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XII. *On the Stability of two Rectilinear Vortices of Compressible Fluid moving in an Incompressible Liquid.* By BIBHUTIBHUSAN DATTA, M.Sc., Lecturer in Applied Mathematics, University of Calcutta, India*.

THE stability of the circular form in two rectilinear vortices has been discussed by Sir J. J. Thomson † for incompressible fluid.

Dr. Chree ‡ attempted to extend his treatment to a compressible fluid but did not succeed except in some special cases. The object of the present paper is to complete the work begun by Dr. Chree.

When the vortex-lines are straight lines parallel to the axis of z , the velocity components u, v at any point (x, y) in the fluid are given by §

$$u = -\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}; \quad . \quad . \quad (1)$$

$$\text{hence} \quad \nabla_1^2 \phi = -\theta, \quad \nabla_1^2 \psi = \zeta, \quad . \quad . \quad . \quad (2)$$

$$\text{where} \quad \theta = \frac{du}{dx} + \frac{dv}{dy}, \quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Then by the theory of attraction,

$$\left. \begin{aligned} \phi &= -\frac{1}{2\pi} \iint \theta' \log r \, dx' dy' + \phi_0, \\ \psi &= \frac{1}{2\pi} \iint \zeta' \log r \, dx' dy' + \psi_0, \end{aligned} \right\} \quad . \quad . \quad (3)$$

θ', ζ' being respectively the value of θ and ζ at the point (x', y') and

$$r = \{(x-x')^2 + (y-y')^2\}^{\frac{1}{2}}, \quad \nabla_1^2 \phi_0 = 0, \quad \nabla_1^2 \psi_0 = 0.$$

If $\frac{d}{dt}$ denote partial differentiation, and $\frac{D}{Dt}$ differentiation

* Communicated by the Author.

† 'Motion of Vortex Rings,' p. 71.

‡ Messenger of Mathematics, vol. xvii. p. 113 (1888).

§ Lamb, 'Hydrodynamics,' 4th ed. p. 213 (1916).

following the fluid, the equation of continuity is

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} = 0,$$

$$\text{or} \quad \frac{1}{\rho} \frac{D\rho}{Dt} + \theta = 0.$$

$$\therefore \quad \phi = \frac{1}{2\pi} \iint \frac{1}{\rho'} \frac{D\rho'}{Dt} \log r \, dx' \, dy' + \phi_0. \quad (4)$$

If ρ_0 be the mean value of the density of the fluid at the cross section σ of a circular vortex, defined by the relation

$$\iint \rho' \, dx' \, dy' = \rho_0 \sigma, \quad (5)$$

we get

$$\phi = \frac{1}{2\pi} \frac{\sigma}{\rho_0} \frac{D\rho_0}{Dt} \log r + \phi_0.$$

Again, the equation of continuity is

$$\frac{D \cdot \rho_0 \sigma}{Dt} = 0,$$

$$\text{or} \quad \frac{1}{\rho_0} \frac{D\rho_0}{Dt} + \frac{1}{\sigma} \frac{D\sigma}{Dt} = 0.$$

$$\text{Hence} \quad \phi = -\frac{1}{2\pi} \frac{D\sigma}{Dt} \log r + \phi_0. \quad (6)$$

Since σ always refers to a definite column of fluid, there is no difference between $\frac{D\sigma}{Dt}$ and $\frac{d\sigma}{dt}$.

Let us consider a truly circular cross section of radius a ; if the vorticity be the same at any instant at all points of the section, we have, when $r > a$,

$$\left. \begin{aligned} \phi &= -\frac{1}{2\pi} \frac{d\sigma}{dt} \log r + C, \\ \psi &= \frac{\kappa}{2\pi} \log r + D; \end{aligned} \right\} \quad (7)$$

where κ is the strength of the vortex, C and D being

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constants. Also when $r < a$,

$$\left. \begin{aligned} \phi &= -\frac{1}{4\pi a^2} \frac{d\sigma}{dt} r^2 + C', \\ \psi &= \frac{\kappa}{4\pi a^2} r^2 + D'. \end{aligned} \right\} \dots \dots \dots (8)$$

Let the equation of the boundary of the cross section at any instant be given by

$$r = a + \Sigma(\alpha_n \cos n\theta + \beta_n \sin n\theta), \quad \dots \dots (9)$$

where α_n and β_n are functions of t , independent of θ , and very small compared with a .

