ON THE HESSIAN CONFIGURATION AND ITS CONNECTION WITH THE GROUP OF 360 PLANE COLLINEATIONS

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THE Hessian configuration is the name given to a set of nine points in a plane which lie three by three on twelve straight lines. Its most familiar form is that given by the nine inflexions of a real cubic curve. The object of the first part of this memoir is to establish the existence of the configuration and to deduce its principal properties, especially the nature of the group of collineations for which the configuration is invariant, from a purely geometrical point of view. This group in its abstract form and in its analytical form as a group of linear substitutions in three variables has formed the subject of several investigations. The earliest is due to M. Jordan (*Traité des Substitutions*, pp. 302–305); while one of the most recent is given by Herr Weber (*Lehrbuch der Algebra*, Vol. II., pp. 400–410). None of these investigations with which I am acquainted, however, approaches the problem from the point of view which most naturally presents itself, namely, as a question of pure projective geometry. This is the point of view here taken, and it is contended that both the properties of the configuration and the nature of the group thereby appear in a clearer light.

In the second part of the memoir it is shewn that, starting from the Hessian configuration, there may be constructed a very remarkable configuration of 45 points of which the following are some of the properties:—

The line joining any two of the points passes through either one, two, or three others. The points lie 5 by 5 on 36 lines, 4 by 4 on 45 lines, and 3 by 3 on 120 lines. From the 45 points just 10 Hessian configurations can be formed, each two of which have just one of the points in common.

Finally, it is shown that such a configuration is invariant for a group of 360 collineations, which is simply isomorphic with the alternating group on six symbols.

The existence of such a group of collineations, which was established by H. Valentiner (*Die endelige Transformations-Gruppen Theorie*, 1889) on analytical grounds, is here shewn to follow from purely geometrical considerations.

As regards notation, some mode of representing collineations in a plane is necessary. Any such collineation is completely determined by the positions of the points A', B', C', D' into which it changes four given points A, B, C, D, no three of which are collinear. The symbol

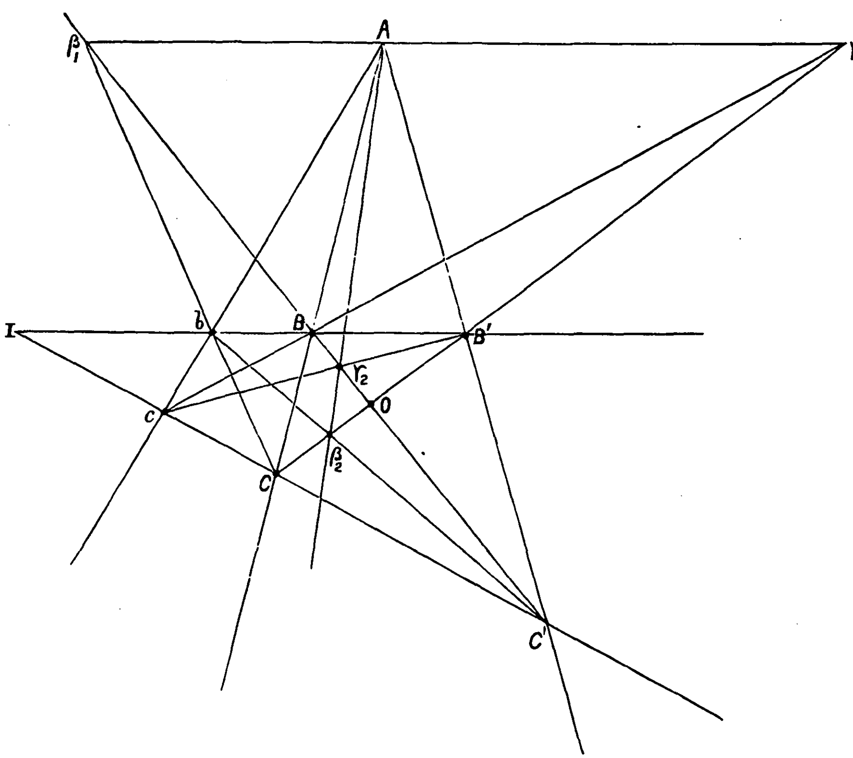
$$\begin{pmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \end{pmatrix}$$

therefore completely specifies such a collineation. If the collineation is one of finite order, it will permute sets of points cyclically. Its effect on such sets of points may be represented in the usual way by the symbol

$$(PQR \ldots T).$$

I.

1. Take four points B, C, B', C', no three of which lie in a line. Let BC and $B'C'$ meet in A. Through A draw Abc, meeting BB' and CC' in b and c. Denote by β_1 and γ_1 the points in which bC and cB meet BC'



and CB' respectively; and by β_2 and γ_2 the points in which bC' and cB' meet $B'C$ and $C'B$.

To the pencil of lines through A, of which Abc is one, correspond the two projective ranges described by b and c on BB' and CC'. At the same time β_1, γ_1, β_2, γ_2 describe projective ranges on BC', CB', $B'C$, $C'B$ respectively.

Particular positions of the four points β_1, γ_1, β_2, γ_2 are determined by the table :—

Abc	β_1	γ_1	β_2	γ_2
ABC	B	C	O	O
$AB'C'$	O	O	B'	C'
AI	C'	B'	C	B

where O is the intersection of BC' with $B'C$, and I that of BB' with CC'. The projective ranges β_1, γ_1 on the lines BC' and CB', having a self-corresponding point, viz., O, are in perspective. Also in two particular positions, viz., those corresponding to the positions ABC and AI of Abc, $\beta_1\gamma_1$ passes through A. Hence the projective ranges β_1, γ_1 are in perspective with respect to A. Similarly, the projective ranges β_2, γ_2 are in perspective with respect to A.

