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The Characteristics of an Asymmetric Optical Combination.

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1. The general properties of optical combinations which are symmetrical round an axis, such as ordinary telescopes and microscopes, have, as is well known, been analysed by Gauss; and the performance of these instruments, when aberrations are left out of account, has been shown to depend simply on three constants—which may be taken geometrically as the coordinates of the two principal points and the two principal foci, between whose mutual distances one linear relation exists.

It is now customary, and it conduces to clearness of view, to throw the theory into a geometrical form in the manner first completely set forth by Maxwell (*Quarterly Journal of Mathematics*, II, 1858). To effect this, without entering into details of the construction of the special instrument, we are confined to the use of only those properties that are *characteristic* of rays of light in general. For the purposes of the problem, when restricted to symmetry round the axis, these properties may be stated in the simple approximate form that (i) all rays proceeding from a point go to form the image of that point, (ii) for all such rays the time of passage from point to image ($\sum \mu ds$) is the same, because they belong to the same wave spreading out from the object point, and finally, after passing through the instrument, converging to the image. The application of the former of these principles requires that the rays are everywhere inclined at a small angle to the axis, as is usually the case in practice.

2. It will be convenient to begin by briefly analysing the parts played by these two fundamental principles in the theory; the following mode of procedure is simple and comprehensive.

Let P_1, P_2 and Q_1, Q_2 be two pairs of conjugate foci on the axis, and let the linear magnification transverse to the axis (appropriately called simply the *magnification*) of small objects at P_1, Q_1 be m_p, m_q , respectively; these are clearly the same in all azimuths round the axis. Draw any ray $q_1 R_1 p_1$ meeting the planes through P_1, Q_1 , transverse to the axis in p_1, q_1 , and meeting the axis in R_1 ; after passing through the instrument let its path be $q_2 R_2 p_2$. Then, by the principle of images, $Q_2 q_2 = m_q \cdot Q_1 q_1, P_2 p_2 = m_p \cdot P_1 p_1$ (if the image were inverted, the sign of m would be negative); so that

$$\frac{Q_2 R_2}{R_2 P_2} = \frac{m_q}{m_p} \cdot \frac{Q_1 R_1}{R_1 P_1} \dots\dots\dots(1).$$

This law of simple proportions determines absolutely the relative positions of conjugate foci R_1, R_2 . To find the corresponding magnification m_r , it is only necessary to draw a ray through Q_1 , meeting the transverse planes through R_1, P_1 in ρ_1, ϖ_1 , and passing out at the other side through Q_2, ρ_2, ϖ_2 . Then

$$m_r = \frac{R_2 \rho_2}{R_1 \rho_1} = m_p \frac{Q_2 R_2}{Q_2 P_2} / \frac{Q_1 R_1}{Q_1 P_1} \dots\dots\dots(2).$$

The principle of images is thus sufficient to determine completely the performance of the instrument.

3. But hitherto no notice has been taken of the nature of the optical media in which the object and image lie. We may introduce this consideration by aid of the second general principle. It can be applied by comparing two similarly situated rays, which have therefore the same value of $\Sigma \mu \delta s$ in passage through the instrument. Two such are the rays through q_1 , one, $q_1 p_1$, through R_1 , the middle point of $Q_1 P_1$, and the other $q_1 \varpi_1$, parallel to the axis. The expression $\Sigma \mu \delta s$ has the same value S from p_1 to p_2 as from ϖ_1 to ϖ_2 , because these points are symmetrically situated above and below the axis. Hence, considering the two rays from q_1 to q_2 , we have

$$\mu_1 \cdot q_1 p_1 + S - \mu_2 \cdot q_2 p_2 = \mu_1 \cdot q_1 \varpi_1 + S - \mu_2 \cdot q_2 \varpi_2,$$

therefore $\mu_1 (q_1 p_1 - q_1 \varpi_1) = \mu_2 (q_2 p_2 - q_2 \varpi_2)$.

