The Pseudo-Definition of the Straight Line

Author(s): George Bruce Halsted

Source: *The Mathematical Gazette*, Vol. 3, No. 56 (Mar., 1906), pp. 291-294

Published by: [Mathematical Association](#)

Stable URL: <http://www.jstor.org/stable/3603477>

Accessed: 18-02-2016 16:44 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

<http://www.jstor.org>

examination of the figure proves, we have a theorem that only one conic of a confocal system has a point on it the centre of curvature at which is a given point A .

The figure has been drawn for the case in which B and C are acute angles. In the case of one of them being an obtuse angle, there is only a slight modification of the figure. For instance the conic in this case which has A for the centre of curvature of a point T on it is seen to be a hyperbola.

The following table follows from an examination of the figure drawn for the case of B and C acute angles:

	Number of real Tangents from A .	Number of real Normals from A .
Ellipses from circle at infinity to ellipse through A ,	0	2
Ellipses from ellipse through A to ellipse through T ,	2	2
Ellipses from ellipse through T to line ellipse,	2	4
Hyperbolas from line $A'D'$ to hyperbola through A ,	0	2
All remaining hyperbolas,	2	2

H. L. TRACHTENBERG.

THE PSEUDO-DEFINITION OF THE STRAIGHT LINE.

WE recognise as decadence the substitution for Euclid's theorem I. 20, "Any two sides of a triangle are together greater than the third side," of the objectionable pseudo-axiom, "A straight line is the shortest line from one point to another."

This phrase, as Hilbert has well said, is of necessity meaningless when the concept "length of a curve" has not been defined. Every book on elementary geometry which has introduced as axiom or definition of a straight line any phrase equivalent to "A straight line is the shortest distance between two points," every book which has at its beginning as axiom or definition of a straight line any phrase equivalent to "A straight line is the shortest distance between two points," has been in error. We believe the advances in regard to the foundations of elementary geometry make timely the rejection of this piece of decay from the body of the elements, and that the time is ripe for the return at this point to the pristine purity of Euclid.

If any additional argument were necessary in reference to a matter so palpable, we have it in the fact that the French themselves have repudiated this pseudo-definition introduced so

unfortunately by Legendre. Of it Laisant says: "This definition, almost unanimously abandoned, represents one of the most remarkable examples of the persistence with which an absurdity can propagate itself throughout the centuries."

"In the first place, the idea expressed is incomprehensible to beginners, since it presupposes the notion of the length of a curve; and further, it is a vicious circle, since the length of a curve can only be understood as limit of a sum of rectilinear lengths; moreover, it is not a definition at all, since, on the contrary, it is a proposition or a theorem."

Reflection shows even a boy how far such a phrase is from being simple. It presupposes all the complex of assumptions necessary for measurement. But more and worse; it presumes beforehand the possibility of measuring all kinds of lines that can be drawn anywhere in space, else how could it be said that a certain one of these lines is the shortest distance between two points? Yet the straight line occurs in non-metrical, in descriptive, in pure projective geometry. All postulates demanded by metrical geometry are presupposed in this pseudo-definition. But worse; before measurement is possible we must have a standard, a unit for measurement, and the standard for length is and has always been itself the straight line. Thus measurement of length or distance presupposes the straight line as a necessary pre-condition of its own possibility, and therefore cannot be taken as a simpler and more ultimate notion with which to define the straight line, nor as a simple assumption upon which to build up the opening demonstrations of geometry. Furthermore, the very existence of a minimum is itself an assumption, and here a gratuitous and unnecessary one. These difficulties are so serious, so obvious, it is little short of a miracle that this absurd crudity has not long ago been ejected from elementary demonstrative geometry.

The common and wide-spread crude notion of the length of a string is a physical, not a geometrical notion. By making fast one end of a string and passing the other through a hole or a ring and observing how more and more of it passes out until the string becomes stretched or straight, we have an example of those experiences or experiments which confuse the physical notion of quantity of string with the geometrical notion of length of a piece of a straight line or a sect, or a combination of sects; and this physical notion is at the basis of a common mental confusion which unconsciously thinks it applicable to the length of the semi-circle. Such a mind would square or rectify the circle by putting a chalk mark on the tire of a waggon wheel, or winding a thread around a silver dollar. Such a procedure might serve to correct Solomon's value for π , but all the measurements since the world began are not

enough to prove the theorem: "Two sides of a triangle are together greater than the third."

