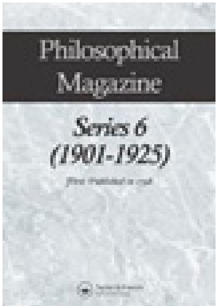

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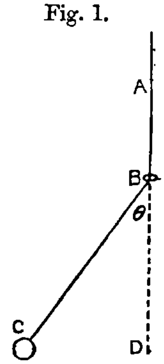
XXXIV. *On the Pressure of Vibrations.*
By Lord RAYLEIGH, *F.R.S.**

THE importance of the consequences deduced by Boltzmann and W. Wien from the doctrine of the pressure of radiation has naturally drawn increased attention to this subject. That æthereal vibrations must exercise a pressure upon a perfectly conducting, and therefore perfectly reflecting, boundary was Maxwell's deduction from his general equations of the electromagnetic field; and the existence of the pressure of light has lately been confirmed experimentally by Lebedew. It seemed to me that it would be of interest to inquire whether other kinds of vibration exercise a pressure, and if possible to frame a general theory of the action.

We are at once confronted with a difference between the conditions to be dealt with in the case of æthereal vibrations and, for example, the vibrations of air. When a plate of polished silver advances against waves of light, the waves indeed are reflected, but the medium itself must be supposed capable of penetrating the plate; whereas in the corresponding case of aerial vibrations the air as well as the vibrations are compressed by the advancing wall. In other cases, however, a closer parallelism may be established. Thus the transverse vibrations of a stretched string, or wire, may be supposed to be limited by a small ring constrained to remain upon the equilibrium line of the string, but capable of sliding freely upon it. In this arrangement the string passes but the vibrations are compressed, when the ring moves inwards.

Fig. 1.



We will commence with the very simple problem of a pendulum in which a mass C is suspended by a string. B is a ring constrained to the vertical line AD and capable of moving along it; $BC=l$, and θ denotes the angle between BC and AD at any time t. If B is held at rest, BC is an ordinary pendulum, and it is supposed to be executing small vibrations; so that $\theta=\Theta\cos nt$, where $n^2=g/l$. The tension of the string is approximately W, the weight of the bob; and the force tending to push B upwards is at time t $W(1-\cos\theta)$. Now this expression is closely related to the potential energy of the pendulum, for which

$$V=Wl(1-\cos\theta).$$

* Communicated by the Author.

The mean upward force upon B is accordingly equal to the mean value of $V \div l$; or since the mean value of V is half the constant total energy E of the system, we conclude that the mean force (L), driving B upwards, is measured by $\frac{1}{2}E/l$.

From the equation

$$L = \tfrac{1}{2}E/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

it is easy to deduce the effect of a *slow* motion upwards of the ring. The work obtained at B must be at the expense of the energy of the system, so that

$$dE = -L\,dl = -\tfrac{1}{2}E\,dl/l.$$

By integration

$$E = E_1 l^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where E_1 denotes the energy corresponding to $l=1$. From (2) we see that by withdrawing the ring B until l is infinitely great, the whole of the energy of vibration may be abstracted in the form of work done by B, and this by a uniform motion in which no regard is paid to the momentary phase of the vibration.

The argument is nearly the same for the case of a stretched string vibrating transversely in one plane. The string itself may be supposed to be unlimited, while the vibrations are confined by two rings of which one may be fixed and one movable.

If the origin of x be at one end of a string of length l, the transverse displacement may be expressed by

$$y = \phi_1 \sin\frac{\pi x}{l} + \phi_2 \sin\frac{2\pi x}{l} + \ldots, \quad . \quad . \quad . \quad (3)$$

$$\dot{y} = \dot{\phi}_1 \sin\frac{\pi x}{l} + \dot{\phi}_2 \sin\frac{2\pi x}{l} + \ldots, \quad . \quad . \quad . \quad (4)$$

where $\phi_1, \phi_2, \ldots$ are coefficients depending upon the time. For the kinetic and potential energies we have respectively ('Theory of Sound,' § 128)

$$T = \tfrac{1}{4}\rho l \sum_{s=1}^{s=\infty} \dot{\phi}_s^2, \qquad V = \tfrac{1}{4}Wl \sum_{s=1}^{s=\infty} \frac{s^2\pi^2}{l^2}\phi_s^2, \quad . \quad . \quad (5)$$

in which W represents the constant tension and ρ the longitudinal density of the string. For each kind of ϕ the sums of T and V remain constant during the vibration; and the same is of course true of the totals given in (5).