Therefore, approximately, since the transverse distances are supposed small compared with the longitudinal, and $P_1 p_1 = P_1 \varpi_1$, we have

$$\begin{aligned}\mu_1 \frac{4Q_1 q_1^2}{2Q_1 P_1} &= \mu_2 \frac{(Q_2 q_2 + P_2 p_2)^2 - (Q_2 q_2 - P_2 \varpi_2)^2}{2Q_2 P_2}, \\ \mu_1 \frac{4}{Q_1 P_1} &= \mu_2 \frac{(m_q + m_p)^2 - (m_q - m_p)^2}{Q_2 P_2}, \\ \frac{Q_2 P_2}{Q_1 P_1} &= \frac{\mu_2}{\mu_1} \cdot m_p m_q \dots\dots\dots (3).\end{aligned}$$

The results (1) and (2) give the position and transverse magnification of the image of a small object anywhere situated. The result (3) forms a very elegant expression (given by Maxwell) for the longitudinal magnification (termed by Maxwell the *elongation*) of any object not confined to be small, viz., it is equal to the ratio of the indices multiplied by the product of the transverse magnifications of its extremities.

It follows from (3) that $m_p = \left(\frac{\mu_2 \alpha_2}{\mu_1 \alpha_1}\right)^{-1}$, where α_1, α_2 are the inclinations of the ray to the axis.

It is also clear that (3) includes (2), so that (1) and (3) form a complete system of fundamental formulæ. The points P_1, Q_1 may be taken as the origins of measurement; and then three observations suffice to determine all the constants of the instrument.

4. The scheme introduced by Gauss makes use of the *principal points* A_1, A_2 , for which $m = +1$, and of the principal foci F_1, F_2 .

The image of a point p_1 may now be constructed by drawing rays $p_1 a_1$ parallel to the axis, and $p_1 F_1 a_1$ through F_1 , and tracing them on from their points of emergence on the other principal plane to their point of intersection at p_2 . Since

$$A_1 a_1 = A_2 a_2, \quad A_1 \alpha_1 = A_2 \alpha_2,$$

(because $m = +1$) it follows that the instrument behaves as if the principal planes through A_1, A_2 were coincident, if we leave out of

account the shifting of the image system along the axis consequent on moving A_2 into coincidence with A_1 . With the exception of this shifting, the instrument therefore behaves as a single thin lens whose principal foci are at distances from it equal to A_1F_1, A_2F_2 . In fact it follows in a well-known manner from the diagram that, if we consider lines measured from a principal point or focus as positive when away from the instrument,

$$F_1P_1 \cdot F_2P_2 = F_1A_1 \cdot F_2A_2 = \text{constant} \dots\dots\dots (4),$$

the generalisation of Newton's formula for a lens; which also leads to

$$\frac{A_1P_1}{A_1F_1} + \frac{A_2P_2}{A_2F_2} = 1 \dots\dots\dots (5).$$

The principle $\Sigma \mu \delta s = \text{constant}$ for all rays from p_1 to p_2 requires in the same way as above that

$$\frac{A_1F_1}{\mu_1} = \frac{A_2F_2}{\mu_2} \dots\dots\dots (6).$$

Thus when $\mu_1 = \mu_2$, $A_1F_1 = A_2F_2 = f$,

where f may be called the focal length of the instrument; for it is the focal length of the *simple equivalent lens*, convex when f is positive, which, as we have seen, is equivalent to the instrument in all respects except as regards the situation of the image system along the axis. The instrument has, in this case, an *optical centre*, which is the middle point of A_1A_2 .

5. To determine experimentally the constants of the instrument, we may proceed by any of the known methods that apply to lenses. (i) The positions of the principal foci F_1, F_2 may be marked by the aid of a parallel beam; then the positions of any pair of conjugate foci yield the value of f^2 by formula (4). To determine whether the positive or negative value of f is to be taken, we must observe whether the image is inverted or erect. (ii) The bright point may be moved along a graduated scale till its conjugate focus approaches it most nearly; the distance between them is then $A_1A_2 + 4f$, and the centre of the instrument is equidistant from them. Another observation will complete the determination. This method requires modification if $A_1A_2 + 4f$ is small or negative. (iii) Two conjugate foci at a distance c apart being selected, the instrument is shifted along its axis through a distance a till they again become conjugate; then, if $A_1A_2 = b$,

$$a^2 = (c-b)(c-b-4f).$$

Another observation will suffice.

6. With an instrument (telescope) so focussed that a system of rays parallel to the axis emerges parallel to the axis, the magnification is obviously constant at all distances. Therefore so also is the elongation, by (3). The value of the magnification is now the essential constant of the instrument; the exact position of the image system along the axis being, as before, a thing inessential.

7. This brief sketch has shown that a symmetrical optical combination may usually be replaced by its simple equivalent lens, so far as regards the relative position and magnitudes of the images formed. It is easy to see that it may be replaced by two lenses so as to give their *exact* positions on the axis.

