SURFACES WITH SPHERICAL LINES OF CURVATURE AND SURFACES WHERE SYSTEMS OF INFLEXIONAL TANGENTS BELONG TO SYSTEMS OF LINEAR COMPLEXES

By J. E. CAMPBELL.

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A CONSIDERABLE amount of attention has been devoted to the study of the system of surfaces on which the lines of curvature are plane or spherical, and the paper which follows is based on the exposition contained in Darboux's *Théorie Générale des Surfaces*, Livre IV, Ch. IX and XI, taken in conjunction with Lie's beautiful contact transformation, by which spheres are transformed into straight lines. Lie's transformation was first presented in his paper, "Über Complexe, inbesondere Linien- und Kugelcomplexe mit Anwendung auf die Theorie partieller Differential-gleichungen," *Math. Annalen*, 5 Bd. (1872), p. 145. A most interesting account of this line geometry and of the famous transformation will be found in the *Geometrie der Berührungstransformationen*, Lie-Scheffers, Vol. I; and I would draw attention to chapters 7, 9, 10, and 14, and in particular to Theorem 12 on p. 373, Theorem 25 on p. 639, and the relations between the line and sphere geometry illustrated on pp. 654 and 655.

In what follows I give a brief account of Lie's transformation; and show how by means of it he connects the geometry of lines of curvature with the geometry of asymptotic lines, and in particular the surfaces, whose lines of curvature are spherical, with surfaces whose asymptotic lines are such that the tangents to them belong to a system of linear complexes. Readers who desire fuller information will refer to the *Berührungstransformationen*. I then consider at much greater length the latter class of surfaces, a knowledge of whose properties has been shown to involve a knowledge of the properties of the first class, and in this investigation I make considerable use of the vectorial notation.

1. Lie's contact transformation has the generating equations

$$x'+\iota y'+xz'+z=0, \quad x(x'-\iota y')-y-z'=0,$$

leading to

$$p'(x-q)+1+qx=0, \quad q'(x-q)+\iota(1-qx)=0, \quad p+z'+q(x'-\iota y')=0.$$

Each element of space $x'y'z'$ can be expressed uniquely in terms of the corresponding element of space xyz. If, however, we wish to express an element of the latter space in terms of the first, we have two alternatives, viz.,

$$x=\frac{p'+\iota q'}{1-\sqrt{1+p'^2+q'^2}} \quad \text{or} \quad \frac{p'+\iota q'}{1+\sqrt{1+p'^2+q'^2}};$$

but we keep to the first of these alternatives.

Eliminating x, y, z between the generating equations and the equations of a straight line

$$a=mz-ny, \quad \beta=nx-lz, \quad \gamma=ly-mx,$$

we have

$$l(x'^2+y'^2+z'^2)-\beta(x'-\iota y')-m(x'+\iota y')+(n+\gamma)z'-a=0.$$

Writing the equation of this sphere in the form

$$x'^2+y'^2+z'^2+2gx'+2fy'+2hz'+c=0,$$

we see that the line whose six coordinates are

$$\frac{l}{1}=\frac{m}{-g+\iota f}=\frac{n}{h-r}=\frac{a}{-c}=\frac{\beta}{-g-\iota f}=\frac{\gamma}{h+r}$$

corresponds to that part of the sphere for which

$$r=(z'+h)\sqrt{1+p'^2+q'^2},$$

where r is the sphere's radius. We call this line the positive correspondent of the sphere. The line deduced from the positive correspondent by changing the sign of r is called the negative correspondent and corresponds to the other hemisphere of the sphere. For a plane, regarded as a sphere of infinite radius, we see that the corresponding lines will be perpendicular to the axis of x.

