V.—SOME PROPOSED REFORMS IN COMMON LOGIC.

By CHRISTINE LADD FRANKLIN.

The whole field of Deductive Logic, even when thought of merely in the terms of common language, has acquired, for the student of Symbolic Logic, a symmetry and a completeness and a simplicity which it is, apparently, far from possessing in the minds of its usual exponents. The natural repugnance which the ordinary logician felt, at first, to seeing processes of deductive reasoning made the subject of a great development by a purely mechanical process, has in great part passed away; it would have been hard for it to survive the eloquent persuasiveness of Mr. Venn's *Symbolic Logic*. It seems, therefore, to be time for the simplified ways of looking at things which prevail in Symbolic Logic to begin to sink into the elementary expositions of the subject. The simple reforms which I am about to propose in the present paper have nothing in the world to do with Symbolic Logic; but they will, nevertheless, be most likely to commend themselves to one who has been in the habit of moving in the orderly region to which that discipline has reduced the field of Thought.

I.

ON THE NAMING OF RELATED PROPOSITIONS.

It need hardly be stated that the object in giving a name to a group of things is to furnish the mind with an artificial contrivance by which it can, mechanically, call up to consciousness the properties which those things have in common, and which mark them out from other things to which the name is not applied. That system of naming is best which sets forth, in itself, the greatest possible number of resemblances and differences; which is such, in other words, that, when the names of which it is composed have once been learned, as little as possible remembering of relations is left to be done by the mind by main force. A subsidiary requirement, but one which it is not too much to say should always be met, is that the same thing, if it is a subject for consideration in several different branches of learning, should receive the same name in all. There is a set of words which the logician uses—'obverse,' 'converse,' 'reciprocal,' 'contrapositive,' 'inverse,' &c.—which do not meet either of these requirements. 'Obverse' has been adopted by the Society for the Improvement of Geometrical Teaching for a different

proposition from that to which it is usually applied in Logic, and the school-boy who learns in one class-room that 'Non-A is non-B' is the obverse of 'All A is B,' has to painfully bear in mind that its obverse is 'A is not non-B' when he enters another. Nor is there any consensus among logicians themselves as to the use of these names. I propose to show that a new set of names, not very different from those now in use, would enable one to set forth the affinities of the things named in a way that is not even aimed at at present.

'All x is y ' and 'No non- y is x ' are propositions which stand to each other in a certain relation. In what terms we shall describe that relation can only be decided after we have fixed upon our definition of proposition. But there are two definitions in common use, and the proposition is a totally different *thing* according as one or the other of them is adopted. According to one, a proposition is "a portion of discourse in which a predicate is affirmed or denied of a subject"; according to the other, "*every portion of knowledge conveyed in language*, everything propounded for belief or disbelief," is a proposition. But by the latter definition, the two statements just made about x and y are *one and the same proposition*. As statements of fact, as descriptions of the universe in terms of x and y , they are identical; each has the effect simply to deny the existence of the combination xy . But they do not affirm or deny the same predicate of the same subject, and hence, in the other sense of proposition, they are not the same. I conceive that the latter, that is, the material, sense of the word is the more useful and convenient, and that it is better to say that

- 'All x is non- y ,'
- 'All y is non- x ,'
- 'No x is y ,'
- 'There is no x which is y ,'
- 'There is no y which is x ,'
- 'The combination xy does not exist,'
- 'Everything is either non- x or y ,'
- 'Everything is either y or non- x ,'
- 'There is nothing but non- x and y ,'
- 'There is nothing but y and non- x ,'

are all *different forms of the same proposition*, than to say that they are different propositions. But if logicians prefer to use the word in the sense of the Conceptualists, then it is incumbent upon them to furnish some other common name for the above group of propositions. There is a very important respect in which they are one and the same *thing*, and it ought to be possible to indicate that fact by a name. It might be said that they are the same statement-of-fact, or the same description-of-the-universe, or, possibly, simply the same statement, but it seems a pity to

introduce a new technical term into Logic, when 'proposition' and 'form in which it is expressed' answer the purpose perfectly well.