The principal motive of this discussion is to inquire how far similar specifications hold for instruments not symmetrical.

The properties *characteristic* of a ray system now take a more complicated form, viz., that elaborated by Sir W. Rowan Hamilton, from the fundamental property that $\Sigma \mu \delta s$ is stationary along a ray, between any two points, whether in the first medium, or in the final medium, or situated in any other way, and that it has the same value, proportional to the time of propagation, for all the rays from one wave front to another.

The general problem before us is, given the path of the central ray of a pencil of light which traverses any system of media which may be heterogeneous and may be doubly-refracting, but passes from an initial homogeneous isotropic medium to a final one of the same character, to determine how many and what kind of observations in the initial and final media would be necessary in order to obtain a complete account of the nature of the change produced by transmission through the system in any pencil proceeding along this path.

The method of discussing the propagation of a narrow pencil on Hamilton's principles has been set forth afresh by Maxwell (*Proc. Lond. Math. Soc.*, vi, 1874 and 1875); in what follows we shall use a similar analysis.

8. Take origins O_1, O_2 on the central ray in the initial and final media, with axes Z_1, Z_2 tangential to the ray and directed *both away from* the refracting system. Let the value of the reduced path ($\Sigma \mu \delta s$) for a ray between points $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ in the transverse planes through these origins be

$$U \equiv \text{const.} + \frac{1}{2}a_1x_1^2 + c_1x_1y_1 + \frac{1}{2}b_1y_1^2 + px_1x_2 + qx_1y_2 + rx_2y_1 + sy_2y_2 \\ + \frac{1}{2}a_2x_2^2 + c_2x_2y_2 + \frac{1}{2}b_2y_2^2, \dots \dots \dots (7).$$

As U is stationary near the axis, because the transverse planes are tangential to the wave fronts, there can be no terms of the first degree in it.

The ten constants in its expression may be called the constants of the optical combination, when referred to these origins.

9. The first aim must therefore be to choose new origins which will reduce this expression to its simplest form. Taking then the planes $Z_1 = \gamma_1^{-1}$, $Z_2 = \gamma_2^{-1}$ as new transverse planes, we have for the reduced distance between the points $(\xi_1, \eta_1, \gamma_1^{-1})$, $(\xi_2, \eta_2, \gamma_2^{-1})$ in these planes the expression

$$V = \mu_1 s_1 + U + \mu_2 s_2 \dots\dots\dots (8),$$

where $s_1 = \{\gamma_1^{-2} + (x_1 - \xi_1)^2 + (y_1 - \eta_1)^2\}^{\frac{1}{2}}$

$$= \gamma_1^{-1} + \frac{1}{2}\gamma_1 (x_1^2 + y_1^2) - \gamma_1 (x_1 \xi_1 + y_1 \eta_1) + \frac{1}{2}\gamma_1 (\xi_1^2 + \eta_1^2) \dots\dots (9_1),$$

$$s_2 = \gamma_2^{-1} + \frac{1}{2}\gamma_2 (x_2^2 + y_2^2) - \gamma_2 (x_2 \xi_2 + y_2 \eta_2) + \frac{1}{2}\gamma_2 (\xi_2^2 + \eta_2^2) \dots\dots (9_2).$$

We have to eliminate x_1, y_1 and x_2, y_2 from V . Now, since $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ are in the free path of this ray, we must have

$$\frac{\partial V}{\partial x_1} = 0, \quad \frac{\partial V}{\partial y_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0, \quad \frac{\partial V}{\partial y_2} = 0;$$

$$\text{therefore} \quad \left. \begin{aligned} (\mu_1 \gamma_1 + a_1) x_1 + c_1 y_1 + p x_2 + q y_2 &= \mu_1 \gamma_1 \xi_1 \\ c_1 x_1 + (\mu_1 \gamma_1 + b_1) y_1 + r x_2 + s y_2 &= \mu_1 \gamma_1 \eta_1 \\ p x_1 + r y_1 + (\mu_2 \gamma_2 + a_2) x_2 + c_2 y_2 &= \mu_2 \gamma_2 \xi_2 \\ q x_1 + s y_1 + c_2 x_2 + (\mu_2 \gamma_2 + b_2) y_2 &= \mu_2 \gamma_2 \eta_2 \end{aligned} \right\} \dots\dots\dots (10).$$

These equations determine the paths in the initial and final media of the ray which goes from $(\xi_1, \eta_1, \gamma_1^{-1})$ to $(\xi_2, \eta_2, \gamma_2^{-1})$.