2. If $l', m', n', a', \beta', \gamma'$ are the coordinates of a linear complex it is said to be special if

$$l'a'+m'\beta'+n'\gamma'=0.$$

If $l_1, m_1, n_1, a_1, \beta_1, \gamma_1$ and $l_2, m_2, n_2, a_2, \beta_2, \gamma_2$ are the coordinates of two linear complexes, and p and q are two parameters, the complexes whose coordinates are

$$p_1 l_1 + p_2 l_2,\ p_1 m_1 + p_2 m_2,\ p_1 n_1 + p_2 n_2,\ p_1 a_1 + p_2 a_2,\ p_1 \beta_1 + p_2 \beta_2,\ p_1 \gamma_1 + p_2 \gamma_2$$

are said to form a pencil. In any pencil of complexes clearly there are two special ones.

Consider now the pencil whose two special complexes are the positive and negative correspondents of a sphere, viz.,

$$l',\ m',\ n',\ a',\ \beta',\ \gamma' \quad \text{and} \quad l',\ m',\ \gamma',\ a',\ \beta',\ n',$$

then the general complex of the pencil has the coordinates

$$l',\ m',\ n' \cos^2 \frac{\theta}{2} + \gamma' \sin^2 \frac{\theta}{2},\quad a',\ \beta',\ \gamma' \cos^2 \frac{\theta}{2} + n' \sin^2 \frac{\theta}{2}.$$

We now see that if $l, m, n, a, \beta, \gamma$ are the six coordinates of any line which is the positive correspondent of a sphere, and if this line belongs to the above general complex, the sphere will cut at an angle θ, the sphere of which $l', m', n', a', \beta', \gamma'$ is the positive correspondent.

3. We know that the two spheres, which touch any surface and whose radii are the principal radii of curvature of the surface at the point, touch the surface at two consecutive points (Salmon, *Solid Geometry*, 4th edition, p. 267). If we apply our contact transformation to the surface and the spheres, we obtain in the other space a surface and two inflectional tangents, and thus the lines of curvature are transformed into asymptotic lines.

If a sphere can be described through a line of curvature it will intersect the surface everywhere along that line at the same angle, and, therefore, all of the spheres of one system at the same angle. This is but a particular case of the well known theorem that two surfaces intersect at a constant angle if their line of intersection is a line of curvature on both surfaces. Conversely it must be shown that if a sphere intersects all of the spheres which have stationary contact with a surface along a line of curvature at the same angle, then the line of curvature is spherical, that is, lies on a sphere.

Let the surface, referred to its lines of curvature, be given vectorially by the equations

$$z_1 = a\lambda_1, \quad z_2 = b\lambda_2,$$

where the surface is traced out by the extremity of the vector z which depends on the two parameters u and v, and where λ is a unit vector parallel to the normal. The suffix 1 denotes that the vector or scalar to which it is attached is the derivative of that vector or scalar with respect to u, and the suffix 2 has a similar meaning; a and b are scalars, which are the principal radii of curvature. The equation of the sphere which has stationary contact with the surface along the $v =$ constant, direction may be written

$$z' = z - a\lambda + a\mu,$$

l' being the vector to any point of it and μ a unit vector.

Let

$$z' = \gamma + c\mu'$$

be a fixed sphere of radius c, γ be the vector to its centre, and μ' a unit vector, and suppose that this sphere cuts all the spheres, obtained by varying u only in the equation

$$z' = z - a\lambda + a\mu,$$

at the same angle θ.

From first principles we have

$$(z - a\lambda - \gamma)^2 + a^2 + c^2 - 2ac\cos\theta = 0,$$

and therefore, differentiating with respect to u and remembering that

$$z_1 = a\lambda_1,$$

we get

$$a_1 S(z - a\lambda - \gamma)\lambda - aa_1 + a_1 c\cos\theta = 0,$$

or

$$S(z - \gamma)\lambda + c\cos\theta = 0.$$

Differentiating again with respect to u and remembering that $Sz_1\lambda$ is zero, we see that

$$S(z - \gamma)\lambda_1 = 0.$$

It follows that a sphere whose centre is at the extremity of γ can be described through the line of curvature along which only u varies.