Again β_2 and γ_1 give projective ranges on the line CB'. To the three positions

$$O,\ B',\ C$$

of β_2 there correspond the three positions

$$C,\ O,\ B'$$

of γ_1. Hence the projective transformation of CB' which changes the first of these projective ranges into the second is

$$\begin{pmatrix} OB'C \\ COB' \end{pmatrix}.$$

If this projective transformation is repeated three times, it leads to

$$\begin{pmatrix} OB'C \\ OB'C \end{pmatrix}$$

which, leaving three distinct points of the line unchanged, leaves every point unchanged. Hence the projective transformation is a projective transformation of order 3.

It has therefore two distinct (imaginary if B, C, B', C' are real) fixed points. The two projective ranges β_2, γ_1 on CB' have therefore two distinct self-corresponding points (which are imaginary when the four original points are real). Denote them by B'' and B''_0. Similarly the two projective ranges γ_2, β_1 on $C'B$ have two self-corresponding points C'' and C''_0. Also, since $A\beta_1\gamma_1$, $A\beta_2\gamma_2$ are straight lines, to each of B'' and B''_0 there must correspond one of C'' and C''_0, such that the lines joining the corresponding pairs pass through A; say $AB''C''$ and $AB''_0C''_0$.

To the positions B'', C'' of β_2, γ_1 and γ_2, β_1 corresponds a definite position $AB'''C'''$ of Abc: and to the positions B''_0, C''_0 there corresponds another definite position $AB'''_0C'''_0$ of Abc.

Consider now the nine points

$$A, B, C, B', C', B'', C'', B''', C'''.$$

It follows immediately from the figure that

$$ABC,\ AB'C',\ AB''C'',\ AB'''C''',$$
$$BB'B''',\ BB''C''',\ BC'C'',$$
$$CC'C''',\ CC''B''',\ CB'B'',$$
$$B'C''C''',\ C'B''B'''$$

are straight lines. In other words, the straight line joining any two of these nine points passes through a third; or the nine points lie three by three on 12 straight lines. The same is also true of the nine points

$$A, B, C, B', C', B''_0, C''_0, B'''_0, C'''_0,$$

where the last four points are distinct from the last four of the previous set.

2. The existence of a Hessian configuration is thus proved, and it is shown that the given construction leads to one or the other of two distinct configurations. The next point to consider is how these two configurations are related. With this object, consider the effect of the projective transformation of order 2 defined by

$$\begin{pmatrix} B\ C\ B'C' \\ C'B'C\ B \end{pmatrix}$$

on either of them. This transformation permutes the five points A, B, C, B', C' among themselves. It leaves the line CB' unchanged, and effects on it the projective transformation of order 2

$$\begin{pmatrix} OB'C \\ OCB' \end{pmatrix}.$$

The previously considered transformation of order 3 on CB', viz.,

$$\begin{pmatrix} OB'C \\ COB' \end{pmatrix},$$

of which B'' and B''_0 are the fixed points, is changed into its inverse by the transformation of order 2; for obviously

$$\begin{pmatrix} OB'C \\ OCB' \end{pmatrix} \begin{pmatrix} OB'C \\ COB' \end{pmatrix} \begin{pmatrix} OB'C \\ OCB' \end{pmatrix} = \begin{pmatrix} OB'C \\ B'CO \end{pmatrix}.$$

Hence B'' and B''_0 are permuted by the projective transformation of the plane of order 2. Similarly, C'' and C''_0 are permuted by it. But when seven of the points of a Hessian configuration are known the remaining two can be determined by drawing straight lines. Hence the projective transformation which changes

$$A,\ B,\ C,\ B',\ C',\ B'',\ C''$$

into

$$A,\ B,\ C,\ B',\ C',\ B''_0,\ C''_0$$

necessarily changes the pair B''', C''' into the pair B'''_0, C'''_0.

The two Hessian configurations are therefore transformed each into the other by the projective transformation

$$\begin{pmatrix} B\ C\ B'C' \\ C'B'C\ B \end{pmatrix}.$$

Returning now again to the construction, it involves not only four points, but also a particular pair of lines through them. Three such pairs may be drawn, viz., BC, $B'C'$, meeting in A; BB', CC', meeting in I; $B'C$, BC', meeting in O. If B, C, B', C' belong to a Hessian configuration of nine points, either A, I, or O must also belong to the configuration. Suppose, in fact, that neither A nor I belongs to it. Then, besides B, C, B', C', the configuration has one distinct point on each of the lines BC, $B'C'$, BB', CC'; so that there is only one remaining point. Hence O, in which $B'C$, BC' intersect, must be a point of the configuration. Now I and O do not belong to the two distinct configurations already determined which contain A. Hence in all just six distinct Hessian configurations can be constructed to contain any given four points, no three of which lie in a line. Moreover the set of 24 plane collineations which permute among themselves the four points B, C, B', C' also obviously permute A, I, O; and it has been seen that one of these collineations which leaves A unchanged permutes the two corresponding Hessian configurations. Therefore the six distinct Hessian configurations which can be constructed to contain four given points (no three of which lie on a straight line) are transitively permuted among themselves by the group of 24 collineations which permutes the four points.

But any four points of a plane, no three of which are collinear, can be projected into any other four. Hence any one Hessian configuration can be projected into any other. It follows from this that the distribution of the nine points on twelve lines given on p. 57 is quite general.