10. By solving these equations (10) we obtain the coordinates of the points in which the ray from $(\xi_1, \eta_1, \gamma_1^{-1})$ to $(\xi_2, \eta_2, \gamma_2^{-1})$ meets the transverse planes through the origins. If, however, the determinant Δ of their left-hand sides vanishes, the system is equivalent to only three independent equations, together with the condition.

$$A_1 \mu_1 \gamma_1 \xi_1 + C_1 \mu_1 \gamma_1 \eta_1 + P \mu_2 \gamma_2 \xi_2 + Q \mu_2 \gamma_2 \eta_2 = 0 \dots\dots\dots (11),$$

in which A_1, C_1, P, Q are the minors of the first column (or any other column) of the determinant. The three independent equations do not now determine the coordinates x_1, y_1 in terms of $(\xi_1, \eta_1, \gamma_1^{-1})$ and $(\xi_2, \eta_2, \gamma_2^{-1})$, but merely lead to a linear relation between these coor-

dinates, of the form

$$Sx_1 + Qy_1 = \text{constant} \dots \dots \dots (12),$$

where Q, S are the minors of the terms q, s in the determinant Δ .

Also

$$Qx_2 + Py_2 = \text{constant}.$$

The interpretation is clearly as follows. By (12) the rays from $(\xi_1, \eta_1, \gamma_1^{-1})$ to $(\xi_2, \eta_2, \gamma_2^{-1})$ now form a singly infinite system, which cut the transverse plane through O_1 along a line, and similarly cut the transverse through O_2 along a line. For different pairs of points subject to the relation (11), on the same transverse planes $Z_1 = \gamma_1^{-1}$, $Z_2 = \gamma_2^{-1}$, these lines form parallel systems. The meaning of this relation (11) itself is, that to the point $(\xi_1, \eta_1, \gamma_1^{-1})$ there correspond a system of points which form a line in the plane $Z_2 = \gamma_2^{-1}$; and that, corresponding to different points in the plane $Z_1 = \gamma_1^{-1}$, these lines are all parallel.

The positions of the two transverse planes $Z_1 = \gamma_1^{-1}$, $Z_2 = \gamma_2^{-1}$ are connected by the relation $\Delta = 0$, which is quadratic in both γ_1 and γ_2 . Thus for a beam proceeding from any point in one of these planes, the focal lines lie in the two conjugate planes determined by the equation $\Delta = 0$. For different points in the same transverse plane γ_1 the focal lines lie in the same pair of conjugate planes γ_2 , and form two parallel systems, each inclined to the corresponding axis x_2 at an angle whose tangent is $(-Q/P)$. For all points in either of the planes γ_2 , one of the focal lines lies in the plane γ_1 , and is inclined to the axis x_1 at an angle whose tangent is $(-S/Q)$; and this line is one of a parallel system in that plane, such that all points on one of them have the same focal line in the plane γ_2 .

11. The equation (11) now shows that all the rays emanating from a luminous line or slit whose equation is

$$\mu_1 \gamma_1 (A_1 \xi_1 + C_1 \eta_1) = H \dots \dots \dots (13),$$

pass through the line whose equation is

$$\mu_2 \gamma_2 (P \xi_2 + Q \eta_2) = -H \dots \dots \dots (14).$$

This remark has an interesting bearing on the working of refracting spectroscopes. It shows, in fact, that without any special adjustment of the refracting prismatic surfaces the slit may always be rotated into such a position that its image for any monochromatic light will be a sharp line. But it will not be an image in the ordinary sense of corresponding point for point with the slit. This position of the slit is the one inclined to the axis of x , at an angle

whose tangent is $(-A_1/C_1)$, in which A_1, C_1 are expressed as above in terms of the position of the image, and the constants of the combination.