Outside the vortex, $r > a$, let

$$\left. \begin{aligned} \phi &= C - \frac{1}{2\pi} \frac{d\sigma}{dt} \log r + \Sigma(E_n \cos n\theta + F_n \sin n\theta) \left(\frac{a}{r}\right)^n, \\ \psi &= D + \frac{\kappa}{2\pi} \log r + \Sigma(A_n \cos n\theta + B_n \sin n\theta) \left(\frac{a}{r}\right)^n; \end{aligned} \right\} \quad (10)$$

inside the vortex, $r < a$,

$$\left. \begin{aligned} \phi' &= C' - \frac{1}{4\pi a^2} \frac{d\sigma}{dt} r^2 + \Sigma(E_n' \cos n\theta + F_n' \sin n\theta) \left(\frac{r}{a}\right)^n, \\ \psi' &= D' + \frac{\kappa}{4\pi a^2} r^2 + \Sigma(A_n' \cos n\theta + B_n' \sin n\theta) \left(\frac{r}{a}\right)^n. \end{aligned} \right\} \quad (11)$$

Since ϕ and ψ must be continuous when crossing the boundary, we at once get, neglecting products of small quantities,

$$A_n' = A_n, \quad B_n' = B_n, \quad E_n' = E_n, \quad F_n' = F_n. \quad \dots (12)$$

The radial and tangential velocities for points just inside and just outside the vortex must be the same. Therefore when

$$\begin{aligned} r &= a + \Sigma(\alpha_n \cos n\theta + \beta_n \sin n\theta) \\ \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial \phi}{\partial r} &= \frac{1}{r} \frac{\partial \psi'}{\partial \theta} + \frac{\partial \phi'}{\partial r} \\ \text{and} \quad -\frac{\partial \psi}{\partial r} + \frac{\partial \phi}{r \partial \theta} &= -\frac{\partial \psi'}{\partial r} + \frac{\partial \phi'}{r \partial \theta}. \end{aligned}$$

Since products of small terms are neglected, these conditions become

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi'}{\partial r}, \quad \frac{\partial \phi}{\partial r} = \frac{\partial \phi'}{\partial r}.$$

Substituting the values of ψ , ψ' , ϕ , ϕ' from (10), (11) and equating coefficients of $\cos n\theta$ and $\sin n\theta$, we get

$$A_n = -\frac{\kappa}{2\pi na} \alpha_n, \quad B_n = -\frac{\kappa}{2\pi na} \beta_n,$$

$$E_n = \frac{1}{2\pi na} \frac{d\sigma}{dt} \alpha_n, \quad F_n = \frac{1}{2\pi na} \frac{d\sigma}{dt} \beta_n. \quad \dots \quad (13)$$

Hence outside the vortex, we have

$$\left. \begin{aligned} \phi &= C - \frac{1}{2\pi} \frac{d\sigma}{dt} \log r + \Sigma \frac{\frac{d\sigma}{dt}}{2\pi na} (\alpha_n \cos n\theta + \beta_n \sin n\theta) \left(\frac{a}{r}\right)^n, \\ \psi &= D + \frac{\kappa}{2\pi} \log r - \Sigma \frac{\kappa}{2\pi na} (\alpha_n \cos n\theta + \beta_n \sin n\theta) \left(\frac{a}{r}\right)^n. \end{aligned} \right\} \quad (14)$$

