755u^3 \text{ or } 5.755u^4 + 1.755u^3.$$

The former is drawn in Fig. 8, with two companion-curves, which are obtained by slightly differing parameters ξ, η .

FIG. 8.

46. When $\xi = \eta = 1$, the quartic (A) is resolved into two conics.

In this case $(xy + gu^2 = 0)$ is a necessary condition, and must be a factor of (C). This is only possible when $g^2 + 1 = 0$. See § 8, where the condition $[(g^2 + 1)(g^2 - 3) = 0]^2$ is shown to be necessary for the imaginary contact of (C) and (D), when $fh = 0$.

The terminal bicuspidal quartic (A),

$$x^2y^2 - 4xyu^2 + u^4 = 0,$$

has two asymptotes parallel to the axes.

47. For values of $g^2 > 3$, (C) and (D) are resolved into two real conics; all points on that portion of (C) which lies between the two portions of (D) are acnodes in (A); and all points in the external portion of (C) are crunodes in (A).

Let $g = -2$. The subsidiary curves (C), (D) are

$$(u^2 - xy)(u^2 - 3xy) = 0 \dots\dots\dots (C),$$

$$(2u^2 - xy)(2u^2 - 3xy) = 0 \dots\dots\dots (D).$$

The (B) curves for the portions of (C) are severally

$$(\xi - 1)(\eta - 1) = 4 \text{ and } \frac{1}{2}\frac{\eta}{\xi}.$$

Ex. 1.—To the point $(x = y = u = 1)$ the line $(\xi = \eta = -1)$ corresponds.

The point is an acnode on the corresponding (A) quartic

$$u^4 + 2u^2 - 4xyu^2 + x^2y^2 = 0.$$

The companion-curves are unipartite, no trace of a separate circuit being found.

Ex. 2.—To the point $(x = y = -u = \frac{1}{3})$ the line $(\xi = \eta = 3)$ [corresponds], and [the point is on] the acnodal (A) quartic

$$3u^4 + 2u^2 + 4u^2xy - x^2y^2 = 0.$$

Ex. 3. $\xi = \eta = 1 - \frac{1}{3}\sqrt{3}$. The point $(x = y = \frac{1}{3}\sqrt{3}u = 2 + \sqrt{3})$ is a crunode on the quartic (A), Fig. 9,

$$u^4 - \frac{1}{3}\sqrt{3}(x + y)u^2 = x^2y^2 - 4u^2xy,$$

or

$$3.0755u^4 - 1.9245u^2 = (xy - 2u^2)^2.$$

The companion-curves are drawn for slightly differing values of the

conjoint-parameters, *i.e.*, as the line (ξ, η) does or does not intersect the discriminating (B) curve. They are both unipartite.

FIG. 9.

Ex. 4. $\xi = \eta = 1 + \frac{10}{9}\sqrt{3}$. The point $(x = y = 2 - \sqrt{3})$ is a crunode on

$$(xy - 2u^2)^2 = 1.9245u^3 + 6.9245u^4 \dots\dots\dots(A).$$

(Fig. 10.) The unipartite and bipartite companion-curves are also drawn.

FIG. 10.

Bicuspidal Quartics, when both cusps are imaginary.

48. The general system of such quartics, and their subsidiary curves, may be thus denoted :

$$u^5 (\xi x + \eta y - 1) = (mx^2 + ny^2)^3 + 2gu^3 (mx^2 + ny^2) \dots\dots\dots (A),$$

$$u^4 + 2u^2 g (mx^2 + ny^2) + 3 (mx^2 + ny^2)^2 = 0 \dots\dots\dots (C),$$

$$(gu^2 + mx^2 + ny^2)(\frac{1}{3}gu^2 + mx^2 + ny^2) = 0 \dots\dots\dots (D).$$

The discriminating curve is found, as in § 43, to consist of two conics

$$\frac{1}{m} (\xi - c)^2 + \frac{1}{n} (\eta - d)^2 = 16G (G + g) \dots\dots\dots (B),$$

where

$$3G^2 + 2gG + 1 = 0.$$

When $g^2 < 3$, the locus of nodes (C) is resolved into two imaginary conics, and the system (A) has no trinodal quartics.

49. When $g^2 = 3$, the subsidiary curves are

$$[u^2 + g (mx^2 + ny^2)]^2 = 0 \dots\dots\dots (C),$$

$$[u^2 + g (mx^2 + ny^2)] [gu^2 + mx^2 + ny^2] = 0 \dots\dots\dots (D).$$

Every point in (C) is also in (D), and therefore is a cusp in (A).