12. To return to § 9, by solving (10) and substituting in the expression for V , we obtain the function characteristic of the combination when referred to the new transverse planes. The work is simplified by making use of the fact that V is a minimum as above, so that

$$\frac{\partial V}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0, \quad \dots$$

By Euler's theorem of homogeneous functions

$$V = \text{const.} + \mu_1 s_1 + \mu_2 s_2 + 2 \left(\frac{\partial U}{\partial x_1} x_1 + \frac{\partial U}{\partial x_2} x_2 + \frac{\partial U}{\partial y_1} y_1 + \frac{\partial U}{\partial y_2} y_2 \right),$$

where

$$\frac{\partial U}{\partial x_1} = \frac{\partial V}{\partial x_1} - \mu_1 \gamma_1 (x_1 - \xi_1),$$

$$\frac{\partial U}{\partial x_2} = \frac{\partial V}{\partial x_2} - \mu_2 \gamma_2 (x_2 - \xi_2),$$

$$\frac{\partial U}{\partial y_1} = \frac{\partial V}{\partial y_1} - \mu_1 \gamma_1 (y_1 - \eta_1),$$

$$\frac{\partial U}{\partial y_2} = \frac{\partial V}{\partial y_2} - \mu_2 \gamma_2 (y_2 - \eta_2).$$

Therefore

$$\begin{aligned} V &= \text{const.} + \mu_1 s_1 + \mu_2 s_2 - \frac{1}{2} \mu_1 \gamma_1 x_1 (x_1 - \xi_1) - \frac{1}{2} \mu_1 \gamma_1 y_1 (y_1 - \eta_1) \\ &\quad - \frac{1}{2} \mu_2 \gamma_2 x_2 (x_2 - \xi_2) - \frac{1}{2} \mu_2 \gamma_2 y_2 (y_2 - \eta_2) \\ &= \text{const.} + \mu_1 \gamma_1^{-1} - \frac{1}{2} \mu_1 \gamma_1 (\xi_1 x_1 + \eta_1 y_1) + \frac{1}{2} \mu_1 \gamma_1 (\xi_1^2 + \eta_1^2) \\ &\quad + \mu_2 \gamma_2^{-1} - \frac{1}{2} \mu_2 \gamma_2 (\xi_2 x_2 + \eta_2 y_2) + \frac{1}{2} \mu_2 \gamma_2 (\xi_2^2 + \eta_2^2) \dots \dots \dots (15). \end{aligned}$$

In this expression, which is remarkable as not involving explicitly any of the constants, the values of $x_1, y_1, \dots$ in terms of $\xi_1, \eta_1, \dots$ are to be substituted.

13. Before proceeding with this substitution it will be convenient to examine how the coefficients p, q, r, s may be simplified by rotation of the axes. Changing to polar coordinates by writing

$$\left. \begin{aligned} x_1 &= \rho_1 \cos \theta_1 \\ y_1 &= \rho_1 \sin \theta_1 \end{aligned} \right\}, \quad \left. \begin{aligned} x_2 &= \rho_2 \cos \theta_2 \\ y_2 &= \rho_2 \sin \theta_2 \end{aligned} \right\},$$

we have

$$px_1x_2 + qx_1y_2 + rx_2y_1 + sy_1y_2 \\ = \frac{1}{2}\rho_1\rho_2 \{ (p+s) \cos \chi - (q-r) \sin \chi + (p-s) \cos \psi + (q+r) \sin \psi \},$$

where

$$\psi = \theta_1 + \theta_2, \quad \chi = \theta_1 - \theta_2, \\ = \frac{1}{2}\rho_1\rho_2 \{ P \cos (\chi + \epsilon) + P' \cos (\psi + \epsilon') \} \dots\dots\dots (16).$$

The expressions

$$P^2 = (p+s)^2 + (q-r)^2,$$

$$P'^2 = (p-s)^2 + (q+r)^2,$$

are therefore invariant for the transformation; that is,

$$(i) \quad p^2 + s^2 + q^2 + r^2 \dots\dots\dots (17),$$

$$(ii) \quad ps - qr \dots\dots\dots (18),$$

are invariant.

We cannot therefore make p, q, r, s all vanish. It is possible to have $q = 0, r = 0$; and then $p = s$ will involve a transformation of origins such that the first invariant (i) is twice the second (ii).

Conversely this latter condition by itself gives

$$(p-s)^2 + (q+r)^2 = 0,$$

so that, as the quantities are to remain real, it amounts to two conditions $p = s, q = -r$, whatever be the directions of the axes.