4. From the theorem just proved it follows that if all the inflectional tangents along an asymptotic line belong to a linear complex whose coordinates are

$$l',\ m',\ n'\cos^2\frac{\theta}{2} + \gamma'\sin^2\frac{\theta}{2}, \qquad \alpha',\ \beta',\ \gamma'\cos^2\frac{\theta}{2} + n'\sin^2\frac{\theta}{2},$$

then, in the transformed surface, the corresponding line of curvature will be spherical. Conversely, from the original theorem, a spherical line of curvature will transform into an asymptotic line, all the tangents to which will belong to a linear complex of the kind we have discussed. If the line of curvature is plane, the corresponding linear complex will have the special mark that l' is zero.

If the line of curvature is circular two spheres can be described through it, and therefore the inflectional tangents in the corresponding asymptotic line will belong to two linear complexes.

Suppose that all the lines of curvature of one system are spherical, then, in the corresponding surface, all the asymptotic lines along which v is constant will have the property that the corresponding inflectional tangents belong to a linear complex whose coordinates are functions of v only, and conversely.

5. Instead, therefore, of investigating the surfaces with plane or spherical lines of curvature, we investigate the surfaces whose systems of inflectional tangents belong to systems of linear complexes, and, on transforming back again, we have the surfaces with plane or spherical lines of curvature.

The equations

$$\alpha = mz - ny, \quad \beta = nx - lz, \quad \gamma = ly - mx,$$

representing a straight line whose six coordinates are $l, m, n, \alpha, \beta, \gamma$, the line belongs to the linear complex whose coordinates are $l', m', n', \alpha', \beta', \gamma'$, when

$$\alpha' l + \beta' m + \gamma' n + l'\alpha + m'\beta + n'\gamma = 0.$$

We can express the theory of the linear complex simply by aid of the vector notation. Let i, j, k be three unit vectors mutually at right angles, and let

$$a' = il' + jm' + kn', \quad b' = i\alpha' + j\beta' + k\gamma',$$

$$a = il + jm + kn, \qquad b = i\alpha + j\beta + k\gamma;$$

then we say that the vectors a and b are the coordinates of the line, and a', b' are the coordinates of the linear complex.

If x and y are any two vectors, I find it more convenient to denote the vectorial part of xy by the symbol $\widehat{xy}$ than by the more usual symbol Vxy, and the scalar part by $\underset{\smile}{xy}$ rather than by Sxy.

The two coordinates of a straight line a and b are connected by the equation

$$\underset{\smile}{ab} = 0,$$

and, if the straight line belongs to the linear complex whose coordinates are a' and b', we have

$$\underset{\smile}{ab'} + \underset{\smile}{ba'} = 0.$$

The equation of the straight line whose coordinates are a, b is

$$\widehat{az'} = b,$$

z' being a vector drawn, as all the other vectors, from the origin, its extremity traces out the line. If the extremity of a vector z traces out any curve in space, we may take, as the two coordinates of any tangent line to this curve,

$$a = z_1, \quad b = \widehat{z_1 z},$$

where u is the parametric coordinate of any point on the curve and z_1 denotes the first derivative of z with respect to u. The equation of the tangent line to the curve will now be

$$\widehat{z_1 z'} = \widehat{z_1 z}.$$

If we now denote by a and b the coordinates of a linear complex, all the tangent lines to the curve traced out by the extremity of the vector z will belong to this linear complex if

$$Sz_1(az+b) = 0.$$

If a and b are fixed, this, then, is the equation of the curve whose tangent lines belong to the given complex.

6. Suppose next that we have a surface and that we choose the parametric coordinates so that the curves

$$v = \text{constant}, \quad u = \text{constant}$$

are the asymptotic lines, then, if the surface is traced out by the extremity of the vector z, we have

$$z_1 = \widehat{ll_1}, \quad z_2 = -\widehat{ll_2},$$

where the suffix 2 attached to any vector denotes the derivative of that vector with respect to v. This is true for any surface, and is for many purposes the most convenient way of studying the properties of the surface. The vector l depends on the two parametric coordinates u and v of the surface and is clearly parallel to the normal to the surface at the point u, v.