From (3)

$$\frac{dy}{dx} = \frac{\pi}{l}\left(\phi_1 \cos\frac{\pi x}{l} + 2\phi_2 \cos\frac{2\pi x}{l} + \ldots\right),$$

so that when $x=l$

$$\frac{dy}{dx}=\frac{\pi}{l}(-\phi_1+2\phi_2-3\phi_3+\ldots).$$

Accordingly the force tending to drive out the ring at $x=l$ is at time t

$$\tfrac{1}{2}W\,.\,\frac{\pi^2}{l^2}(-\phi_1+2\phi_2-3\phi_3+\ldots)^2,$$

or in the mean taken over a long interval,

$$\tfrac{1}{2}W\,.\,\text{Mean}\,\Sigma\frac{s^2\pi^2}{l^2}\phi_s^2.$$

Comparing with (5), we see that the mean force L has the value $2l\times$ mean V; or since mean V = mean T = $\frac{1}{2}$E, E denoting the constant total energy,

$$L=E/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The force driving out the ring is thus numerically equal to the *longitudinal density of the energy*.

This result may readily be extended to cases where the vibrations are not limited to one plane; and indeed the case in which the plane of the string uniformly revolves is especially simple in that T and V are then constant with respect to time.

If the ring is allowed to move out slowly, we have

$$dE=-L\,dl=-E\,dl/l,$$

or on integration

$$E=E_1 l^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

analogous to (5), though different from it in the power of l involved. If l increase without limit, the whole energy of the vibrations may be abstracted in the form of work done on the ring.

We will now pass on to consider the case of air in a cylinder, vibrating in one dimension and supposed to obey Boyle's law according to which $p=a^2\rho$. By the general hydrodynamical equation ('Theory of Sound,' § 253 *a*),

$$\varpi=\int\frac{dp}{\rho}=-\frac{d\phi}{dt}-\tfrac{1}{2}U^2, \quad . \quad . \quad . \quad . \quad (8)$$

where ϕ denotes the velocity-potential and U the resultant velocity at any point; so that in the present case, if we integrate over a long interval of time,

$$a^2\int\log p\,dt+\tfrac{1}{2}\int U^2\,dt \quad . \quad . \quad . \quad . \quad . \quad (9)$$

retains a constant value over the length of the cylinder. If p_0 denote the pressure that would prevail throughout, had there been no vibrations, $p-p_0$ is small and we may replace (9) by

$$a^2\int\left\{\frac{p-p_0}{p_0}-\frac{1}{2}\frac{(p-p_0)^2}{p_0^2}\right\}dt+\frac{1}{2}\int U^2\,dt. \quad . \quad . \quad (10)$$

The expression (10) has accordingly the same value at the piston as for the mean of the whole column of length l. Now for the mean of the whole column

$$\int(p-p_0)\,dx=0;$$

and thus if p_1 denote the value of p at the piston where $x=l$,

$$a^2\int\left\{\frac{p_1-p_0}{p_0}-\frac{1}{2}\frac{(p_1-p_0)^2}{p_0^2}\right\}dt$$

$$=-\frac{a^2}{2l}\iint\frac{(p-p_0)^2}{p_0^2}\,dx\,dt+\frac{1}{2l}\iint U^2\,dx\,dt. \quad . \quad (11)$$