Suppose there are two vortices of strengths κ , κ' ; let their cross sections at time t be σ and σ' respectively, the distance between their centres being c . Let the radii of their cross sections be given by

$$\left. \begin{aligned} R &= a + \Sigma (\alpha_n' \cos n\theta + \beta_n' \sin n\theta), \\ R' &= b + \Sigma (\alpha_n' \cos n\theta' + \beta_n' \sin n\theta'). \end{aligned} \right\} \quad \dots \quad (15)$$

Let ϕ , ψ be the functions at an external point for the first and ϕ' , ψ' for the second. If (r, θ) and (r', θ') be the coordinates of a point referred to the centres of the two vortices, ϕ and ψ will be given by (14), and

$$\left. \begin{aligned} \phi' &= C' - \frac{1}{2\pi} \frac{d\sigma'}{dt} \log r' + \Sigma \frac{\frac{d\sigma'}{dt}}{2\pi nb} (\alpha_n' \cos n\theta' + \beta_n' \sin n\theta') \left(\frac{b}{r'}\right)^n, \\ \psi' &= D' + \frac{\kappa'}{2\pi} \log r' - \Sigma \frac{\kappa'}{2\pi nb} (\alpha_n' \cos n\theta' + \beta_n' \sin n\theta') \left(\frac{b}{r'}\right)^n. \end{aligned} \right\} \quad (16)$$

We shall fix our attention on the second vortex. If $\mathfrak{R}$ be the radial velocity of a point on it and $b\Theta$ the velocity perpendicular to the radius vector, both relative to the centre

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of the vortex, we have from (15)

$$\mathfrak{K} = \frac{db}{dt} + \Sigma \left(\frac{d\alpha_n'}{dt} \cos n\theta' + \frac{d\beta_n'}{dt} \sin n\theta' \right) \\ - \Sigma n \Theta (\alpha_n' \sin n\theta' - \beta_n' \cos n\theta'). \quad \dots \quad (17)$$

Since products and powers of α_n' , β_n' are to be neglected, in (17) we may put

$$\Theta = \frac{\kappa'}{2\pi b^2}. \quad \dots \quad (18)$$

Again

$$\mathfrak{K} = -\frac{\partial \psi'}{r' \partial \theta'} - \frac{\partial \phi'}{\partial r'} - \frac{\partial \psi}{r' \partial \theta'} - \frac{\partial \phi}{\partial r'} - \frac{\kappa}{2\pi c} \sin(\theta' - \epsilon) \\ - \frac{1}{2\pi c} \frac{d\sigma}{dt} \cos(\theta' - \epsilon), \quad \dots \quad (19)$$

where r' must be put equal to $b + \Sigma(\alpha_n' \cos n\theta' + \beta_n' \sin n\theta')$ after differentiation, and ϵ is the angle made by the line joining the centres of the two vortices with its initial position. If b/c be small, ϕ and ψ can be expressed in terms of r' , θ' , and ϵ , such as

$$\phi = C - \frac{1}{2\pi} \frac{d\sigma}{dt} \left[\log c + \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} \left(\frac{r'}{c} \right)^s \cos s(\theta' - \epsilon) \right] \\ + \Sigma \frac{1}{2\pi n} \frac{d\sigma}{dt} \frac{a^{n-1}}{c^n} \left[(\alpha_n \cos n\epsilon + \beta_n \sin n\epsilon) \sum_{s=0}^{\infty} (-1)^s \right. \\ \times \frac{n(n+1) \dots (n+s-1)}{s!} \left(\frac{r'}{c} \right)^s \cos s(\theta' - \epsilon) \\ + (\beta_n \cos n\epsilon - \alpha_n \sin n\epsilon) \sum_{s=1}^{\infty} (-1)^{s-1} \\ \times \frac{n(n+1) \dots (n+s-1)}{s!} \left(\frac{r'}{c} \right)^s \sin s(\theta' - \epsilon) \left. \right], \\ \psi = D + \frac{\kappa}{2\pi} \left[\log c + \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} \left(\frac{r'}{c} \right)^s \cos s(\theta' - \epsilon) \right] \\ - \Sigma \frac{\kappa}{2\pi n} \cdot \frac{a^{n-1}}{c^n} \left[(\alpha_n \cos n\epsilon + \beta_n \sin n\epsilon) \sum_{s=0}^{\infty} (-1)^s \right. \\ \times \frac{n(n+1) \dots (n+s-1)}{s!} \left(\frac{r'}{c} \right)^s \cos s(\theta' - \epsilon) \\ + (\beta_n \cos n\epsilon - \alpha_n \sin n\epsilon) \sum_{s=1}^{\infty} (-1)^{s-1} \\ \times \frac{n(n+1) \dots (n+s-1)}{s!} \left(\frac{r'}{c} \right)^s \sin s(\theta' - \epsilon) \left. \right].$$