Since $$z_{12} = z_{21},$$

we must have $$\widehat{ll_{12}} = 0;$$

and therefore the vector l must satisfy the Laplacian equation

$$l_{12} = pl,$$

where p is a scalar.

This equation is said to be of the first rank if p is zero. If it is not of the first rank let

$$p' = p - \frac{\partial^2 \log p}{\partial u \, \partial v},$$

and let

$$l' = l_1,$$

then we see that

$$l'_{12} = \frac{p_1}{p} l'_2 + pl'.$$

The Laplacian invariants of this new equation are p and p'. If p' is zero we can solve the original equation by quadratures, and it is said to be of the second rank. If it is neither of the first or second rank, let

$$l'' = l'_1 - \frac{p_1}{p} l' = l_{11} - \frac{p_1}{p} l_1,$$

then we see that

$$l''_2 = p'l', \quad l''_{12} = p'_1 l' + p'l'_1,$$

so that

$$l''_{12} - \frac{\partial}{\partial u} \log (pp') \, l''_2 - p'l'' = 0.$$

The invariants of this Laplacian equation in l'' are

$$p' \quad \text{and} \quad p' - \frac{\partial^2 \log (pp')}{\partial u \partial v}.$$

If the second invariant is zero, the original equation

$$l_{12} = pl$$

is said to be of the third rank.

The above is a brief sketch of Laplace's transformations in so far as they will be required for the purpose of this paper. The method is fully explained in Darboux's *Theory of Surfaces*, II, p. 23 and onwards, as also in Forsyth's *Theory of Differential Equations*, Part IV, § 191, &c.

The fact that we are dealing with a vectorial equation

$$l_{12} = pl$$

rather than with a scalar one makes no real difference, for if

$$l = ix + jy + kz,$$

the vectorial equation is equivalent with the three scalar ones

$$x_{12} = px, \quad y_{12} = py, \quad z_{12} = pz.$$

7. Suppose now that the surface has the property that the inflectional tangents which touch the asymptotic line $v =$ constant belong to a linear complex whose coordinates a and b are functions of v, and that this is true for all values of v, we shall prove that the equation

$$l_{12} = pl$$

is of the first, second, or third rank. We have for any surface which satisfies our condition

$$Sz_1(az+b) = 0;$$

and therefore, differentiating with respect to u,

$$Sz_{11}(az+b).$$

Now the parametric lines being asymptotic, we have

$$z_{11} = mz_1+nz_2,$$

where m and n are scalars.

If n is zero we see at once that

$$z = r\alpha+\beta,$$

where r is a scalar and α and β are vectors depending only on v; that is, the surface is a ruled one. A ruled surface obviously satisfies the condition of the question, and the coordinates a and b of the linear complex only need to satisfy one condition. It is also easily seen that for a ruled surface

$$l_{12} = pl$$

must be of the first or second rank.

Again, if

$$l_{11} = ml_1+sl,$$

where m and s are scalars, we have, since

$$z_1 = \widehat{ll_1},$$

$$z_{11} = mz_1;$$

and therefore the surface is ruled.

We shall not further consider the case of ruled surfaces, and therefore we may assume in future that

$$l_{11} \neq ml_1+sl \quad \text{and} \quad l_{11} \neq mz_1.$$

8. Leaving aside the particular case of ruled surfaces, we shall prove that not only must

$$l_{12} = pl$$

be of one of the first three ranks, but that, when the surface is given, the coordinates of the complex a and b are determined.

We shall also see that, if the equation is of the necessary rank, we cannot take any value of l which satisfies the equation, but only those which satisfy a particular further condition.