It is not difficult to prove that the right-hand member of (11) vanishes. Thus, expressing the motion in terms of ϕ, suppose that

$$\phi=\cos\frac{s\pi x}{l}\cos\frac{s\pi at}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Then

$$p-p_0=\rho_0\,d\phi/dt, \qquad U=d\phi/dx;$$

and since $p_0=a^2\rho_0$, we get

$$\frac{1}{2l}\iint\left\{\left(\frac{d\phi}{dx}\right)^2-\frac{1}{a^2}\left(\frac{d\phi}{dt}\right)^2\right\}dx\,dt,$$

and this vanishes by (12). Accordingly

$$\int(p_1-p_0)\,dt=\int\frac{(p_1-p_0)^2}{2p_0}\,dt. \quad . \quad . \quad . \quad . \quad (13)$$

Again by (12)

$$\int\left(\frac{d\phi}{dt}\right)_l^2 dt=\frac{2}{l}\iint\left(\frac{d\phi}{dt}\right)^2 dx\,dt,$$

so that

$$\int(p_1-p_0)\,dt=\frac{1}{p_0 l}\iint(p_1-p_0)^2\,dx\,dt = \frac{\rho_0}{l}\iint U^2\,dx\,dt.$$

Now $\rho_0\iint U^2\,dx\,dt$ represents twice the mean total kinetic energy of the vibrations or, what is the same, the constant total energy E. Thus if L denote the mean additional force due to the vibrations and tending to push the piston out,

$$L=E\,l^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

As in the case of the string, the total force is measured by the longitudinal density of the total energy; or, if we prefer so to express it, the additional *pressure* is measured by the volume-density of the energy.

In the last problem, as well as in that of the string, the vibrations are in one dimension. In the case of air there is no difficulty in the extension to two or three dimensions. Thus, if aerial vibrations be distributed equally in all directions, the pressure due to them coincides with *one-third* of the volume-density of the energy. In the case of the string, where the vibrations are transverse, we cannot find an analogue in three dimensions; but a membrane with a flexible and extensible boundary capable of slipping along the surface, provides for two dimensions. If the vibrations be equally distributed in the plane, the force outwards per unit length of contour will be measured by one-half of the superficial density of the total energy.

A more general treatment of the question may be effected by means of Lagrange's theory. If l be one of the coordinates fixing the configuration of a system, the corresponding equation is

$$\frac{d}{dt}\left(\frac{dT}{dl'}\right)-\frac{dT}{dl}+\frac{dV}{dl}=L, \quad . \quad . \quad . \quad . \quad (15)$$

where T and V denote as usual the expressions for the kinetic and potential energies. On integration over a time t_1

$$\int\frac{L\,dt}{t_1}=\frac{1}{t_1}\left[\frac{dT}{dl'}\right]+\frac{1}{t_1}\int\left(\frac{dV}{dl}-\frac{dT}{dl}\right)dt.$$

If dT/dl' remain finite throughout, and if the range of integration be sufficiently extended, the integrated term disappears, and we get

$$\int\frac{L\,dt}{t_1}=\frac{1}{t_1}\int\left(\frac{dV}{dt}-\frac{dT}{dl}\right)dt. \quad . \quad . \quad . \quad (16)$$

On the right hand of (16) the differentiations are partial, the coordinates other than l and all the velocities being supposed constant.

We will apply our equation (16) in the first place to the simple pendulum of fig. 1, l denoting the length of the vibrating portion of the string BC. If x, y be the horizontal and vertical coordinates of C,

$$x = l \sin\theta, \qquad y = l - l\cos\theta;$$

and accordingly if the mass of C be taken to be unity,

$$T = \tfrac{1}{2}l'^2(2 - 2\cos\theta) + l'\theta' \,.\, l\sin\theta + \tfrac{1}{2}\theta'^2 l^2, \quad . \quad . \quad (17)$$

l', θ' denoting dl/dt, $d\theta/dt$. Also

$$V = gl\,(1 - \cos\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

From (17), (18)

$$\frac{dV}{dl} = g(1 - \cos\theta), \quad \frac{dT}{dl} = l'\theta' \sin\theta + \theta'^2 l. \quad . \quad . \quad (19)$$