Substituting in (19) we get

$$\begin{aligned}
 \mathfrak{K} = & -\Sigma \frac{\kappa'}{2\pi b^2} (\alpha_n' \sin n\theta' - \beta_n' \cos n\theta') + \frac{1}{2\pi b} \frac{d\sigma'}{dt} \\
 & + \frac{\kappa}{2\pi} \sum_{s=1}^{\infty} (-1)^{s-1} \sin s(\theta' - \epsilon) \frac{b^{s-1}}{c^s} \\
 & \quad \times \left[1 + \frac{s-1}{b} \Sigma (\alpha_n' \cos n\theta' + \beta_n' \sin n\theta') \right] \\
 & + \Sigma \frac{\kappa}{2\pi n} \frac{a^{n-1}}{c^n} \left[(\alpha_n \cos n\epsilon + \beta_n \sin n\epsilon) \sum_{s=1}^{\infty} (-1)^{s-1} \right. \\
 & \quad \times \frac{n(n+1) \dots (n+s-1)}{(s-1)!} \frac{b^{s-1}}{c^s} \sin s(\theta' - \epsilon) \\
 & \quad + (\beta_n \cos n\epsilon - \alpha_n \sin n\epsilon) \sum_{s=1}^{\infty} (-1)^{s-1} \\
 & \quad \times \frac{n(n+1) \dots (n+s-1)}{(s-1)!} \frac{b^{s-1}}{c^s} \cos s(\theta' - \epsilon) \left. \right] \\
 & + \frac{1}{2\pi} \frac{d\sigma}{dt} \sum_{s=1}^{\infty} (-1)^{s-1} \cos s(\theta' - \epsilon) \frac{b^{s-1}}{c^s} \\
 & \quad \times \left[1 + \frac{s-1}{b} \Sigma (\alpha_n' \cos n\theta' + \beta_n' \sin n\theta') \right] \\
 & - \Sigma \frac{1}{2\pi n} \frac{d\sigma}{dt} \frac{a^{n-1}}{c^n} \left[(\alpha_n \cos n\epsilon + \beta_n \sin n\epsilon) \sum_{s=1}^{\infty} (-1)^s \right. \\
 & \quad \times \frac{n(n+1) \dots (n+s-1)}{(s-1)!} \frac{b^{s-1}}{c^s} \cos s(\theta' - \epsilon) \\
 & \quad + (\beta_n \cos n\epsilon - \alpha_n \sin n\epsilon) \sum_{s=1}^{\infty} (-1)^{s-1} \\
 & \quad \times \frac{n(n+1) \dots (n+s-1)}{(s-1)!} \frac{b^{s-1}}{c^s} \sin s(\theta' - \epsilon) \left. \right] \\
 & - \frac{\kappa}{2\pi c} \sin(\theta' - \epsilon) - \frac{1}{2\pi c} \frac{d\sigma}{dt} \cos(\theta' - \epsilon).
 \end{aligned}
 \tag{20}$$

The two values of $\mathfrak{K}$ as given by (17) and (20) must be the same; so that we can equate the terms independent of θ' in either expression as well as the coefficients of the sines and cosines of multiples of θ' . The terms independent of θ' simply verify the known result that $\frac{d\sigma'}{dt} = 2\pi b \frac{db}{dt}$.