[*February*, 1906.—The six Hessian configurations each of which contains the four points B, C, B', C', while each also contains either A, I, or O, contain in all just 12 other points, each of which occurs in two of the configurations. These 12 points lie three by three on 8 lines, two of which pass through each of the four points B, C, B', C'. The two lines through B, containing 6 of the 12 points, are the two (imaginary) fixed lines of the collineation which, leaving B unchanged, permutes B', C, C' cyclically; and the 6 points are the points in which these two fixed lines of the collineation meet $B'C$, CC', and $C'B'$. That a set of 12, and not 24, points arises in this way follows from the fact that the (imaginary) fixed lines of the collineations which leave B (or B') unchanged and permute cyclically B', C, C' (or B, C, C') meet CC' in the same pair of points.]

3. From the twelve lines just four sets of three may be formed such that each set contains all nine points. These sets are:—

$$ABC,\quad B'C''C''',\quad C'B''B''';$$
$$AB'C',\quad BB''C''',\quad CB'''C'';$$
$$AB''C'',\quad BB'B''',\quad CC'C''';$$
$$AB'''C''',\quad BC'C'',\quad CB'B''.$$

The lines of any one set intersect those of any other in a point belonging to the configuration. The three lines of any one set intersect in three points which do not belong to the configuration. There thus arises a set of twelve points, whose relations to the configurations will be determined.

The collineation of order 2 defined by

$$\begin{pmatrix} BCB'C' \\ CBC'B' \end{pmatrix}$$

is a perspective with A for its vertex (or fixed point), and IO in the figure for its axis or fixed line. Since it leaves A unchanged and permutes B, C, B', C', it must either leave unchanged or permute the two configurations which have these five points in common. Now neither B''_0, C''_0, B'''_0 nor C'''_0 lies on AB''. Hence the collineation leaves unchanged each of the configurations. It therefore permutes B'' with C'' and B''' with C'''. Similarly, there is a collineation of order 2 with any other one of the nine points for its vertex which permutes the remaining eight points of the configuration in pairs. The axes of these perspectives corresponding to

$$A,\ B,\ C,\ B',\ C',\ B'',\ C'',\ B''',\ C''',$$

as vertices will be denoted by

$$a,\ b,\ c,\ b',\ c',\ b'',\ c'',\ b''',\ c'''.$$

The perspective of order 2 with A for its vertex and a for its axis—say Aa —leaves four of the twelve lines, viz., ABC, $AB'C'$, $AB''C''$, $AB'''C'''$, unchanged, and permutes the remaining pairs which belong to the four sets, viz.,

$$B'C''C''',\ C'B''B''';\quad BB''C''',\ CB'''C'';$$

$$BB'B''',\quad CC'C''';\quad BC'C'',\ CB'B''.$$

So each other perspective such as Bb leaves unchanged one of the twelve lines in each of the four sets of three and permutes the remaining two. But the only points which are unchanged for a perspective other than its vertex are the points on its axis. Hence the set of twelve points which arise from the intersections of the twelve lines lie four by four on the set of nine lines

$$a,\ b,\ c,\ b',\ c',\ b'',\ c'',\ b''',\ c'''.$$

These nine lines then conversely pass three by three through the twelve points. In fact, each of the three perspectives Aa, Bb, Cc permutes $B'C''C'''$ and $C'B''B'''$; so that a, b, c pass through that one of the twelve points which is determined by the intersection of $B'C''C'''$ and $C'B''B'''$. This point may be conveniently denoted by abc; and then to each of the twelve lines such as ABC will correspond uniquely one of the twelve points, viz., abc; just as to each of the nine points such as A there corresponds uniquely one of the nine lines, viz., a.

4. The configuration has hitherto been regarded as consisting of the original nine points. The phrase may now be used in a more extended sense as including :—

(α) a set of nine points, (γ) a set of twelve points,

(β) a set of twelve lines, (δ) a set of nine lines.

The points of (α) lie three by three on the lines of (β). The intersections of the lines of (β) other than the points of (α) are the points of (γ), and these lie four by four on the lines of (δ). Moreover there is a unique one-to-one correspondence between the elements of (α) and (δ), and also between the elements of (β) and (γ).

5. It has been seen that the configuration is unchanged by the nine perspectives of order 2 of which Aa is typical. It will now be shown that there is also a system of perspectives of order 3, of which the points of (γ) and the corresponding lines of (β) are the vertices and axes, for which also the configuration is invariant.

Consider the collineation defined by

$$\begin{pmatrix} B'C'B''C'' \\ C''B''B'''C''' \end{pmatrix}.$$

This collineation changes $B'C'$ and $B''C''$ into $B''C''$ and $B'''C'''$; and therefore leaves A unchanged. It also leaves $B'C''C'''$ and $C'B''B'''$, two lines intersecting in abc, unchanged. Hence it leaves every line through abc unchanged; *i.e.*, it is a perspective of which abc is the vertex. Now b is a line through abc, and $AB'''C'''$, $CB'B''$; $AB'C'$, $CC''B'''$; $AB''C''$, $CC'C'''$; are the three pairs of the twelve lines which intersect on b. But the collineation in question changes $B'B''$ into $C''B'''$, $B'C'$ into $B''C''$, $B''C''$ into $B'''C'''$. Hence it permutes cyclically the three points in which the above three pairs of lines meet b. The collineation, when repeated three times, leaves therefore every line through abc and every point on b unaltered; and therefore it leaves every point in the plane unaltered. It is therefore a collineation of order 3, and changes C''' and B''' into B' and C' respectively. Since it leaves A unchanged and permutes B', C', B'', C'', B''', C''' among themselves, it must leave the configuration unchanged; and therefore it must leave the two remaining points B and C unchanged. The collineation is therefore a perspective of order 3 of which abc is the vertex and ABC is the axis. Similarly, the configuration is invariant for each of the perspectives of order 3 for which one of the twelve points is the vertex, and the corresponding one of the twelve lines is the axis.