Since $$z_{11} = mz_1 + nz_2,$$

and n is not zero, we have also

$$Sz_2(az+b) = 0.$$

Since, if $$\widehat{az}+b = 0,$$

the surface is ruled, we need only consider the other alternative that it is parallel to the normal, and therefore

$$\widehat{az}+b = kl,$$

where k is some scalar which cannot be zero. Differentiate this equation with respect to u, and we obtain

$$Va\widehat{ll}_1 = kl_1 + k_1 l,$$

that is, $$l_1 \underset{\smile}{al} - l a l_1 = k_1 l + k_1 l,$$

so that $$\underset{\smile}{al} = k, \quad k_1 = -\underset{\smile}{al_1}.$$

Differentiating the first of these equations with respect to u, we see that

$$k_1 = \underset{\smile}{al_1} = -\underset{\smile}{al_1};$$

and therefore k_1 must be zero; that is, k is a function of v only. As k is not zero we may by a transformation of the form

$$v' = \phi(v)$$

take it to be minus unity, and we thus have

$$\underset{\smile}{al}+1 = 0, \quad l_{12} = pl.$$

By differentiation we now obtain

$$\underset{\smile}{l_1 a} = 0, \quad \underset{\smile}{l_1 a_2} = p, \quad p\,\underset{\smile}{la_2} + \underset{\smile}{l_1 a_{22}} = p_2.$$

9. We first consider the possibility that

$$a_2 = Aa,$$

where A is a scalar function of u. The vector a would then be fixed in direction and p would be zero, so that the equation would be of the first rank. We then have

$$l = \alpha + \beta,$$

where α is a vector depending on u only and v a vector depending on v only.

Since a must be a vector fixed in direction we may say that

$$a = mk,$$

where i, j, k are unit vectors, mutually at right angles, and m a scalar function of v. If $f(u)$ and $\phi(v)$ are respectively the coefficients of k in α and β, then, in order that we may have

$$\underset{\smile}{al} + 1 = 0,$$

it is necessary and sufficient that

$$m\,[f(u) + \phi(v)] = 1.$$

This involves $f(u)$ being a mere constant and defines m in terms of v, and thus the vector a.

We now leave aside the particular hypothesis as regards a. If, however, the equation is still of rank 1 we have

$$l_1 = m\overset{\frown}{a a_2};$$

and therefore $$l_{11} = \frac{m_1}{m}\,l_1,$$

which leads to a ruled surface.

10. Differentiating with respect to u, the equation

$$p\,\underset{\smile}{la_2} + \underset{\smile}{l_1 a_{22}} = p\,\frac{\partial}{\partial v}\log p,$$

we get $$pp' = \frac{p_1}{p}\,\underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}},$$

where $$p' = p - \frac{\partial^2 \log p}{\partial u\,\partial v}.$$

If the equation is of the second rank so that p' is zero, then, from what we have said about Laplace's transformation, we know that

$$l_{11}-\frac{p_1}{p}l_1=\zeta,$$

when ζ is a vector depending on u only. We therefore have, from what we have just proved, the further fact about this vector ζ, viz., that

$$\underbrace{\zeta a_{22}}=0.$$

But $\underbrace{l_1 a}=0,\quad \underbrace{l_{11}a}=0,\quad \underbrace{l_{11}a_2}=p_1,\quad \underbrace{l_1 a_2}=p;$

and therefore also $\underbrace{\zeta a}=0,\quad \underbrace{\zeta a_2}=0.$

We leave aside the case where ζ is zero as we have already considered it, since then

$$l_{11}=\frac{p_1}{p}l_1.$$

The vectors a, a_2, and a_{22} being each perpendicular to ζ must be coplanar, and therefore

$$a_{22}=Aa_2+Ba,$$

where A and B are scalars. It follows that the vector $\widehat{aa_2}$ is fixed *in direction* in space; for this direction does not depend on u, and we have now seen that in the particular case before us it does not depend on v either.