Equating the coefficients of $\cos \theta'$, we have

$$\frac{d\alpha_1'}{dt} = \frac{\kappa}{2\pi c^2} (\beta_1 \cos 2\epsilon - \alpha_1 \sin 2\epsilon) + \frac{1}{2\pi c^2} \frac{d\sigma}{dt} (\alpha_1 \cos 2\epsilon + \beta_1 \sin 2\epsilon);$$

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 so that to our degree of approximation $\frac{d\alpha_1'}{dt}=0$; similarly
 we get $\frac{d\beta_1'}{dt}=0$. Hence, so far as these terms are con-
 cerned, compressibility of fluid makes no change in the
 motion or shape of the vortices.

Equating the coefficients of $\cos 2\theta'$ and $\sin 2\theta'$, we get to
 the same degree of approximation

$$\frac{d\alpha_2'}{dt} + \frac{\kappa'}{2\pi b^2}\beta_2' = \frac{\kappa b}{2\pi c^2}\sin 2\epsilon - \frac{b}{2\pi c^2}\frac{d\sigma}{dt}\cos 2\epsilon, \quad (21)$$

$$\frac{d\beta_2'}{dt} - \frac{\kappa'}{2\pi b^2}\alpha_2' = -\frac{\kappa b}{2\pi c^2}\cos 2\epsilon - \frac{b}{2\pi c^2}\frac{d\sigma}{dt}\sin 2\epsilon. \quad (22)$$

About these equations Dr. Chree remarks: "It is scarcely
 likely that these equations admit of a complete solution."*
 But the solution of them is of essential importance, for on
 it depends the determination of the exact change in the
 form of the cross section. I shall presently show that they
 can be solved. The terms in α_3, β_3 will involve $\frac{a^3}{c^3}$, and thus
 will be relatively unimportant, so we shall not determine
 them here.

Let
$$\begin{aligned} u &= \alpha_2' + i\beta_2', \\ v &= \alpha_2' - i\beta_2', \end{aligned} \quad \text{where } i = \sqrt{-1}. \quad (23)$$

Multiply the equation (22) by i : the two equations (21),
 (22) become, by addition and subtraction,

$$\left. \begin{aligned} \frac{du}{dt} - i\frac{\kappa'}{2\pi b^2}u &= -i\frac{\kappa b}{2\pi c^2}e^{2i\epsilon} - \frac{b}{2\pi c^2}\frac{d\sigma}{dt}e^{2i\epsilon}, \\ \frac{dv}{dt} + i\frac{\kappa'}{2\pi b^2}v &= i\frac{\kappa b}{2\pi c^2}e^{-2i\epsilon} - \frac{b}{2\pi c^2}\frac{d\sigma}{dt}e^{-2i\epsilon}. \end{aligned} \right\} \quad (24)$$

These are linear differential equations. On integration
 we get

$$\left. \begin{aligned} ue^{-i\kappa' \int \frac{dt}{2\pi b^2}} &= u_0 - \int_0^t e^{i(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2})} \left\{ i\frac{\kappa b}{2\pi c^2} + \frac{b}{2\pi c^2}\frac{d\sigma}{dt} \right\} dt, \\ ve^{i\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}} &= v_0 + \int_0^t e^{-i(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2})} \left\{ i\frac{\kappa b}{2\pi c^2} - \frac{b}{2\pi c^2}\frac{d\sigma}{dt} \right\} dt, \end{aligned} \right\} \quad (25)$$

where u_0 and v_0 are initial values of u and v . If ${}_0\alpha_2', {}_0\beta_2'$ be
 the values of α_2', β_2' respectively when $t=0$, we must have

$$u_0 = {}_0\alpha_2' + i{}_0\beta_2', \quad v_0 = {}_0\alpha_2' - i{}_0\beta_2'. \quad (26)$$

* *L. c.* pp. 115, 118.