These expressions are general; but for our present purpose it will suffice if we suppose that l' is zero, that is that the ring is held at rest. Accordingly

$$\frac{dV}{dl} = \frac{V}{l}, \qquad \frac{dT}{dl} = \frac{2T}{l},$$

and (16) gives

$$\int \frac{L\,dt}{t_1} = \frac{1}{t_1}\int \frac{V - 2T}{l}\,dt. \quad . \quad . \quad . \quad . \quad (20)$$

On the right hand of (20) we find the mean values of V and and of T. But these mean values are equal. In fact

$$\int V dt = \int T\,dt = \tfrac{1}{2}E t_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

if E denote the total energy. Hence, if L now denote the mean value,

$$L = -\tfrac{1}{2}E/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the negative sign denoting that the mean force necessary to hold the ring at rest must be applied in the direction which tends to diminish l, i. e. downwards. In former equations (1), (6), (14), L had the reverse sign.

We will now consider more generally the case of one dimension, using a method that will apply equally whether for example the vibrating body be a stretched string, or a rod vibrating flexurally. All that we postulate is homogeneity of constitution, so that what can be said about any part of the length can be said equally about any other part. In applying Lagrange's method the coordinates are l the length of the vibrating portion, and ϕ_1, ϕ_2, &c. defining, as in (3), the displacement from equilibrium during the vibrations. As functions of l, we suppose that

$$V \propto l^m, \qquad T \propto l^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Thus, if L be the force corresponding to l, we get by (16)

$$\int \frac{\mathrm{L}\,dt}{t_1} = \frac{1}{t_1}\int\left(\frac{m\mathrm{V}}{l} - \frac{n\mathrm{T}}{l}\right)dt,$$

in which

$$\int \mathrm{V}\,dt = \int \mathrm{T}\,dt = \tfrac{1}{2}\mathrm{E}\,.\,t_1,$$

E representing as before the constant total energy. Accordingly, L now representing the mean value,

$$\mathrm{L} = \frac{(m-n)\mathrm{E}}{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

In the case of a medium, like a stretched string, propagating waves of all lengths with the same velocity, $m=-1$, $n=1$, and $\mathrm{L}=-\mathrm{E}/l$, as was found before.

In the application to a rod vibrating flexurally, $m=-3$, $n=1$, so that

$$\mathrm{L} = -2\mathrm{E}/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

If $m=n$, L vanishes. This occurs in the case of the line of disconnected pendulums considered by Reynolds in illustration of the theory of the group velocity *, and the circumstance suggests that L represents the tendency of a group of waves to spread. This conjecture is easily verified. If in conformity with (13) we suppose that

$$\mathrm{V} = \mathrm{V}_0\, l^m\, \phi_1^2, \qquad \mathrm{T} = \mathrm{T}_0\, l^n\, \dot{\phi}_1^2,$$

and also that

$$\phi_1 = \sin\frac{2\pi t}{\tau}, \qquad \dot{\phi}_1 = \frac{2\pi}{\tau}\cos\frac{2\pi t}{\tau},$$

τ being the period of the vibration represented by the coordinate ϕ_1, we obtain, remembering that the sum of T and V must remain constant,

$$\mathrm{V}_0\, l^m = \mathrm{T}_0\, l^n\,.\, 4\pi/\tau^2.$$

This gives the relation between τ and l. Now v, the wave-velocity, is proportional to l/τ; so that

$$v \propto l^{1-\frac{1}{2}n+\frac{1}{2}m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Thus, if u denote the group-velocity, we have by the general theory

$$u/v = \tfrac{1}{2}n - \tfrac{1}{2}m\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

and in terms of u and v by (24)

$$\mathrm{L} = -\frac{u\mathrm{E}}{vl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

* See Proc. Math. Soc. ix. p. 21 (1877); Scientific Papers, i. p. 322. Also Theory of Sound, vol. i. Appendix.