14. We proceed to work out the transformation to new origins. It is easy to see that

$$V = \text{const.} + \frac{1}{2}\mu_1\gamma_1 (\xi_1^2 + \eta_1^2) + \frac{1}{2}\mu_2\gamma_2 (\xi_2^2 + \eta_2^2) + \frac{1}{2}W \dots\dots (19),$$

where

$$\begin{vmatrix} \mu_1\gamma_1 + a_1 & c_1 & p & q \\ c_1 & \mu_1\gamma_1 + b_1 & r & s \\ p & r & \mu_2\gamma_2 + a_2 & c_2 \\ q & s & c_2 & \mu_2\gamma_2 + b_2 \end{vmatrix} W \\ = \begin{vmatrix} \mu_1\gamma_1 + a_1 & c_1 & p & q & \mu_1\gamma_1\xi_1 \\ c_1 & \mu_1\gamma_1 + b_1 & r & s & \mu_1\gamma_1\eta_1 \\ p & r & \mu_2\gamma_2 + a_2 & c_2 & \mu_2\gamma_2\xi_2 \\ q & s & c_2 & \mu_2\gamma_2 + b_2 & \mu_2\gamma_2\eta_2 \\ \mu_1\gamma_1\xi_1 & \mu_1\gamma_1\eta_1 & \mu_2\gamma_2\xi_2 & \mu_2\gamma_2\eta_2 & . \end{vmatrix} \dots\dots (20).$$

The coefficients of the terms

$$p'\xi_1\xi_2 + q'\xi_1\eta_2 + r'\xi_2\eta_1 + s'\eta_1\eta_2$$

on the right-hand side are twice the corresponding minors in Δ , the coefficient of W , multiplied by $\mu_1\gamma_1\mu_2\gamma_2$; they involve, if common multipliers are laid aside, only γ_1 , γ_2 , and $\gamma_1\gamma_2$, and these linearly.

We notice that

$$(p's' - q'r')\Delta = (ps - qr)(\mu_1\gamma_2\mu_2\gamma_1)^2.$$

The conditions $p' = s'$, $q' = -r'$, lead to a quadratic equation to determine γ_1 , and then a linear equation for γ_2 . If the roots of the quadratic are real, there are determined in this way two pairs of principal points on the central ray. We cannot then, by turning both sets of axes through the same angle round that ray, make $q' = 0$, so that also $r' = 0$ by the relation of invariance in § 13; but this can be done by rotation through different angles.

The condition for the reality of these principal points is that a certain function of the coefficients, which it is unnecessary to write out, should be positive.

When this condition is not satisfied, we might attempt to make the reduction in another way. By turning the axes through different angles and moving the origins, we can make $p' = s'$, $q' = 0$, $r' = 0$ in an infinite number of ways. The considerations now to be given will show that this process leads to the same result as the above.

The case of imaginary roots may, however, be reduced to the real one by the addition of a known astigmatic lens or obliquely-placed simple lens, orientated to the proper azimuth, whose presence may be afterwards allowed for.

15. Resuming the original notation, there are therefore, with the above restriction, two pairs of origins and axes for either of which the function characteristic of the combination assumes the form

$$\begin{aligned} U \equiv \text{const.} + \frac{1}{2}a_1x_1^2 + c_1x_1y_1 + \frac{1}{2}b_1y_1^2 \\ + p(x_1x_2 + y_1y_2) \\ + \frac{1}{2}a_2x_2^2 + c_2x_2y_2 + \frac{1}{2}b_2y_2^2 + \dots \dots (21). \end{aligned}$$

This may in a manner be taken as the canonical form of the characteristic.

We can now express very simply in Maxwell's manner the equations which determine the form of the emergent beam. Let the characteristic functions of the same beam in the neighbourhoods of

the two origins O_1, O_2 be

$$V_1 = \mu_1 z_1 + \frac{1}{2} A_1 x_1^2 + C_1 x_1 y_1 + \frac{1}{2} B_1 y_1^2 + \dots \dots \dots (22_1),$$

$$V_2 = \mu_2 z_2 + \frac{1}{2} A_2 x_2^2 + C_2 x_2 y_2 + \frac{1}{2} B_2 y_2^2 + \dots \dots \dots (22_2).$$

The other possible quadratic terms are absent because we must have identically satisfied the characteristic equation,

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \mu^2 \dots \dots \dots (23)$$