Before proceeding to invent names for the (universal) propositions related to $x \prec y$ ('All x is y '), it will be well to set forth the entire lot of propositions to be named. They are fifteen in number (sixteen including the original proposition); viz., there are four different descriptions of the universe in terms of x and y , according as one or another of the combinations, xy , $\bar{x}\bar{y}$, $x\bar{y}$, $\bar{x}y$ is said not to exist, and each of these statements can be expressed in four different ways. The following table will represent them.

TABLE I.

	o	c	0	∞
A	$x \prec y$	$\bar{y} \prec \bar{x}$	$x\bar{y} \prec 0$	$\infty \prec \bar{x} + y$
V	$\bar{x} \prec \bar{y}$	$y \prec x$	$\bar{x}y \prec 0$	$\infty \prec x + \bar{y}$
E	$x \prec \bar{y}$	$y \prec \bar{x}$	$xy \prec 0$	$\infty \prec \bar{x} + \bar{y}$
I	$\bar{x} \prec y$	$\bar{y} \prec x$	$\bar{x}\bar{y} \prec 0$	$\infty \prec x + y$

The propositions in any row are different expressions for the same state of facts, and the propositions in any column are different facts in the same form of expression. The propositions in the principal diagonal (they are indicated by being inclosed in heavier lines) may be looked upon as the fundamental forms of expression for the four different statements of fact. The first and third, 'All x is y ' and 'There is no x which is y ,' are the Aristotelian propositions, and I shall indicate them, as is usual, by the letters A and E. The second and fourth are the propositions introduced by De Morgan. I propose to indicate them by the same letters inverted, $\bar{v}$ and $\bar{e}$, and to call these symbols *ad* and *ed* (the *d* in the name carrying an allusion to De Morgan). $y \prec x$ is got from $x \prec y$ by turning it about, and that suggests turning about the symbol; and so with E and $\bar{e}$. $x \prec y$ and $y \prec x$ are together equivalent to the identity $x = y$ (or $\bar{x} = \bar{y}$), and in the same way $x \prec \bar{y}$ and $\bar{y} \prec x$ are together equivalent to the identity $x = \bar{y}$ (or $\bar{x} = y$). As the representative of an identity, I shall use Θ ; it symbolises the flowing together of the two propositions A and $\bar{v}$, or E and $\bar{e}$.¹ (For

¹ Any other two distinct propositions whatever out of this scheme are equivalent to the denial of the existence of some one of the terms x , y , $\bar{x}$, $\bar{y}$. For instance, $\bar{v}$ and $\bar{e}$, $\bar{x} \prec \bar{y}$ and $\bar{x} \prec y$, assert together that 'Non- x is neither y nor non- y ,' and hence that 'There is no non- x '. It

the corresponding particular propositions, the contradictories of these universals, I propose to use the corresponding small letters.) The four fundamental universal propositions expressed in the forms most appropriate to each,—expressed as it would be necessary to express them if our language did not permit the application of the modifier; *not*, to terms,—are,

A	$x \prec y.$	All x is $y.$
V	$y \prec x.$	All y is $x.$
E	$xy \prec 0.$	There is no xy (or, No x is $y).$
I	$\infty \prec x + y.$	Everything is either x or $y.$ ¹

The capital letters by themselves shall indicate the substance of the propositions; when it is desirable to specialise the form in which they are expressed, subscript characters shall be attached to them. The subscript o shall indicate that the subject and predicate of the proposition have their original position, c that they have both been contraposed, 0 that they have both been put back behind the copula, and ∞ that they have both been put forward in front of it. We shall then have, as regards form, for any four identical propositions,

A	$x \prec y.$	All x is $y.$
A _c	$\bar{y} \prec \bar{x}.$	All non- y is non- $x.$
A ₀	$x\bar{y} \prec 0.$	No x is non- $y.$
A _{∞}	$\infty \prec \bar{x} + y.$	Everything is either non- x or $y.$

All the sixteen propositions which are related to $x \prec y$ can then be *symbolically* named without any difficulty; the character of the letter marks the content of the proposition and the character of the subscript marks its form. It remains to invent a series of *names* which shall have the advantages possessed by this set of symbols, and the main fact to be set forth in the names is that some of these related propositions are, in a certain respect, the *same* as the original proposition, and that others are not. $\bar{y} \prec \bar{x}$

is common among logicians to say that two such propositions are incompatible; but that is not true, they are simply together incompatible with the existence of x . When the school-boy has proved that the meeting-point of two lines is not on the right of a certain transversal and that it is not on the left of it, we do not tell him that his propositions are incompatible and that one or the other of them must be false, but we allow him to draw the natural conclusion that there is no meeting-point, or that the lines are parallel.