Now, substituting the values of u and v from (23) in (25), we get by addition and subtraction,

$$\begin{aligned} \alpha_2' \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) + \beta_2' \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \\ = {}_0\alpha_2' + \int_0^t \left[\frac{\kappa b}{2\pi c^2} \sin\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ \left. - \frac{b}{2\pi c^2} \frac{d\sigma}{dt} \cos\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt, \\ \alpha_2' \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) - \beta_2' \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \\ = -{}_0\beta_2' + \int_0^t \left[\frac{\kappa b}{2\pi c^2} \cos\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ \left. + \frac{b}{2\pi c^2} \frac{d\sigma}{dt} \sin\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt. \end{aligned}$$

Solving these equations for α_2' and β_2' , we get

$$\begin{aligned} \alpha_2' = {}_0\alpha_2' \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) - {}_0\beta_2' \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \\ + \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa \sin\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ \left. - \frac{d\sigma}{dt} \cos\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt \\ + \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa \cos\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ \left. + \frac{d\sigma}{dt} \sin\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt, \end{aligned} \quad (27)$$

$$\begin{aligned} \beta_2' = {}_0\alpha_2' \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) + {}_0\beta_2' \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \\ + \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa \sin\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ \left. - \frac{d\sigma}{dt} \cos\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt \\ - \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa \cos\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ \left. + \frac{d\sigma}{dt} \sin\left(2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt. \end{aligned} \quad (28)$$

In our further investigations, we should remember that if $\epsilon=0$ when $t=0$, it can be proved that*

$$\epsilon = \frac{\kappa + \kappa'}{2\pi} \int_0^t \frac{dt}{c_0^2 + (\sigma + \sigma' - \sigma_0 - \sigma_0')/\pi}, \quad \dots \quad (29)$$

where c_0 , σ_0 , σ_0' are the values of c , σ , σ' respectively when $t=0$.

The equations (27) and (28) show that there are two distinct types of vibrations of α_2' and β_2' of gradually varying periods given by

$$\frac{\kappa'}{2\pi} \int_0^t \frac{dt}{b^2} = 2n\pi, \quad \dots \quad (30)$$

$$\frac{\kappa + \kappa'}{\pi} \int_0^t \frac{dt}{c_0^2 + (\sigma + \sigma' - \sigma_0 - \sigma_0')/\pi} = 2n\pi, \quad \dots \quad (31)$$

where n is any integer. Evidently the period given by (30) is shorter than the period given by (31).

Suppose ${}_0\alpha_2' = {}_0\beta_2' = 0$; since $2\epsilon - \frac{\kappa'}{2\pi} \int \frac{dt}{b^2}$ is always negative, we get

$$\begin{aligned} \alpha_2' = & -\cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2} - 2\epsilon\right) \right. \\ & \left. + \frac{d\sigma}{dt} \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2} - 2\epsilon\right) \right] dt \\ & + \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2} - 2\epsilon\right) \right. \\ & \left. - \frac{d\sigma}{dt} \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2} - 2\epsilon\right) \right] dt, \quad (32) \end{aligned}$$

so that vibrations are becoming more and more prominent as the vortices are approaching one another; since

$$\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2} - 2\epsilon \right)$$

is always real and b/c is also small, the value of α_2' cannot increase indefinitely so that the motion is stable†.

We shall now proceed to consider some interesting particular cases.

* Chree, *l. c.* p. 109.

† This point will be clearer from equations (25) which are of the type

$$ue^{if_1(t)} = u_0 - \int_0^t e^{if_2(t)} F\left(\frac{b}{c^2}\right) dt.$$

Since $\frac{dc^2}{dt} = 2\frac{d}{dt}b^2$ (Chree, p. 108), the function $F\left(\frac{b}{c^2}\right)$ continually decreases as t increases so that in the limit the integral must be finite.