It has been seen that, given 5 of the 9 points belonging to the configuration, such as A, B, C, B', C', they determine one of the 9 lines, viz., a; and then from these (which may be all real) just two configurations may be formed.

Suppose now that three collinear points A, B, C of the 9 are given, and the corresponding three concurrent lines a, b, c. These again may be all real. Containing these an infinite number of configurations may be formed, any two of which are in perspective, with abc for the vertex and ABC for the axis of the perspective.

The three points A, B, C and the three lines a, b, c are permuted among themselves by the collineations Aa, Bb, Cc, as also must be every Hessian configuration containing them. Now Aa followed by Bb is a collineation of order 3 of which abc is a fixed point and ABC a fixed line, A, B, C being permuted cyclically on it.

This collineation has two (imaginary) fixed lines i, j through abc. Since $B'C''C'''$ and $C'B''B'''$ are invariant for it, they must coincide with i, j. Hence, to construct the configuration, B' may be taken to be *any* point on i, and the remaining 5 points are then determinate. Any two such Hessian configurations which have A, B, C, a, b, c in common must clearly have either all the rest of the 9 points and 9 lines in common (*i.e.*, be identical) or have none of them in common. In the latter case the remaining 6 of the 9 points of each configuration lie in threes on the lines i and j.

6. From the perspectives of order 3 arise the whole of the collineations for which the configuration is invariant. There are just four, with their inverses, which leave the point A unchanged. These are

$$abcABC,\quad ab'c'AB'C',\quad ab''c''AB''C'',\quad ab'''c'''AB'''C'''.$$

The permutations which they give of the eight points, other than A, are respectively

$$(B'C''C''')(C'B''B'''),\quad (BB''C''')(CC''B'''),$$
$$(BB'B''')(CC'C'''),\quad (BC'C'')(CB'B''),$$

and their inverses, the unchanged points being in each case unwritten, those written being permuted cyclically.

From the collineations written it is clear that collineations arise giving all possible even permutations of the four lines ABC, $AB'C'$, $AB''C''$, $AB'''C'''$. It remains to determine whether any collineation for which the configuration is invariant can give an odd permutation of the lines; and, secondly, what collineations leave the configuration and each of the four lines invariant.

Since all possible even permutations of the lines occur, it is sufficient to consider a collineation which interchanges ABC, $AB'C'$. Such a collineation must be either

$$\begin{pmatrix} BCB'C' \\ C'B'CB \end{pmatrix},\quad \begin{pmatrix} BCB'C' \\ B'C'BC \end{pmatrix},\quad \begin{pmatrix} BCB'C' \\ B'C'CB \end{pmatrix},\quad \text{or}\quad \begin{pmatrix} BCB'C' \\ C'B'BC \end{pmatrix}.$$

Of these it is shewn in § 2 that the first does not leave the configuration invariant. The second arises by combining the first with

$$\begin{pmatrix} BCB'C' \\ CBC'B' \end{pmatrix},$$

for which the configuration is invariant. The configuration therefore is not invariant for the second. The third and fourth are inverses of each other, and it is sufficient to consider one of them. But the third arises on combining the two perspectives of order 3, $(BB''C''')(CC''B''')$ and $(BC''C')(CB''B')$, of which $ab'c'$, $ab'''c'''$ are the vertices and $AB'C'$, $AB'''C'''$ the axes. It therefore gives an even permutation of the four lines. There are therefore no collineations for which the configuration is invariant that give an odd permutation of the four lines.

A collineation which leaves each of the four lines unchanged must permute or leave unchanged the members of each B, C pair. Consider then the collineation

$$\begin{pmatrix} BCB'C' \\ BCC'B' \end{pmatrix}$$

which permutes one pair and leaves unchanged the members of another. If this left $AB''C''$ and $AB'''C'''$ unchanged, it would leave every line through A unchanged, and would be a perspective with A for its vertex, which it is not. Hence the only collineation, other than identity, which leaves each of the four lines and the configuration unchanged is the perspective of order 2, Aa.

The number of even permutations of four symbols is twelve. Hence the order of the greatest group of collineations for which the configuration and the point A are invariant is 24. This sub-group contains the perspective of order 2, Aa, as a self-conjugate operation, and in respect of it is simply isomorphic with a tetrahedral group. Moreover, it contains no collineations of order 2 except the perspective Aa. Now A may be changed into any one of the other eight points by collineations for which the configuration is invariant. Hence the order of the greatest group for which the configuration is invariant is 216. It may be noticed, as following obviously from the present point of view, that the only collineations of order 2 in the group are the nine perspectives of the set Aa.

7. For the sequel two sub-groups of the G_{216}, which leaves the configuration invariant, are of special importance. The first is the sub-group generated by the 9 perspectives of order 2. Since these are the only perspectives of order 2, this sub-group is an invariant sub-group of the G_{216}. Its order is 18, and besides the perspectives of order 2 it contains 8 collineations (not perspectives) of order 3 and identity. This is at once verified by the permutations of the 9 points that the perspectives of order 2 give rise to. The fixed points of the 8 collineations of order 3 (each occurring with its inverse) are the 12 γ-points of the configuration. This sub-group will be called the G_{18}. In respect of it the G_{216} has been shewn to be simply isomorphic with a tetrahedral group. The latter has three sub-groups of order 2, forming a conjugate set. The G_{216} has therefore three sub-groups of order 36 (each containing the G_{18}) which form a conjugate set. Any one of these will be denoted by G_{36}. They arise by combining the G_{18} with any one of the collineations of order 4 belonging to the G_{216}. Such a collineation of order 4, arising by combining the two perspectives of order 3, $ab'c'AB'C'$ and $ab''c''AB''C''$, gives

the permutation

$$(BB''CC'')(B'B'''C'C''')$$

of the 8 points other than A; and the particular G_{36}, which is made use of in the sequel, is the group that is generated by the G_{18} and this collineation of order 4.