It follows that γ denoting a unit vector in this fixed direction

$$\zeta=m\gamma,$$

where m is a scalar depending on u only. By a transformation of the form

$$du'=\sqrt{m}\,du$$

we can therefore bring the equations

$$l_{11}-\frac{p_1}{p}l_1=m\gamma \quad\text{and}\quad l_{12}=pl,$$

to the respective forms

$$l_{11}-\frac{p_1'}{p'}l_1=\gamma \quad\text{and}\quad l_{12}=p'l,$$

where $$p'=pm^{-\frac{1}{2}}.$$

We now omit accent on p and have to find the most general common integral of

$$p\frac{\partial}{\partial u}\frac{l_1}{p}=\gamma \quad\text{and}\quad l_{12}=pl,$$

where
$$p = \frac{\partial^2 \log p}{\partial u \partial v}.$$

Let
$$\theta_1 = \frac{1}{p},$$
then the first equation gives us
$$l_1 \theta_1 = \theta\gamma + \eta,$$
where η is a vector depending on v only.

Differentiating with respect to v and remembering that
$$l_{12} = pl,$$
we see that
$$l = \left(\frac{p_2}{p}\theta + \theta_2\right)\gamma + \frac{p_2}{p}\eta + \eta_2,$$
and it is easily verified that this value of l satisfies the condition of being the common integral of the two equations
$$l_{11} - \frac{p_1}{p} l_1 = \gamma \quad \text{and} \quad l_{12} = pl.$$
In order that we may also have
$$\underset{\smile}{al} + 1 = 0,$$
where a depends on v only, we see that it is necessary and sufficient that the vector a should satisfy the three conditions
$$\underset{\smile}{a\gamma} = 0, \quad \underset{\smile}{a\eta} = 0, \quad \underset{\smile}{a\eta_2} + 1 = 0;$$
for we saw from the definition of γ that the first of these equations must be true, and the second follows from the fact that $\underset{\smile}{al_1} = 0$. We thus have the solution required of the important particular case of
$$l_{12} = pl$$
being of the second rank.

It may be noticed in passing that it is only when the equation is of the first or second rank that the vectors a, a_2, a_{22}, and therefore all the other derived vectors a_{222}, ..., are coplanar.

11. We now assume that p' is not zero, and by differentiating with respect to v the equation
$$pp' = \frac{p_1}{p} \underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}}$$

(obtained at the beginning of § 10), and remembering that

$$l_{112} = p_1 l + p l_1,$$

we have
$$pp' \frac{\partial}{\partial v} \log pp' = \frac{p_1}{p} \underset{\smile}{l_1 a_{222}} - \underset{\smile}{l_{11} a_{222}} - p' \underset{\smile}{l_1 a_{22}}.$$

Let
$$a_{222} = A a_{22} + B a_2 + C a,$$

where A, B, C are scalars depending on v only, then since

$$\frac{p_1}{p} \underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}} = pp',$$

$$\frac{p_1}{p} \underset{\smile}{l_1 a_2} - \underset{\smile}{l_{11} a_2} = 0,$$

$$\frac{p_1}{p} \underset{\smile}{l_1 a} - \underset{\smile}{l_{11} a} = 0,$$

we see that
$$\frac{\partial}{\partial v} \log (pp') = A - \frac{\underset{\smile}{l_1 a_{22}}}{p}.$$

Differentiating once more with respect to u, we immediately deduce from the equation

$$pp' = \frac{p_1}{p} \underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}},$$

that
$$p' = \frac{\partial^2 \log (pp')}{\partial u \partial v},$$

so that the equation must be of the third rank.