As Dr. E. W. Hobson says: "It is a very significant fact that the operation of counting, in connection with which numbers, integral and fractional, have their origin, is the one and only absolutely exact operation of a mathematical character which we are able to undertake upon the objects which we perceive. On the other hand, all operations of the nature of measurement which we can perform in connection with the objects of perception contain an essential element of inexactness. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition."

This assumption or definition, a straight line is the shortest of lines, can therefore have no meaning until first has been determined what geometric meaning can be given to the length of a curve.

Moreover, underneath all this is the fact that even the theorem, Euclid I. 20: "Any two sides of a triangle are together greater than the third side"; a proposition which the Sophists said even donkeys knew, simple as it is, depends upon a complex of preceding assumptions. Hilbert has thrown brilliant new light upon it in the *Proceedings of the London Mathematical Society*, 1902, pages 50-68, where he creates a geometry in which two sides of a triangle may be together less than the third side, exhibiting as a specific and definite example a right triangle in which the sum of the two sides is less than the hypotenuse.

In general the length of a curve is defined by the aggregate formed by the lengths of the perimeters of a proper sequence of inscribed polygons. This definition in ordinary cases creates, fixes for the curve a length; but in case no such aggregate is convergent, the curve is regarded as not rectifiable. Even our creative definition has failed to endow it with length, and thus lengthless it remains.

If, however, it can be shown that the lengths of the perimeters of these inscribed polygons form a convergent aggregate which is independent of the particular choice of the polygons of the sequence, the curve is rectifiable, its length being defined by the number given by the aggregate. Always the recondite idea of limit has been found necessary for defining what we are to mean by length of a curve. How visible then the blunder in trying to define the straight by a metric comparison with all curves!

What geometric meaning has the phrase: "A straight line is the shortest line that can be drawn from one point to another"? A relation of equality or inequality between two magnitudes must have some foundation, and be capable of some intelligible test. In the traditional geometry the foundation of all proof

by Euclid's method consists in establishing the congruence of magnitudes.

To make the congruence evident, the geometrical figures are supposed to be applied to one another, of course without changing their form and dimensions. But since no part of a curve can be congruent to any piece of a straight, so, for example, no part of a circle can be equivalent to any sect from the definition of equivalent magnitudes as those which can be cut into pieces congruent in pairs.

In any comparison of size by congruence, we must be able to place one of the magnitudes or portions of it in complete or partial coincidence with the other. No such direct comparison can be instituted between a straight and a line no piece of which is straight. Thus the whole of Euclid's Elements fails utterly to institute or prove any relation as regards size between a sect and an arc joining the same two points. The operation of measurement we cannot effect, rigorously speaking, either for curves or for curved surfaces, since the unit for length is a sect, and the unit for area the square on that sect. In fact, however little may be the parts of a curve, they do not cease to be curved, and consequently they cannot be compared directly with a sect; just as parts of a curved surface are not directly comparable with portions of a plane.

We cannot even affirm that any ratio exists between a circle and its diameter until after we have made some extra-Euclidean and post-Euclidean assumption.

Therefore when Phillips and Fisher, of Yale, give as their definition of a straight (1898, p. 4, No. 7, Def.): "A straight line is a line which is the shortest path between any two of its points," they pass through and beyond Euclid's Elements to give us his simplest element; they institute a metrical comparison not only with circular arcs, but also with all curves known and unknown; they presuppose a foreknowledge of all lines in a definition of the simplest line. Is it still needful to say this is grossly bad logic, bad pedagogy, bad mathematics?

There being no good reason for retaining longer this old bit of decadence, we desire the removal of this blemish, this parasite which has clung so persistently, so obstinately to what Sir Henry Savile called "the beauteous body of geometry."

GEORGE BRUCE HALSTED.

REVIEWS.

PRACTICAL (OR EXPERIMENTAL) MATHEMATICS.

ONE result of the criticisms on the teaching of Mathematics that originated in Professor Perry's address to the British Association in 1901 has been a great increase in the attention paid to practical