II.

8. I consider now sets of points which are permuted by the G_{18} that arises from the nine collineations of order 2. In general, such a set of points will consist of 18 members; but, if one of the points is on one of the nine lines of the Hessian configuration, the set will have only 9 members. In this case a uniform notation will be used,

$$A_i,\ B_i,\ C_i,\ B'_i,\ C'_i,\ B''_i,\ C''_i,\ B'''_i,\ C'''_i,$$

denoting the set of 9 permuted points lying respectively on

$$a,\ b,\ c,\ b',\ c',\ b'',\ c'',\ b''',\ c'''.$$

When the Hessian configuration and one point of such a set is given, the others are determined by drawing straight lines: *e.g.*, B'_i is the point of intersection of b' with the line joining A_i to C'.

Four such sets, with suffixes 1, 2, 3, 4, are formed as follows:—

Through A draw $AB_1B'_2$, meeting b, b' in B_1 and B'_2; and construct the sets 1 and 2. Join A to B''_1, and let it meet b' in B'_3. Form the set 3. Join A to B'''_2, and let it meet b in B_4. Form the set 4. As $AB_1B'_2$ turns round A, AB'''_3 and AB''_4 describe superposed projective pencils, which must have two self-corresponding rays. Let $AB_1B'_2$ and $A\bar{B}_1\bar{B}'_2$ be the positions of the original line which lead to the self-corresponding rays in these two projective pencils. Then the four sets of 9 points each which arise from A_1, A_2, A_3, A_4 are such that

$$AB_1C_1B'_2C'_2,\quad AB''_1C''_1B'_3C'_3,\quad AB_4C_4B'''_2C'''_2,\quad AB''_4C''_4B'''_3C'''_3,$$

and the other 32 symbols that arise from them by the collineations of the G_{18} represent straight lines.

The same statement is true for the four sets of 9 points each which arise from $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$.

Moreover, these are the only two sets of 36 points (consisting each of 4 sets of 9 points conjugate in respect of the G_{18}) for which the statement is true.

The G_{18} is contained self-conjugately in a G_{72} formed by combining it with the collineations of order 4 denoted by the permutations

$$(BB''CC'')(B'B'''C'C'''),$$
$$(BB'CC')(B''C'''C''B'''),$$
$$(BB'''CC''')(B'C''C'B'').$$

Each of these collineations then must either leave unchanged or permute the two sets of 36 points. It is easily verified that the second and third collineations permute the sets ; and therefore the first must change each set with itself. If io be used to denote the set into which the collineation $(BB''CC'')(B'B'''C'C''')$ changes the set i, then it changes the four lines

$$AB_1C_1B'_2C'_2, \quad AB''_1C''_1B'_3C'_3, \quad AB_4C_4B'''_2C'''_2, \quad AB''_4C''_4B'''_3C'''_3$$

into the lines

$$AB''_{10}C''_{10}B'''_{20}C'''_{20}, \quad AC_{10}B_{10}B'''_{30}C'''_{30}, \quad AB''_{40}C''_{40}C'_{20}B'_{20}, \quad AC_{40}B_{40}C'_{30}B'_{30},$$

and, apart from sequence, the one set of lines is identical with the other. Hence, on comparison, the set 10 is 4, 40 is 1, 20 is 3, and 30 is 2; in other words, the collineation $(BB''CC'')(B'B'''C'C''')$ permutes the two sets 1 and 4 together and the two sets 2 and 3.

Considering then one set of 36 points, in respect of the G_{36} that arises on combining the G_{18} with the collineation $(BB''CC'')(B'B'''C'C''')$, it consists of two sets of 18 points each, each set being transitively permuted among themselves by the G_{36}. Moreover, the 36 points lie four by four on the set of 36 lines given in the following table, which themselves are permuted transitively in two sets of 18 each by the G_{36}. Through each of the nine original points of the Hessian configuration just 4 of the 36 lines pass. Hence the original 9 points and the 36 constructed from them form a set of 45, which lie five by five on a set of 36 lines that pass four by four through each of them.

TABLE I.

$AB_1C_1B'_2C'_2$,	$AB_4C_4B'''_2C'''_2$,	$AB''_1C''_1B'_3C'_3$,	$AB''_4C''_4B'''_3C'''_3$,
$BA_1C_1B''_2C'''_2$,	$BA_4C_4C''_2C'_2$,	$BB'_1B'''_1B''_3C'''_3$,	$BB'_4B'''_4C''_3C'_3$,
$CB_1A_1B'''_2C''_2$,	$CB_4A_4B'_2B''_2$,	$CC'''_1C'_1B'''_3C''_3$,	$CC'''_4C'_4B'_3B''_3$,
$B'C''_1C'''_1A_2C'_2$,	$B'C''_4C'''_4B''_2C_2$,	$B'B'''_1B_1A_3C'_3$,	$B'B'''_4B_4B''_3C_3$,
$C'B'''_1B''_1B'_2A_2$,	$C'B'''_4B''_4B_2C''_2$,	$C'C_1C'''_1B'_3A_3$,	$C'C_4C'''_4B_3C''_3$,
$B''C'_1B'''_1C'''_2B_2$,	$B''C'_4B'''_4C_2B'_2$,	$B''A_1C''_1C'''_3B_3$,	$B''A_4C''_4C_3B'_3$,
$C''C'''_1B'_1C_2B'''_2$,	$C''C'''_4B'_4C'_2B_2$,	$C''B''_1A_1C_3B'''_3$,	$C''B''_4A_4C'_3B_3$,
$B'''B''_1C'_1C''_2C_2$,	$B'''B''_4C'_4A_2C'''_2$,	$B'''B_1B'_1C''_3C_3$,	$B'''B_4B'_4A_3C'''_3$,
$C'''B'_1C''_1B_2B''_2$,	$C'''B'_4C''_4B'''_2A_2$,	$C'''C'_1C_1B_3B''_3$,	$C'''C'_4C_4B'''_3A_3$.