12. Now we have seen that if

$$l'' = l_{11} - \frac{p_1}{p} l_1,$$

$$l''_{12} - \frac{\partial}{\partial u} \log (pp') l''_2 - p' l'' = 0;$$

and therefore if the original equation is of the third rank a first integral of this equation is

$$l''_1 - \frac{\partial \log pp'}{\partial u} l'' = \zeta,$$

where ζ is a vector depending on u only. But we have just seen that

$$\underset{\smile}{l'' a} = 0, \quad \underset{\smile}{l'' a_2} = 0, \quad \underset{\smile}{l'' a_{22}} + pp' = 0;$$

and therefore $\underset{\smile}{\zeta a} = 0, \quad \underset{\smile}{\zeta a_2} = 0, \quad \underset{\smile}{\zeta a_{22}} = 0.$

It follows, since $$Saa_2a_{22} \neq 0,$$

that ζ must be zero.

We can now find l, for

$$l = \frac{p_2}{p^2}\, l_1 + \frac{\partial}{\partial v}\,\frac{l_1}{p} = \frac{p_2}{p^2}\, l' + \frac{\partial}{\partial v}\,\frac{l'}{p} = \frac{p_2}{p}\,\frac{l_2''}{pp'} + \frac{\partial}{\partial v}\,\frac{l_2''}{pp'}\,;$$

and, since $$pp'\,l_1'' = \frac{\partial}{\partial u}\,(pp')\,l'',$$

$$l'' = pp'\zeta,$$

where ζ is a vector depending on v only; we therefore have

$$l = \frac{p_2}{p}\left(\zeta_2 + \zeta\,\frac{\partial}{\partial v}\log pp'\right) + \frac{\partial}{\partial v}\left(\zeta_2 + \zeta\,\frac{\partial}{\partial v}\log pp'\right),$$

or say $$l = \zeta_{22} + A\zeta_2 + B\zeta,$$

where A and B are scalars defined by

$$A = \frac{\partial}{\partial v}\log p^2p', \qquad B = \frac{\partial}{\partial v}\log p\,\frac{\partial}{\partial v}\log pp' + \frac{\partial^2}{\partial v^2}\log pp'.$$

We have seen that

$$\underset{\smile}{l''a} = 0, \qquad \underset{\smile}{l''a_2} = 0, \qquad \underset{\smile}{l''a_{22}} + pp' = 0\,;$$

and therefore $\underset{\smile}{\zeta a} = 0, \quad \underset{\smile}{\zeta a_2} = 0, \quad \underset{\smile}{\zeta a_{22}} + 1 = 0.$

It follows that the vector a is determined by the equations

$$\underset{\smile}{a\zeta} = 0, \qquad \underset{\smile}{a\zeta_2} = 0, \qquad \underset{\smile}{a\zeta_{22}} + 1 = 0.$$

We have now seen how to determine the vectors a and l which satisfy the three equations $$a_1 = 0, \qquad \underset{\smile}{al} + 1 = 0, \qquad l_{12} = pl,$$

and seen that the latter equation must be of rank 1, 2 or 3 that this may be possible, and that this necessary condition is also sufficient.

Having obtained a and l we can find z by quadratures from

$$z_1 = \widehat{ll_1}, \qquad z_2 = -\widehat{ll_2}.$$

We now see that $$\widehat{az} + l$$

is a vector which depends on v only, since it vanishes on differentiation with respect to u; it is, therefore equal to $-b$, and we have thus found the vectors a and b uniquely which fit the particular surface. We notice that for a ruled surface, and only for a ruled surface, are these indeterminate.

13. We now pass on to consider the properties of surfaces, both of of whose systems of inflectional tangents belong to linear complexes, and we shall slightly change our notation for the sake of symmetry.

$$\text{Let} \quad A = \frac{\partial}{\partial u} \log p^2 p', \quad A' = \frac{\partial}{\partial u} \log p \frac{\partial}{\partial u} \log pp' + \frac{\partial^2}{\partial u^2} \log pp',$$

$$B = \frac{\partial}{\partial v} \log p^2 p', \quad B' = \frac{\partial}{\partial v} \log p \frac{\partial}{\partial v} \log pp' + \frac{\partial^2}{\partial v^2} \log pp'.$$