The parameters

$$\xi = c + \frac{1}{Ug} 8mX, \quad \eta = d + \frac{1}{Ug} 8nY$$

correspond to a point (X, Y, U) on the conic

$$U^2 + g (mX^2 + nY^2) = 0,$$

and

$$\frac{1}{m} (\xi - c)^2 + \frac{1}{n} (\eta - d)^2 + \frac{64}{g^3} = 0 \dots\dots\dots (B).$$

The corresponding tricuspidal quartic (A),

$$u^4 + \frac{1}{Ug} 8u^3 (mX + nY) - 2gu^2 (mx^2 + ny^2) - (mx^2 + ny^2)^3 = 0,$$

may be written, to show the third cusp (X, Y, U),

$$\begin{aligned} & [uU + g (mX + nY)]^3 \left[uU - \frac{g}{3} (mX + nY) \right] \\ & - g^3 mn (XY - YX)^2 [(mx^2 + ny^2)(mX^2 + nY^2) + (mX + nY)^2 - 2u^2 U^2] \\ & = 0. \end{aligned}$$

Ex. $g = -\sqrt{3}$, $m = n = 1 = c = d$. The quartic

$$u^4 - 8(108)^{-\frac{1}{3}} u^3 (x+y) = (x^2+y^2)^2 - 2\sqrt{3} u^2 (x^2+y^2)$$

is tricuspidal, whose real cusp is $U : X : Y :: \sqrt[3]{12} : 1 : 1$.

50. For higher values of g^2 , (C) and (D) are resolved into pairs of conics.

Ex. $g = -2$. Of the pairs of conics

$$(x^2+y^2-u^2)(x^2+y^2-\frac{1}{3}u^2) = 0 \dots\dots\dots (C),$$

the hyperbola is interior, the ellipse exterior, to the pair

$$(x^2+y^2-2u^2)(x^2+y^2-\frac{2}{3}u^2) = 0 \dots\dots\dots (D).$$

All points in the former conic are acnodes, in the latter crunodes, in (A).

51. If (u) is constant, the (A) system consists of Cartesians. The locus of nodes (C) takes the modified form

$$u^3 = 3(mx^3 + ny^3)^2 + 2gu^2(mx^3 + ny^3).$$

The subsidiary curve (D) is unaltered.

52. Binodal quartics, when both nodes are imaginary, are not adapted for this process of classification. In order to classify bicircular quartics with collinear foci, I selected different parameters (*Proceedings of Lond. Math. Soc.*, Vol. XIV.).

53. Binodal quartics, when both nodes are real, and one of them is at infinity, are classified, as in the preceding paragraphs, by writing $dy-1$ for (u), and equating (C) to zero; but the forms must be modified when both nodes are at infinity.

Binodal Quartics, with two nodes at infinity.

54. If the same form, as in § 1, [be taken] when (u) is constant, for such a system

$$u^3 (\xi x + \eta y - 1) = x^3 y^3 - u^3 (fx^3 + 2gxy + hy^3) \dots\dots\dots (A).$$

$$\text{For critical values } \left. \begin{aligned} u^3 \xi &= 2xy^3 - 2u^3 (fx + gy) \\ u^3 \eta &= 2x^3 y - 2u^3 (gx + hy) \end{aligned} \right\} \dots\dots\dots (B),$$

$$u^3 = 3x^3 y^3 - u^3 (fx^3 + 2gxy + hy^3) \dots\dots\dots (C),$$

and, without alteration,

$$u^4 (g^3 - fh) + u^3 (fx^3 - 4gxy + hy^3) + 3x^3 y^3 = 0 \dots\dots\dots (D).$$

These last intersect in the hyperbola

$$(2xy - u^2g)^2 = \frac{1}{3}u^3 [u(g^2 + 2fh) + 2].$$

55. If these hyperbolæ coincide, (C) and (D) have quadruple contact, and the corresponding (A) quartic is resolved into a straight line and cubic.