Boltzmann's theory is founded upon the application of Carnot's cycle to the radiation inclosed within movable reflecting walls. If the pressure (p) of a body be regarded as a function of the volume v *, and the absolute temperature θ, the general equation deduced from the second law of thermodynamics is

$$\frac{dp}{d \log \theta} = \mathrm{M}, \quad \ldots \ldots \quad (29)$$

where $\mathrm{M}\,dv$ represents the heat that must be communicated while the volume alters by dv and $d\theta = 0$. In the application of (29) to radiation we have evidently

$$\mathrm{M} = \mathrm{U} + p, \quad \ldots \ldots \quad (30)$$

where U denotes the density of the energy—a function of θ only. Hence †

$$\frac{dp}{d \log \theta} = \mathrm{U} + p. \quad \ldots \ldots \quad (31)$$

If further, as for radiation and for aerial vibrations,

$$p = \tfrac{1}{3}\mathrm{U}, \quad \ldots \ldots \quad (32)$$

it follows at once that

$$d \log \mathrm{U} = 4\, d \log \theta,$$

whence

$$\mathrm{U} \propto \theta^4, \quad \ldots \ldots \quad (33)$$

the well-known law of Stefan. It may be observed that the existence of a pressure is demanded by (31), independently of (32).

If we generalize (32) by taking

$$p = \frac{1}{n}\mathrm{U}, \quad \ldots \ldots \quad (34)$$

where n is some numerical quantity, we obtain as the generalization of (33)

$$\mathrm{U} \propto \theta^{n+1}. \quad \ldots \ldots \quad (35)$$

It is an interesting question whether any analogue of the second law of thermodynamics can be found in the general theory of the pressure of vibrations, whether for example the energy of the vibrations of a stretched string is partially unavailable in the absence of appliances for distinguishing *phases*. It might appear at first sight that the conclusion already given, as to the possibility of recovering the whole energy by mere retreat of the inclosing ring, was a proof to

* Now with an altered meaning.

† Compare Lorentz, Amsterdam Proceedings, Ap. 1901.

the contrary. This argument, however, will not appear conclusive, if we remember that a like proposition is true for the energy of a gas confined adiabatically under a piston. The residual energy of the molecules may be made as small as we please, but the completion of the cycle by pushing the piston back will restore the molecular energy unless we can first abolish the infinitesimal residue remaining after expansion, and this can only be done with the aid of a body at the absolute zero of temperature. It would appear that we may find an analogue for temperature, so far as the vibrations of *one* system are concerned; but, so far as I can see, the analogy breaks down when we attempt a general theory.

XXXV. *On the Physical Peculiarities of Solutions of Gases in Liquids.* By J. Alfred Wanklyn, *Corresponding Member of the Royal Bavarian Academy of Sciences*.*

THOSE who are acquainted with Bunsen's methods of manipulating gases, and especially those chemists who have enjoyed the privilege of Bunsen's personal instruction, will be familiar with Bunsen's zealous care in so using his "Absorptiometer" that every drop of the liquid solvent should come into repeated contact with the gas to be dissolved.

Bunsen's arrangements presuppose that only those particles of the solvent brought into actual contact with the gas become charged with the gas, and that actual passage of the gas from one stratum of liquid to an adjacent stratum either does not take place at all, or is exceedingly slow.

The question may be fairly asked:—Given a quantity of gas in contact with the surface of pure water in a state of absolute quiescence, will that gas penetrate below the surface of the water except with a degree of slowness calling to mind the slow passage of the less diffusive salts which do not traverse a space of 100 millimetres in a fortnight? An experiment on the action of carbonic acid has been made in my laboratory, which I will now describe.

First of all there was the very simple observation that carbonic acid, confined in a tube over mercury, is capable of being absorbed by distilled water kept at rest, that is to say, without being subjected to the shaking up which is usually resorted to in order to bring about such absorptions. The rate of absorption was also noted, and found to be about one

* Communicated by the Author.