The value of the reduced path from ξ_1, η_1, ζ_1 to ξ_2, η_2, ζ_2 is

$$\begin{aligned} V = & -(\mu_1 \zeta_1 + \frac{1}{2} A_1 \xi_1^2 + \dots) + \mu_1 z_1 + \frac{1}{2} A_1 x_1^2 + \dots \\ & + U + \mu_2 z_2 + \frac{1}{2} A_2 x_2^2 + \dots - (\mu_2 \zeta_2 + \frac{1}{2} A_2 \xi_2^2 + \dots) \dots \dots \dots (24), \end{aligned}$$

in which, as V is stationary, we must have

$$\frac{\partial V}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0, \quad \frac{\partial V}{\partial y_1} = 0, \quad \frac{\partial V}{\partial y_2} = 0,$$

so that

$$\left. \begin{aligned} (a_1 + A_1) x_1 + (c_1 + C_1) y_1 + p x_2 &= 0 \\ (c_1 + C_1) x_1 + (b_1 + B_1) y_1 + p y_2 &= 0 \\ p x_1 + (a_2 + A_2) x_2 + (c_2 + C_2) y_2 &= 0 \\ p y_1 + (c_2 + C_2) x_2 + (b_2 + B_2) y_2 &= 0 \end{aligned} \right\} \dots \dots \dots (25).$$

These equations determine the point x_2, y_2 which corresponds to any point x_1, y_1 , in two ways. On solving the first pair for x_1, y_1 , and comparing with the second pair, there result the relations,

$$\frac{a_2 + A_2}{b_1 + B_1} = \frac{b_2 + B_2}{a_1 + A_1} = -\frac{c_2 + C_2}{c_1 + C_1} = \frac{p^2}{\Delta'} \dots \dots \dots (26),$$

where

$$\Delta' = (a_1 + A_1)(b_1 + B_1) - (c_1 + C_1)^2.$$

These equations determine the constants of the emergent beam.

When the incident pencil comes from a focus, $A_1 = B_1, C_1 = 0$. It is easy to verify that $(A_2 - B_2)/C_2$ is constant for all positions of that focus, only provided

$$(a_2 - b_2)/c_2 = (a_1 - b_1)/c_1;$$

it is only under this condition that the focal lines corresponding to all foci are parallel.

16. By changing U to polar coordinates, we see that the conditions that the combination is symmetrical round the axis, or acts as such a one, are $a_1 = b_1$, $c_1 = 0$, $a_2 = b_2$, $c_2 = 0$.

If we add to the instrument a thin astigmatic lens at O_1 , of index μ and thickness t_1 , whose effective thickness $(\mu - \mu_1) t_1$ is equal to

$$\frac{1}{2} (a_1 + \lambda_1) x_1^2 + c_1 x_1 y_1 + \frac{1}{2} (b_1 + \lambda_1) y_1^2,$$

and a thin astigmatic lens at O_2 whose effective thickness $(\mu - \mu_2) t_2$ is equal to

$$\frac{1}{2} (a_2 + \lambda_2) x_2^2 + c_2 x_2 y_2 + \frac{1}{2} (b_2 + \lambda_2) y_2^2,$$

the total resulting combination will be given by

$$U = \text{const.} - \lambda_1 (x_1^2 + y_1^2) + p (x_1 x_2 + y_1 y_2) - \lambda_2 (x_2^2 + y_2^2) \dots (27),$$

and therefore it will act as if it were symmetrical round the axis.

A pencil from O_1 as focus will not be affected by the first lens. It will therefore reach the second lens with a circular cross section if it has started with a circular section in the first medium, and has not been so wide as to be partially stopped out somewhere in the instrument. The (so-called) circle of least confusion of the emergent pencil will therefore be at the second lens.

Now it is to be remarked that a pencil of light rays, though it always passes through two focal lines, does not in general possess a circular cross-section at any point on its course. It is evident in fact that the condition for its having a circular section is that it should diverge symmetrically from each focal line.

Hence to find the position of O_1 , place a stop in the path of the incident beam so as to make it circular; and move along the axis the luminous point which emits it until the emergent beam is symmetrical with respect to either of its focal lines; as may be tested by focussing a telescope on the line, covering it with a cross-wire, and then putting the telescope out of focus without rotating the wire. The position of O_2 is then at the corresponding circular cross-section of the emergent beam; or it may be similarly determined by placing the luminous point on the other side of the instrument. The result incidentally appears that if a conical pencil from O_2 has a circular cross-section at O_1 ,* then a conical pencil from O_1 has a circular section at O_2 .

Find now by Prof. Stokes'† or any other method, the astigmatic lens

* This theorem of reciprocity may be proved directly from consideration of the general function of $x_1 y_1 z_1$ and $x_2 y_2 z_2$ characteristic of the combination. It is a case of the proposition formulated and discussed in Thomson and Tait's *Natural Philosophy*, §§ 334, 335.