In this case $2xy = u^2g$, $u(g^2 + 2fh) + 2 = 0$,
and the four points of contact

$$x : y : u :: g : 2f : 2f^{\frac{1}{2}}, \quad x : y : u :: 2h : g : 2h^{\frac{1}{2}}.$$

For the first pair of points the corresponding complexes are

$$2u^2xf^{\frac{1}{2}}g + u^2y(ufh - 1)f^{-\frac{1}{2}} + u^3 - u^2(fh^2 + 2gxy + hy^2) + x^2y^2 = 0 \dots (A),$$

or $(y - uf^{\frac{1}{2}}) [x^2y + ux^2f^{\frac{1}{2}} - u^2(2gx + hy) - u^2f^{-\frac{1}{2}}] = 0.$

The lines touch the cubics in the given points of contact.

For the second pair of points the complexes are

$$(x - uh^{\frac{1}{2}}) [xy^2 + uy^2h^{\frac{1}{2}} - u^2(fx + 2gy) - u^2h^{-\frac{1}{2}}] = 0.$$

COR.—Acnodal quartics cannot be resolved into complexes.

56. Double contact of the third order occurs between (C) and (D), when points of contact unite. When this happens

$$g^2 = 4fh, \quad 3ufh + 1 = 0.$$

The preceding quartics unite to form the complexes

$$(x - uh^{\frac{1}{2}})(y - uf^{\frac{1}{2}}) [xy + u(xf^{\frac{1}{2}} + yh^{\frac{1}{2}}) - 3u^2f^{\frac{1}{2}}h^{\frac{1}{2}}] = 0.$$

The lines do not touch the hyperbola, but their point of intersection is on it.

With these constants (C) and (D) assume the same forms as in § 7.

All other points in (C), except the points of contact with (D), are crunodes in (A).

57. Two vectors may unite, without necessary contact of (C) and (D).

The quartics (C) and (D) may be combined into the forms

$$6x^2y^2 - 6gxyu^2 + u^4(g^2 - fh) - u^3 = 0 \dots \dots \dots (1),$$

$$2(fh^2 - gxy + hy^2)^2 + u^3(g^2 - fh) + u = 0 \dots \dots \dots (2).$$

Make (2) homogeneous in $(x : y)$ by the aid of (1); the condition

that this quadratic has equal roots is found to be

$$\begin{aligned} \{-u^2g^4 + 2ug^2(3ufh-1) + 3 - 14ufh - 5u^2f^2h^2\}^2 \\ = 16u^2g^2fh(ug^2 + 5 - ufh)^2. \end{aligned}$$

This quartic in (ug^2) , when expanded, takes the form found in § 8,

$$\begin{aligned} u^4g^8 - 4u^3g^6(7ufh-1) + 2u^2g^4(39u^2f^2h^2 - 78ufh-1) \\ - 4ug^2(19u^2f^3h^3 - 3u^2f^2h^2 + 77ufh+3) + (3-ufh)^3(1+5ufh)^3 = 0. \end{aligned}$$

58. The discriminant of this quartic is $(3ufh+1)^3(4ufh+1)$.

When $3ufh+1=0$, $(g^2-4fh)^3(g^2-16fh)^3=0$.

Case 1, when $g^2=4fh$, is considered in § 56.

Case 2, when $g^2=16fh$, and $3ufh+1=0$, (C) and (D) take the forms

$$\begin{aligned} (xy-3u^2f^4h^4)(3xy-u^2f^4h^4) - u^2(xf^4-yh^4)^2 = 0 \dots\dots\dots (C), \\ (xy-3u^2f^4h^4)(3xy-5u^2f^4h^4) + u^2(xf^4-yh^4)^2 = 0 \dots\dots\dots (D). \end{aligned}$$

They touch in the points

$$xf^4 = yh^4 = \pm u(3fh)^4.$$

The corresponding quartics are resolved into conics

$$u^2\{xf^4+yh^4 \mp 2u(3fh)^4\}^2 - (xy-3u^2f^4h^4)^2 = 0 \dots\dots\dots (A).$$

59. When $4ufh+1=0$, the quartic of § 57 becomes

$$(g^2-9fh)^2[g^2-(2\sqrt{3}\pm 1)^2fh].$$

Case 1, when $g^2=9fh$, and $4ufh+1=0$: (C) and (D) touch each other in the points

$$xf^4 = yh^4 = \pm u(2fh)^4,$$

[and take the forms]

$$\begin{aligned} (xy-2u^2f^4h^4)(3xy-2u^2f^4h^4) - u^2(xf^4-yh^4)^2 = 0 \dots\dots\dots (C), \\ (xy-2u^2f^4h^4)(3xy-4u^2f^4h^4) + u^2(xf^4-yh^4)^2 = 0 \dots\dots\dots (D). \end{aligned}$$

The corresponding quartics (A) intersect in the given points of contact

$$u^2(xf^4+yh^4-2\sqrt{2}uf^4h^4)^2 - (xy-2u^2f^4h^4)^2 = 0 \dots\dots\dots (A).$$

Case 2. The other values of g^2 , viz., $(2\sqrt{3}\pm 1)^2fh$, not being coincident, indicate six points of intersection of (C) and (D), which are cusps in the corresponding (A) quartics. See § 11.