¹ I use ∞ as an abbreviation for 'everything,' because it is easily read in that sense. For those students who are afterwards to study Symbolic Logic, it would be as well at once to write 1 as the symbol for the universe of discourse.

is usually called the 'contrapositive' proposition to $x < y$. I propose to call it the *contraposed* proposition (or, as I prefer to say, the *contraposed form* of the same proposition), where *contra* shall mean that the subject and predicate have both changed places. The participial termination in *contraposed* carries with it the intimation that the thing to which it applies is merely a modified form of expression for the same description of the universe. For the proposition $y < x$ (considered in its relation to $x < y$), geometers use the word 'converse' and logicians the word 'inverse'. I suggest the word *contraverse* as a ground for compromise, and the reason for it is that the *contra* indicates (the same as in the word *contraposed*) that the subject and the predicate have both changed places, the non-participial ending indicates that the proposition is *not* the same as the original one, and the last syllable indicates that the latter has suffered a complete inversion (or version) in sense. The word 'obverse,' logicians apply to the proposition 'No x is non- y ' (or, ' x which is non- y is non-existent,' $x\bar{y} < 0$), and geometers to the proposition $\bar{x} < \bar{y}$. If the word be retained in the latter sense, then the *verse* in the words *contraverse* and *obverse* can serve the student as a mechanical reminder of the fact that the propositions to which they refer are, in meaning, equivalent. I therefore propose the following table of names for the sixteen forms of statement that can be made in terms of x and y and their negatives:—

TABLE II.

	o	c	0	∞
A	original $x < y$	contraposed $\bar{y} < \bar{x}$	retroposed $x\bar{y} < 0$	pro-posed $\infty < \bar{x} + y$
V	obverse $\bar{x} < \bar{y}$	contraverse $y < x$	retroverse $\bar{x}y < 0$	proverse $\infty < x + \bar{y}$
E	oblate $x < \bar{y}$	contralate $y < \bar{x}$	retrolate $xy < 0$	prolate $\infty < \bar{x} + \bar{y}$
F	offert $\bar{x} < y$	contrafert $\bar{y} < x$	retrofert $\bar{x}\bar{y} < 0$	profert $\infty < x + y$

The upper left-hand quadrant contains the propositions which there is most frequent occasion to refer to; they are of constant use in geometry, and it would doubtless conduce to clear thinking (and not be a feat impossible of accomplishment) if they were introduced into common life,—at least, in that stratum of society which has once studied geometry. The names in the upper

right-hand quadrant are constructed mechanically, or nearly so, after those of the first quadrant have been fixed upon. The *retroposed* proposition, $xy < 0$, is that in which the predicate has been brought back and joined to the subject, and the *proposed* (the name is not good, but there seems to be nothing else which fits in with the entire scheme) is that in which the subject has been brought forward and joined to the predicate. As a general name for bringing over any term from one side of this copula to the other, the word *transposing* may be used.¹ It is the word which already means, in Algebra, bringing from either side of the sign of equality to the other. The fact that in the passage what was a positive term becomes a negative term, and conversely, does not prevent our thinking of the process, in Algebra, as the bringing over of a term, nor should it do so in Logic.

The names in the third and fourth rows are mere nonsense-words, but they suffice to complete the scheme. They may be of use at some future time, when the human mind has become much more nimble than it is now in logical forms. And, in any event, the facility with which they can be made up serves as ground for the definitive adoption of the names in the first quadrant. *Retroposed* is not so good a name for the proposition $xy < 0$ when it is put in words as 'no x is non- y ,' as when it is expressed ' x which is non- y is nothing,' but in the former case it may at least be looked upon as a nonsense-word. The main point which I wish to insist upon is that, since the young person has got to learn three or four of these names in any case, he should, out of humanity, since it can easily be done, be furnished with names such that the very form of the word (as the termination 'ed') carries with it the signification of *necessary and sufficient condition* that the derived proposition to which it is applied is the exact equivalent of the original one.