† *Brit. Assoc. Report*, 1849, p. 10; *Collected Papers*, Vol. II., p. 172.

which placed in the final medium at O_2 , would convert a parallel beam into the actual emergent beam; this will be a lens whose principal focal lengths are equal and of opp site signs to the distances of the focal lines from the circular cross-section of the pencil, measured in the direction the light is travelling.

Proceed similarly with light emitted from the point O_2 , and determine the corresponding lens in the medium at O_1 .

The combination is equivalent to these two lenses, together with an instrument symmetrical with respect to the axis, whose principal foci are at O_1, O_2 . The principal point P_1 of the latter on the side O_1 is at once determined as the point at which a transverse plane must be placed that the beam from O_1 may mark out on it a circle equal to its circular cross-section at O_2 on emergence; and in the same way the other principal point P_2 may be determined,—or else by the relation $O_1 P_1 / \mu_1 = O_2 P_2 / \mu_2$.

17. The experimental determinations here sketched amount to the specification of an instrument which is equivalent to the combination; that is, they give the optical constants of the combination.

The special case in which O_1 or O_2 is at an infinite distance may be noticed. The effect of the lens at O_1 is then to alter a pencil coming from a focus to one coming from two focal lines in fixed directions at constant distances from that focus.

A more special case still would be that of a *quasi-telescope* for which all parallel incident pencils emerge parallel.

18. The positions of these principal points O_1, O_2 become indeterminate, in the case of an instrument with a straight axis, when the planes of the principal curvatures of the pair of astigmatic lenses are parallel; and in the more general case, when c_1 and c_2 vanish or can be made to vanish by rotating the axes on both sides through the same angle. This leads to the condition

$$(a_1 - b_1) / c_1 = (a_2 - b_2) / c_2,$$

as at the end of § 15.

This is the single condition necessary that the combination should behave as one having a straight axis and symmetrical with respect to two perpendicular planes through the axis. In the general notation of § 9, it is therefore the condition that $P/Q = \text{constant}$, when $\Delta = 0$.

When the instrument is of this simple character the course of any ray may be constructed by finding those of its traces on the two pairs of corresponding principal planes in the initial and final media, passing

through the axis of the combination. For it is clear, as Maxwell remarks, from the form

$$U = \text{const.} + \frac{1}{2}a_1x_1^2 + px_1x_2 + \frac{1}{2}a_2x_2^2 \\ + \frac{1}{2}b_1y_1^2 + sy_1y_2 + \frac{1}{2}b_2y_2^2 \dots\dots\dots(27),$$

that in the determination of the path of a ray by the method of § 15, the terms containing x_1 , x_2 are now separated from the terms containing y_1 , y_2 .

Therefore the projections of any ray on the planes of x_1z_1 and x_2z_2 are now related by a construction similar to that which applies to instruments symmetrical round an axis, *i.e.*, there are two pairs of *quasi*-principal foci in these planes, and two pairs of *quasi*-principal points, by means of which these projections may be constructed. And having thus obtained the traces of a given emergent ray on the planes x_2z_2 , y_2z_2 , in terms of those of the same ray when incident on the planes x_1z_1 , y_1z_1 , the problem is solved.

This also follows more directly from the remark that the instrument is essentially the same as a straight one with the principal planes of its two terminal aplanatic lenses parallel, and therefore planes of symmetry.

It remains to show how these planes, and their cardinal points, may be identified experimentally. All rays incident in the plane x_1z_1 emerge in the plane x_2z_2 . Therefore the plane x_2z_2 must be that containing the axis z_2 , and a focal line corresponding to any focus on the axis z_1 ; there are therefore two such planes, which are the same whatever point on the axis z_1 is taken for focus, as has been already seen. In the same way the two planes x_1z_1 may be found by taking an incident focus on the axis z_2 . These planes may be grouped into the two corresponding pairs in obvious ways. The positions of the cardinal points in a corresponding pair of planes may be determined by methods exactly analogous to those briefly sketched in § 5. The optical combination in question will then have been completely explored.

It might at first sight be imagined that the four possible adjustments of the axes would be sufficient to make c_1 , c_2 , g , r vanish always, and so reduce every combination to this type. But from the above, these conditions must be inconsistent with each other, unless a special relation between the constants is satisfied.