60. The preceding theorems must be modified to suit the six species of binodal quartics; but as they have been examined at length when the nodes are at finite distances, it is not thought necessary to discuss them when the nodes are at infinity.

On the conditions for Folium-points in Binodal Quartics.

61. It has been stated (§ 2), that the maximum number of inflexions in a binodal quartic is four, and that there may be two folia, containing two inflexions each. It is proposed to find the conditions when two inflexions of a folium unite at a folium-point, or point of undulation.

At such a point both $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2} = 0$, if $p = \frac{dy}{dx}$.

62. By taking the previous binodal quartic system, wherein

$$u \equiv cx + dy - 1,$$

and ξ, η are parameters,

$$\xi x + \eta y - 1 = -\frac{1}{u} (fx^2 + 2gxy + hy^2) + \frac{1}{u^3} x^2 y^2 \dots\dots\dots (A);$$

and, differentiating thrice,

$$\begin{aligned} \xi + \eta p = & -\frac{2}{u} [fx + g(y + xp) + h yp] + \frac{1}{u^3} (fx^2 + 2gxy + hy^2)(c + dp) \\ & + \frac{2}{u^3} xy(y + xp) - \frac{3}{u^4} x^2 y^2 (c + dp) \dots\dots\dots (2); \end{aligned}$$

$$\begin{aligned} 0 = & -\frac{2}{u} (f + 2gp + hp^2) + \frac{4}{u^3} [fx + g(y + xp) + h yp](c + dp) \\ & - \frac{2}{u^3} (fx^2 + 2gxy + hy^2)(c + dp)^2 + \frac{2}{u^3} [2xyp + (y + xp)^2] \\ & - \frac{12}{u^4} xy(y + xp)(c + dp) + \frac{12}{u^5} x^2 y^2 (c + dp)^2 \dots\dots\dots (3); \end{aligned}$$

$$\begin{aligned} 0 = & \frac{6}{u^2} (f + 2gp + hp^2)(c + dp) - \frac{12}{u^3} [fx + g(y + xp) + h yp](c + dp)^2 \\ & + \frac{6}{u^4} (fx^2 + 2gxy + hy^2)(c + dp)^3 + \frac{12}{u^3} p(y + xp) \\ & - \frac{18}{u^4} [2xyp + (y + xp)^2](c + dp) + \frac{72}{u^5} xy(y + xp)(c + dp)^2 \\ & - \frac{60}{u^6} x^2 y^2 (c + dp)^3 \dots\dots\dots (4). \end{aligned}$$

Multiply equation (3) by $\frac{3}{u} (c + dp)$, and add to (4); the terms involving f, g, h disappear, and the residue is the product.

$$\left[p - \frac{y}{u} (c + dp) \right] \left[1 - \frac{x}{u} (c + dp) \right] \left[y + xp - \frac{2}{u} xy (c + dp) \right] = 0.$$

The two first factors give by-solutions, the third factor the main solution.

63. When the two first factors are substituted in (3) and (4), we obtain $(y - xp)^3 (y^3 - fu^3)$ and $(y - xp)^2 (x^2 - hu^2)$.

Now $(y - xp = 0)$ is not a condition; for, if it be coupled with either of these two factors, $u = cx + dy$, which is a false conclusion. The conditions $y^2 - fu^2 = 0$, $x^2 - hu^2 = 0$, are true conditions, and imply that the quartics degenerate into four complexes, of which the tangents at the nodes are factors.

They are inapplicable to biacnodal quartics.

64. Ex. $x = uh^{\frac{1}{2}}$, $c + dp = \frac{u}{x}$; whence $d(y - px) = 1$.

The parameters are found from

$$\xi x + \eta y - 1 = -\frac{x}{u}(fx + 2gy) = -h^{\frac{1}{2}}(fx + 2gy) \dots \dots \dots (A),$$

and after reducing from

$$\xi + \eta p = -\frac{x}{u}(f + 2gp) = -h^{\frac{1}{2}}(f + 2gp).$$

Hence $\xi = c - (1 + fh)h^{-\frac{1}{2}}$, $\eta = d - 2gh^{\frac{1}{2}}$.