For such trivial changes as

from	to
There is no x which is y ,	There is no y which is x ;
No x is y ,	No y is x ;
xy is non-existent,	yx is non-existent ;
Everything is either x or y ,	Everything is either y or x ;
There is nothing but p and q ,	There is nothing but q and p ;

I would propose the word *commutation*. It is the word already in use in Algebra for expressing the fact that the product ab and the sum $a + b$ are equal respectively to ba and $b + a$. But, without referring to its technical mathematical signification, it is a word which may readily be taken as meaning a change in order

¹ Prof. Bain has already felt the necessity for this word when speaking of compound propositions (*Logic*, p. 119).

without any change in value. What I particularly wish to insist upon is that

‘No x is y ’ and ‘No y is x ’

differ from each other just as much as, and no more than,

‘There is no xy ’ and ‘There is no yx ,’

and that the difference is very different in degree from that between

‘No x is y ’ and ‘All x is non- y ’.

These require the application of one of the laws of thought to prove their equivalence; those do not. If one is thinking of the intent (or import) of the propositions ‘No x is y ,’ ‘No y is x ,’ that is, if one is concerned with the fact that such a proposition affirms that none of the individuals marked out by the subject have the qualities represented by the predicate, then the propositions are different; but if one is thinking of the terms merely as names of classes, they are not different—the exclusion of p from q is the same as the exclusion of q from p . The difference which does exist is just the same as that between—

‘There are no citizen-students,’ and ‘There are no student-citizens’.

In the one case, the students are looked over, and it is found that none of them has the quality of being a citizen; in the other, the citizens are examined and found to be deficient in the quality of being a student. But Logic has no difficulty in considering citizen-student and student-citizen as, for all purposes of drawing conclusions from them, one and the same thing.¹ If it were permitted to call this trivial transformation—which is all that can take place after subject and predicate have lost their distinctness through being both (virtually) in the subject or both in the predicate—by the trivial name of ‘commutation,’ then ‘conversion’ might be reserved for the ‘conversion *per accidens*’ of the logician.

The formal definitions for these related propositions should then be the following:—

CONTRAVERSE:—a different proposition, got by taking the same terms, both in opposite places.

OBVERSE:—a different proposition, got by taking the contradictory terms, each in the same place.

CONTRAPOSED FORM:—the same proposition, expressed with contradictory terms, both in opposite places.

RETROPOSED FORM (or negative form):—the same proposition, expressed with the same subject and the contradictory of the predicate, both in the subject (or with the negative copula).

¹ The difference is symbolised by Wundt by distinguishing between cS and sC . It is, of course, sufficiently important in one “universe of discussion,” but not in that in which the sole object is the drawing of conclusions.

PRO-POSED FORM:—the same proposition, expressed with the contradictory of the subject and the same predicate, both in the predicate.

If a proposition is already in the *retroposed* form, it cannot, of course, be *retroposed* ; it can only be partly or wholly *pro-posed*.

II.

THE PARTICULAR PROPOSITION.

If a person wishes to deny the proposition ‘All *a* is *b*’ in the most straightforward way possible, he says, ‘Not all *a* is *b*’. The copula $\prec$, which means, when positive, ‘is wholly’ (‘*a* is wholly *b*,’ or, ‘All *a* is *b*’), will mean, when the sign of negation is placed over it, ‘is not wholly’. $a \succcurlyeq b$ may be read, ‘*a* is not wholly *b*,’ or, ‘Not all *a* is *b*,’ or in the words which the traditional Logic has consecrated, ‘Some *a* is-not *b*’. ‘Not all men are virtuous’ is the same thing as ‘Some men are not virtuous,’ but the former is the more direct and immediate way in which to deny that ‘all men are virtuous,’ and therefore ought to have been chosen as the standard form of words for the contradictory of that expression. The copula, $\prec$, and its negative are the copulas of Mr. Peirce’s Symbolic Logic, but that is not a sufficient reason for not using them in common Logic as a mere printer’s abbreviation for ‘is wholly’ and ‘is not wholly’. (The objection to Mr. Maccoll’s sign for the same thing is that a non-symmetrical relation, such as that between the subject and the predicate of the universal affirmative or the particular negative, ought not to be indicated by a symmetrical sign.)