9. Although the foregoing geometrical construction is formally sufficient to determine the system of points and lines, it cannot be actually carried out. Indeed, of the nine points and nine lines of the Hessian configuration not more than either (i) three points and three lines, (ii) five points and one line, or (iii) one point and five lines, can be real. An actual specification of the points and lines is necessarily analytical. The formulæ are as follows. Taking ABC, $B'C'''C''$, $C'B''B'''$ as the sides of the triangle of reference, the nine collineations of order 2 of the G_{18} with their fixed points and fixed lines are given by the table:—

$(\omega^3 = 1.)$

	Substitution.	Fixed Point.	Fixed Line.
Aa	(x, z, y)	$x = 0, \quad y+z = 0$	$y-z = 0$
Bb	$(x, \omega z, \omega^2 y)$	$x = 0, \; y+\omega z = 0$	$y-\omega z = 0$
Cc	$(x, \omega^2 z, \omega y)$	$x = 0, \; y+\omega^2 z = 0$	$y-\omega^2 z = 0$
$B'b'$	(z, y, x)	$y = 0, \quad z+x = 0$	$z-x = 0$
$C'''c'''$	$(\omega^2 z, y, \omega x)$	$y = 0, \; z+\omega x = 0$	$z-\omega x = 0$
$C''c''$	$(\omega z, y, \omega^2 z)$	$y = 0, \; z+\omega^2 x = 0$	$z-\omega^2 x = 0$
$C'c'$	(y, x, z)	$z = 0, \quad x+y = 0$	$x-y = 0$
$B''b''$	$(\omega y, \omega^2 x, z)$	$z = 0, \; x+\omega y = 0$	$x-\omega y = 0$
$B'''b'''$	$(\omega^2 y, \omega x, z)$	$z = 0, \; x+\omega^2 y = 0$	$x-\omega^2 y = 0$

With this table there is no difficulty in carrying out the calculation, which presents no point of interest. The result is that the four points A_1, A_4, A_2, A_3 are

$$\lambda, 1, 1; \quad \frac{\lambda}{(\omega-\omega^2)\lambda-\omega^2}, 1, 1; \quad -\frac{1+\lambda}{\lambda}, 1, 1; \quad \omega^2-1+\omega\lambda, 1, 1,$$

where λ is an assigned root of

$$\lambda^2+(3+4\omega)\lambda-2\omega^2 = 0,$$

the other root giving the four points $\bar{A}_1$, $\bar{A}_4$, $\bar{A}_2$, $\bar{A}_3$. From the coordinates of the A's those of the B's, &c., can be calculated by the previous table.

This analytical specification of the points may be used to verify the existence of further collinearities among them in a very simple manner. The fact that an adequate figure cannot be drawn, owing to most of the points being imaginary, renders a geometrical treatment of this point very difficult to carry out.

The coordinates of B_4''', which is changed into A_4 by the collineation $C'''c'''$, are

$$\omega^2, 1, \frac{\lambda\omega}{\lambda(\omega-\omega^2)-\omega^2},$$

and, since

$$\begin{vmatrix} \omega^2, & 1, & \dfrac{\lambda\omega}{\lambda(\omega-\omega^2)-\omega^2} \\ 1, & 1, & -1 \\ \lambda, & 1, & 1 \end{vmatrix} = 0,$$

B_4''' lies in a line with A and A_1. Now $AB_4'''C_4'''$ is a line; and hence $AA_1B_4'''C_4'''$ is a line. The collineation

$$(BB''CC'')(B'B'''C'C''')$$

changes A, A_1, B_4''', C_4''' into A, A_4, C_1', B_1' respectively. Hence $AA_4C_1'B_1'$ is also a line. It may be similarly verified that $AA_2B''_3C''_3$ and $AA_3B_2C_2$ are lines. Since B'''_4 is changed into C'''_4 by Aa, which leaves A and A_1 unchanged, $B'''_4C'''_4$ divide AA_1 harmonically; and similarly $B''_3C''_3$ divide AA_2 harmonically. It also follows directly from the coordinates of A_1, A_4, A_2, A_3 that A_2A_3 divide A_1A_4 harmonically.

10. The table of the 36 lines containing the 45 points five by five may now be supplemented by one of 45 lines which contain the same 45 points four by four.

TABLE II.