The corresponding quartic (A),

$$u^4 - u^3 [x(1 + fh)h^{-\frac{1}{2}} + 2gh^{\frac{1}{2}}y] = x^2y^2 - u^2(fx^2 + 2gxy + hy^2),$$

becomes the complex

$$(x - uh^{\frac{1}{2}})[u^3h^{-\frac{1}{2}} - u^2(fx + 2gy) + y^2(x + uh^{\frac{1}{2}})] = 0.$$

The line does not touch the nodal cubic, unless $g^2 = 2 - 2fh$, when the subsidiary curves (C) and (D) have quadruple contact with each other (§ 6).

The equations to three other complexes are similarly found.

65. The last factor of § 62 contains the main solution

$$y + xp - \frac{2}{u}xy(c + dp) = 0.$$

When this condition is substituted in (3) and (4) of § 62, we obtain the locus of all folium-points in the binodal system (A),

$$2x^2y^2 + u^2(fx^2 - 2gxy + hy^2) = 0 \dots \dots \dots (E),$$

whether (u) be at a finite distance, or at infinity.

Ex.—Let $u \equiv x + y - 1$ and $f = -g = h = -2$.

Assume the folium-point in the locus $x^2y^2 = u^2(x + y)^2$, (E), to be $x = y = 2u = \frac{2}{3}$, so that $p + 1 = 0$.

By the aid of (E), (A) becomes $\xi + \eta = \frac{1}{3}u$; also $\xi - \eta = 0$.

The corresponding quartic hyperbola

$$u^3 \left[\frac{1}{3}u(x + y) - 1 \right] = x^2y^2 + 2u^2(x - y)^2,$$

when written in the form

$$(xy-4u^2)^2 = \frac{1}{4}u^2(u-\frac{1}{3})(u+\frac{8}{3}),$$

shows the tangent at the folium-point ($u = \frac{1}{3}$), and a conjugate or acu-bitangent, representing together four inflexions.

66. In the case of bicuspidal quartics $f = h = 0$; and the locus of folium-points (E) degenerates into the coordinate axes and the conic

$$xy - gu^2 = 0 \dots \dots \dots (E).$$

The equation to the discriminating curve is readily obtained

$$\xi x + \eta y - 1 = \frac{1}{u^3} x^2 y^2 - \frac{2}{u} gxy = -g^2 u \dots \dots \dots (A),$$

$$\xi + \eta p = -\frac{1}{2u} g(y+xp) = -g^2(c+dp) \dots \dots (2).$$

We have to eliminate x, y from the three equations

$$(\xi + g^2 c)x + (\eta + g^2 d)y = 1 + g^2,$$

$$[(1-2gcd)\xi + 2gc^2\eta + cg^2]x - [(1-2gcd)\eta + 2gd^2\xi + dg^2]y = 2g(c\eta - d\xi),$$

$$xy = g(cx + dy - 1)^2.$$

The eliminant is the class-quartic

$$\begin{aligned} & [\xi + 2g(\xi - c)(d\xi - c\eta) + g^2(\xi + c) + g^4c] \\ & \times [\eta + 2g(\eta - d)(c\eta - d\xi) + g^2(\eta + d) + g^4d] \\ & = g[2g^2cd + (1-g^2)(d\xi + c\eta) - 2\xi\eta]^2. \end{aligned}$$

For values of ξ, η , thus correlated, bicuspidal quartics with folium-points correspond; and for values of ξ, η , external to, or intersecting this discriminating quartic, bicuspidal quartics, with or without folia, correspond.

Ex. $g = 4$; $u \equiv x + y - 1$. Assume the folium-point ($x=y=2u=\frac{2}{3}$) in the locus $xy = 4u^2$ (E).

The parameters are found from the equations $\xi + \eta + \frac{1}{2}u^3 = 0$, $\xi - \eta = 0$. The corresponding bicuspidal quartic,

$$(xy-4u^2)^2 = \frac{1}{4}u^3(3u-1),$$

is bipartite; and its folium-point lies on the detached unifolium circuit.

67. This classification is equally applicable to spherical binodal curves of the fourth order, and to doubly bitangential class-quartics, both plane and spherical.