*Case I.—Two columns of fluids of varying density
possessing no vorticity.*

In this case $\kappa = \kappa' = \epsilon = 0$,
so that the equations (27) and (28) become

$$\alpha_2' = {}_0\alpha_2' - \int_0^t \frac{b}{2\pi c^2} \frac{d\sigma}{dt} dt,$$

$$\beta_2' = {}_0\beta_2';$$

which agree with the result obtained by Dr. Chree. Since $\frac{d\sigma}{dt} = 2\pi a \frac{da}{dt}$, α_2' would increase or decrease according as a was decreasing or increasing. "Thus, if both columns were diminishing in cross section, and so approaching, there would be a decided tendency in both cross sections to assume an elliptical sort of outline, the major axes coinciding with the line joining their centres."

*Case II.—Two rectilinear vortices in an
incompressible fluid.*

As previously mentioned, this case has been studied by Sir J. J. Thomson. We shall simply show that his results can be deduced from our more general results. Let ${}_0\alpha_2' = {}_0\beta_2' = 0$; also if n be the angular velocity of the central line, $\epsilon = nt$. Then putting $\frac{d\sigma}{dt} = 0$ in (27) we get, since $\kappa' = \zeta' \pi b^2$, $\kappa = \zeta \pi a^2$,

$$\alpha_2' = \cos\left(\frac{\zeta'}{2}t\right) \int_0^t \frac{\zeta a^2 b}{2c^2} \sin\left(2nt - \frac{\zeta'}{2}t\right) dt$$

$$+ \sin\left(\frac{\zeta'}{2}t\right) \int_0^t \frac{\zeta a^2 b}{2c^2} \cos\left(2n - \frac{\zeta'}{2}\right)t dt,$$

$$\text{or } \alpha_2' = \frac{\zeta' a^2 b}{2c^2 \left(\frac{\zeta'}{2} - 2n\right)} \left\{ \cos 2nt - \cos\left(\frac{\zeta'}{2}t\right) \right\};$$

similarly from (28),

$$\beta_2' = \frac{\zeta' a^2 b}{2c^2 \left(\frac{\zeta'}{2} - 2n\right)} \left\{ \sin 2nt - \sin\left(\frac{\zeta'}{2}t\right) \right\}.$$

These will be found identical with the results obtained by Sir J. J. Thomson* when we remember that $\zeta = 2\omega$, and $\zeta' = 2\omega'$.

* 'Motion of Vortex Rings, p. 77.

Case III.—A Vortex-pair.

A combination of two equal and opposite vortices is called a "vortex-pair." Put $\kappa = -\kappa'$, $a = b$; then $\epsilon = 0$: we have from (32)

$$\begin{aligned} \alpha_2' = & \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa' \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ & \left. - \frac{d\sigma'}{dt} \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt \\ & - \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{2\pi c^2} \left[\kappa' \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ & \left. + \frac{d\sigma'}{dt} \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt; \end{aligned}$$

so that in this case, the vortices have a single type of disturbance of period given by

$$\frac{\kappa'}{2\pi} \int \frac{dt}{b^2} = 2\pi n.$$

As a vortex-pair is the two-dimensional analogue of a circular vortex-ring some of the properties of the latter may be deduced from it.

Case IV.—A single vortex parallel to an infinite wall.

This is really a subcase of Case III. For the velocity of the fluid at all points of the plane of symmetry is wholly tangential, so that this plane may be supposed to form a rigid boundary of the fluid on either side of it; thus we obtain the case of a single rectilinear vortex in the neighbourhood of a fixed plane wall to which it is parallel. If h be the distance of the vortex from the wall we get

$$\begin{aligned} \alpha_2' = & \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{8\pi h^2} \left[\kappa' \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ & \left. - \frac{d\sigma'}{dt} \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt \\ & - \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \int_0^t \frac{b}{8\pi h^2} \left[\kappa' \cos\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right. \\ & \left. + \frac{d\sigma'}{dt} \sin\left(\frac{\kappa'}{2\pi} \int \frac{dt}{b^2}\right) \right] dt. \end{aligned}$$

Hence the diameter parallel to the wall is greater if b is increasing, *i. e.* the vortex is receding from the wall and *vice versa*.

University of Calcutta,
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