$AA_1B'''_4C'''_4$,	$AA_4C'_1B'_1$,	$AA_2B''_3C''_3$,	$AA_3B_2C_2$,	$A_1A_4A_2A_3$,
$BB_1C''_4C'_4$,	$BB_4C'''_1B''_1$,	$BB_2B'_3B'''_3$,	$BB_3C_2A_2$,	$B_1B_4B_2B_3$,
$CC_1B'_4B''_4$,	$CC_4C''_1B'''_1$,	$CC_2C'''_3C'_3$,	$CC_3A_2B_2$,	$C_1C_4C_2C_3$,
$B'B'_1B''_4C_4$,	$B'B'_4C'_1A_1$,	$B'B'_2B'''_3B_3$,	$B'B'_3C''_2C'''_2$,	$B'_1B'_4B'_2B'_3$,
$C'C'_1B_4C''_4$,	$C'C'_4A_1B'_1$,	$C'C'_2C_3C'''_3$,	$C'C'_3B'''_2B''_2$,	$C'_1C'_4C'_2C'_3$,
$B''B''_1C_4B'_4$,	$B''B''_4B_1C'''_1$,	$B''B''_2A_3C''_3$,	$B''B''_3C'_2B'''_2$,	$B''_1B''_4B''_2B''_3$,
$C''C''_1C'_4B_4$,	$C''C''_4B'''_1C_1$,	$C''C''_2B''_3A_3$,	$C''C''_3C'''_2B'_2$,	$C''_1C''_4C''_2C''_3$,
$B'''B'''_1A_4C'''_4$,	$B'''B'''_4C_1C''_1$,	$B'''B'''_2B_3B'_3$,	$B'''B'''_3B''_2C'_2$,	$B'''_1B'''_4B'''_2B'''_3$,
$C'''C'''_1B'''_4A_4$,	$C'''C'''_4B''_1B_1$,	$C'''C'''_2C'_3C_3$,	$C'''C'''_3B'_2C''_2$,	$C'''_1C'''_4C'''_2C'''_3$.

Each line of this table, the formation of which from the previous data is obvious, contains 4 of the 45 points, and in each the first pair divide the second pair harmonically. Further, there are just 4 of the set of lines passing through any one of the 45 points. This and the previous table contain implicitly all the properties of the configuration of 45 points which has been constructed.

11. An inspection of the two tables shews that the set of 45 points is invariant for collineations which do not belong to the G_{36} in connection with which the set arises.

Consider in particular the perspective of order 2, defined by

$$\begin{pmatrix} BCB''_2C''_2 \\ B''_2C''_2BC \end{pmatrix}.$$

Since ABC, $AB''_2C''_2$, $BA_1C_1B''_2C'''_2$, $CB_1A_1B'''_2C''_2$, $BA_4C_4C''_2C'_2$, $CB_4A_4B'_2B''_2$ are straight lines, A, A_1, A_4 are unchanged by the perspective; and three lines through A_1, being unchanged, every line through A_1 must be unchanged. Hence A_1 is the fixed point and AA_4 the fixed line of the perspective; and B'_1, C'_1, being points on AA_4, are unchanged. Further,

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since $A_1A_4A_2A_3$, $AA_1B_4'''C_4'''$, $B'B_4'C_1'A_1$, $C'C_4'A_1B_1'$ are lines in each of which the first pair of points divide the second pair harmonically, A_2, A_3 are permuted by the perspective, as also are B_4''', C_4'''; B', B_4'; and C', C_4'. The perspective therefore leaves A, A_1, A_4, B_1', C_1' unchanged, and gives the permutations

$$(A_2A_3)(BB_2'')(CC_2'')(B'B_4')(C'C_4')(B_4'''C_4''').$$

Further permutations are obtained from the tables by taking a pair of lines intersecting in a common point (belonging to the 45) and determining the lines into which they are changed by the perspective. Thus $BB_3A_2C_2$ and $C'C_4C_4'''B_3C_3''$ become $B''B_2''A_3C_3''$ and $B''C_4'B_4'''C_2B_2'$; so that B_3 and B'' are permuted by the perspective. Continuing in this way, it may be very easily verified that the whole of the 45 points are permuted by the perspective of which A_1 is the fixed point and AA_4 the axis, the actual permutations being :—

$$A,\ A_1,\ A_4,\ B_1',\ C_1'\ \text{unchanged},$$

$$(BB_2'')(CC_2'')(B'B_4')(C'C_4')(B''B_3)(C''C_3)(B'''C_1''')(C'''B_1'''),$$

$$(A_2A_3)(B_1B_2''')(C_1C_2''')(B_1''B_3''')(C_1''C_3''')(B_2B_3'')(C_2C_3''),$$

$$(B_2'C_4)(C_2'B_4)(B_3'B_4'')(C_3'C_4'')(B_4'''C_4''').$$

Similarly, it may be shewn that the perspective of order 2 of which A_2 is the fixed point and AA_3 the fixed line permutes the 45 points among themselves.

Now, for the G_{36}, A_1 and A_2 are each one of a set of 18 conjugate points; and AA_4, AA_3 each belong to a set of 18 conjugate lines. From the two perspectives of which A_1 and A_2 are the fixed points, and AA_4, AA_3 the fixed lines, there thus arises a set of 36 perspectives of order 2, for every one of which the configuration of 45 points is invariant. These, with the original nine perspectives of order 2, belonging to the G_{36}, give a set of 45, each with one of the 45 points for fixed point and one of the 45 lines (of Table II.) for fixed line. A set of five is

$$A(A_1A_4A_2A_3),\quad A_1(AA_4),\quad A_4(AA_1),\quad A_2(AA_3),\quad A_3(AA_2),$$

and the remainder are formed by replacing A by B, C, B', C', B'', C'', B''', or C'''.

An inspection of the permutations of the 45 points given by $A_1(AA_4)$ shews that they form a single conjugate set with respect to the group G of collineations generated by the G_{36} and the perspective of order 2, $A_1(AA_4)$. Hence the set of 45 perspectives of order 2 forms a single conjugate set of collineations for G. For a group of plane collineations of finite order cannot contain two distinct perspectives of order 2 with a common vertex.

12. The four lines

$$ABC, \quad AB'C', \quad AB''C'', \quad AB'''C'''$$

are changed by $A_1(AA_4)$ into

$$AB''_2C''_2, \quad AB'_4C'_4, \quad AB_3C_3, \quad AC'''_1B'''_1.$$

Thus the 16 points of the configuration which do not lie on either of the 4 lines of Table I., or of the 4 lines of Table II., that pass through A, lie by pairs on 8 other lines through A. Of these lines which contain the 45 points three by three there are 120, and through each point of the 45 eight of these lines pass. From this it follows that the straight line joining any two points of the configuration passes through either one, two, or three others.