And let α denote a vector depending on u only, and β a vector depending on v only. We are assuming, for the sake of brevity from this onwards, that the surface is not ruled and that the equation

$$l_{12} = pl$$

is of the third rank.

$$\text{If} \qquad l = \alpha_{11} + A\alpha_1 + A'\alpha,$$

then the surface, given by

$$z_1 = \widehat{ll_1}, \quad z_2 = -\widehat{ll_2},$$

will have the property that its inflectional tangents of the u system will belong to a linear complex system which we can determine. If l can also be written in the form

$$l = \beta_{22} + B\beta_2 + B'\beta;$$

then the inflectional tangents of the v system will also belong to a linear complex system which we can determine.

$$\text{Let} \qquad \xi \equiv \alpha_{11} + A\alpha_1 + A'\alpha,$$

we must try if a vector β can be found such that

$$\beta_{22} + B\beta_2 + B'\beta = \xi,$$

$$\beta_1 = 0.$$

$$\text{Noticing that} \qquad B_1 = p, \quad B_1' = p \frac{\partial}{\partial v} \log pp',$$

$$\text{we see that} \qquad \xi_1 = p\beta_2 + p \frac{\partial}{\partial v} \log pp' \beta;$$

and therefore $$p'\beta = \frac{\partial}{\partial u}\frac{\xi_1}{p},$$

since $$p' = \frac{\partial^2 \log pp'}{\partial u \partial v}.$$

We must therefore have $$\frac{\partial}{\partial u}\frac{1}{p'}\frac{\partial}{\partial u}\frac{\xi_1}{p} = 0;$$

that is, we see that $\xi_{111} - A\xi_{11} + \xi_1(A' - A_1) = 0$.

If we can find a ξ to satisfy this equation, then taking

$$p'\beta = \frac{\partial}{\partial u}\frac{\xi_1}{p},$$

we see that β is a vector depending on v only; and, remembering that

$$\xi_{12} = p\xi,$$

we see that $$\xi_1 = p\beta_2 + p\frac{\partial}{\partial v}\log pp'\beta,$$

and differentiating this with respect to v, we see that

$$\xi = \beta_{22} + B\beta_2 + B'\beta.$$

We have therefore only to see if we can choose α, so that

$$\xi_{111} - A\xi_{11} + (A' - A_1)\xi_1 = 0,$$

where $$\xi \equiv \alpha_{11} + A\alpha_1 + A'\alpha.$$

Expanding the first equation we see that it is necessary and sufficient that a vector α depending on u only should be found such that

$$\alpha_{1111} + (2A_1 + 2A' - A^2)\alpha_{111} + (3A_{11} + 3A_1' - 3AA_1)\alpha_{11}$$
$$+ (A_{111} + 3A_{11}' - AA_{11} - 2AA_1' + A'^2 - A_1^2)\alpha_1$$
$$+ (A_{111}' - AA_{11}' + A'A_1' - A_1A_1')\alpha = 0.$$

This equation may be written

$$\alpha_{1111} + 2a\alpha_{111} + 3a_1\alpha_{11} + (a_{11} + 2a')\alpha_1 + a_1'\alpha = 0,$$

where $$2a = 2A_1 + 2A' - A^2, \quad 2a' = 2A_{11}' - 2AA_1' + A'^2.$$

It may easily be verified that, from the definitions of A and A' and the fact that the equation is of the third rank, a and a' are functions of u only. The only additional condition necessary in order that the surface generated from

$$l = \alpha_{11} + A\alpha_1 + A'\alpha,$$

by

$$z_1 = \widehat{ll_1}, \quad z_2 = -\widehat{ll_2},$$

may have both systems of inflectional tangents belonging to systems of linear complexes, is that α, instead of being an arbitrary vectorial function of u, must be one which satisfies the equation

$$\alpha_{11111} + 2a\alpha_{111} + 3a_1\alpha_{11} + (a_{11} + 2a')\alpha_1 + a_1'\alpha = 0.$$

When α has been obtained, β is given uniquely, as we have seen.