13. The Hessian configuration

$$A, B, C, B', C', B'', C'', B''', C''',$$

formed of 9 out of the 45 points, is changed by $A_1(AA_4)$ into the Hessian configuration $A, B''_2, C''_2, B'_4, C'_4, B_3, C_3, C'''_1, B'''_1,$

containing only one point in common with the previous one. Moreover, no one of the set of 12 (β) lines of the first coincides with one of the set of 12 of the second; and each of these sets of 12 belongs to the set of 120 of the last paragraph.

Any Hessian configuration into which the first is changed by a collineation belonging to G must have for its (β) lines 12 from the set of 120 just mentioned. If two configurations have one of these lines, say ABC, in common, the three points A, B, C on it belong to each, and the three lines a, b, c are (δ) lines for each. But then, by § 5, the remaining 12 points making up the configurations would lie on two lines, 6 on each. Now no 6 of the 45 points lie on a line. Hence no two Hessian configurations, formed from the 45 points by the collineations of G, can have a (β) line in common. There are then at most 10 such Hessian configurations. On the other hand,

$$A, B, C, B', C', B'', C'', B''', C'''$$

is changed into another Hessian configuration having any single one of the 9 points written in common with it, by a suitably chosen one of the 45 perspectives of order 2. For instance, $B''_1(B''B''_4)$ changes it into one having B'' in common with it. There are then at least 10 such Hessian configurations. Combining the two results, it follows that from the original Hessian configuration just 10 Hessian configurations can be formed by the collineations of G, having for their (β) lines the lines of the set of 120. Each two of these Hessian configurations have just one point in common; and, conversely, each point of the 45 enters in just two Hessian configurations. Moreover, the 10 Hessian configurations are transitively permuted by the collineations of G; and any collineation which changes each Hessian configuration into itself changes each of the 45 points into itself, and is therefore the identical collineation. But it has already been seen that the greatest sub-group of the G_{216} for which a Hessian configuration is invariant, which leaves invariant the set of 45 points, is a G_{36}. Hence the order of G is 360. It follows immediately that G is the greatest group of collineations for which the 45 points are invariant.

14. The six lines in the upper left-hand corner of Table I., viz.,

$$AB_1C_1B_2'C_2', \quad AB_4C_4B_2'''C_2''', \quad \text{say} \quad l_1, \ l_4,$$

$$BA_1C_1B_2''C_2''', \quad BA_4C_4C_2''C_2', \quad \text{say} \quad l_2, \ l_5,$$

$$CB_1A_1B_2'''C_2'', \quad CB_4A_4B_2'B_2'', \quad \text{say} \quad l_3, \ l_6,$$

contain just 15 of the 45 points, which constitute their complete intersection. The collineations Aa, Bb, Cc, and $A_1(AA_4)$ permute these lines among themselves, giving in fact the permutations

$$(l_2l_3)(l_5l_6), \quad (l_3l_1)(l_6l_4), \quad (l_1l_2)(l_4l_5), \quad (l_1l_4)(l_5l_6).$$

The collineation of order 3, $abcABC$, gives the permutation

$$(l_1l_2l_3)(l_4l_5l_6);$$

and therefore this collineation followed by the perspective $A_1(AA_4)$ leaves l_5 unchanged, and gives the permutation

$$(l_1l_2l_3l_4l_6)$$

of the other five. The six lines are therefore permuted among themselves by an icosahedral group of 60 collineations, and they hence form one of not more than six such sets of six lines which are permuted transitively by G. Now the perspective $A_2(AA_3)$ leaves A unchanged, and changes the lines l_1, l_2, l_3, l_4, l_5, l_6 into another set of six. Every point of the 45 then occurs in at least two such conjugate sets of 15, which constitute the complete intersection of a set of 6 lines. Hence there are not less than six sets of six lines transitively permuted by the group. The icosahedral group of 60 collineations is therefore the greatest sub-group of G for which the set of lines l_1, l_2, l_3, l_4, l_5, l_6 is invariant; and by the collineations of G just six such sets arise which are permuted transitively. Each of the 45 points occurring in just two sets, any collineation which leaves each set unchanged must leave each point of the 45 unchanged, and is the identical collineation. The group of collineations G is therefore simply isomorphic with a group of permutations of six symbols. Hence, the order of G being 360, it is simply isomorphic with the alternating group of six symbols.

Of the eight lines which pass through A and contain just three of the 45 points, two, viz., ABC, $AB_2''C_2''$, occur in connection with the 15 points which form the complete intersection of l_1, l_2, l_3, l_4, l_5, l_6. Two more must occur in connection with the complete intersection of the six lines into which l_1, l_2, l_3, l_4, l_5, l_6 are changed by the perspective $A_2(AA_3)$. There cannot therefore be more than one other set of six lines, containing l_1, and having 15 of the 45 points for their complete intersection. An inspection of Table I. shews that there is just one other such set, viz.,

$$AB_1C_1B_2'C_2', \qquad AB_1''C_1''B_3'C_3',$$
$$B_1'''B_1''C'A_2B_2', \qquad B_1'''B'B_1C_3'A_3,$$
$$C_1'''B'C_1''C_2'A_2, \qquad C_1'''C'C_1A_3B_3'.$$

The 36 lines can therefore be divided in just two distinct ways into six sets of six each, such that the complete intersection of any set of six is 15 of the 45 points; and each of these two sets of six are permuted transitively among themselves by the collineations of the group.