With the aid of this phonetic mark, brief expression can be given to the four different forms of the four different particular propositions as in Table III. Contradictories will be found in corresponding compartments of this and the previous Table.

TABLE III.

	o	c	0	∞
a	original $x \prec y$	contraposed $\bar{y} \prec \bar{x}$	retroposed $x\bar{y} \prec 0$	pro-posed $\infty \prec \bar{x} + y$
v	obverse $x \prec \bar{y}$	contraverse $y \prec x$	retroverse $\bar{x}y \prec 0$	proverse $\infty \prec x + \bar{y}$
e	oblate $y \prec \bar{y}$	contralate $y \prec \bar{x}$	retrolate $xy \prec 0$	prolate $\infty \prec \bar{x} + \bar{y}$
ə	offert $\bar{x} \prec y$	contrafert $\bar{y} \prec x$	retrofert $\bar{x}\bar{y} \prec 0$	profert $\infty \prec x + y$

The youth who is forced to tread the stony path of Logic would have his way made smoother if, since he must learn to attach some letter to the proposition 'Some x is not y ,' he could be provided with a letter which should carry with it the meaning that the proposition is the simple denial of 'All x is y '. This purpose would be secured if the contradictory of A were called either $\bar{A}$ or a . (I think Mr. Venn says that a has been used by Gergonne with this meaning, but I cannot find the reference at this moment.) The letter alone would then indicate the statement of fact, and if it were desired to particularise the mode of expression, a subscript o , c , 0 , or ∞ might be added. The four particular statements that can be made, expressed in only positive terms, are—

- a $x \supsetneq y$. Not all x is y , or Some x is-not y .
- e $y \supsetneq x$. Not all y is x , or Some y is-not x .
- e $xy \supsetneq 0$. xy is not wholly wanting, or There is some x which is y .
- e $\infty \supsetneq x + y$. There is something besides x and y .

The four forms of statement of one and the same fact are—

- o $x \supsetneq y$. Some x is-not y .
- c $\bar{y} \supsetneq \bar{x}$. Some non- y is-not non- x .
- 0 $x\bar{y} \supsetneq 0$. There is some x which is non- y .
- $\infty \infty \supsetneq \bar{x} + y$. The world is not made up of y 's and non- x 's (of the happy and the vicious).

The relations of every proposition to any proposition which may be thought of as the starting-point can be described in exactly the same terms as in the case of universal propositions. (It should be remembered that these terms are merely relative; $\bar{x} \supsetneq \bar{y}$ is not in itself an obverse proposition, but merely the obverse of $x \supsetneq y$.) The identity of x with y is affirmed by the combination of two propositions A and $\forall$, as $x < y$ and $y < x$. It is simply denied by the alternation of two propositions a and ν ; to say that the x 's are not identical with the y 's is to say that either there are some x 's which are not y 's or else some y 's that are not x 's. As any other two universal propositions amount to the simple denial of the existence of some term, in the same way, to say that either there is some y which is non- x or else there is some y which is x , is merely to say that in any case there is some y , that is, to simply affirm the existence of y .

III.

THE EIGHT COPULAS.

The four modes of expression that are given to the starting-proposition in Table I. are not the only possible modes of expression. They are the forms got by preserving the copula the same, and varying the signs and the positions of the terms. By giving different meanings to that connecting-link, the four different things may be said, both in the universal and the particular, without any change in the terms. For the sake of simplicity in the rules, a slight change may be made in Mr. Peirce's copula; namely, the horizontal line may be inverted and allowed to fall within the angle to the right, thus: $\leq$. For the other universal proposition which is essentially affirmative, but which is symmetrical, we can take the same sign turned up; as, $x \vee y$, 'All but x is y '. The ordinary negative proposition, 'None of x is y ,' or, ' x is wholly excluded from y ,' is naturally written with a completed wedge, $\bar{\vee}$; and for the remaining universal proposition, which is also essentially negative, 'None but x is y ,' we may use the same sign with the angle turned down, thus: $\bar{\leq}$. The particular propositions which contradict these universals will then naturally be written with the same symbols made negative or made affirmative as the case may be. The Table will then stand thus for the four different statements that can be made in both quantities :—

TABLE IV.

All of x is y . A $x \leq y$	None but x is y . V $x \bar{\leq} y$	None of x is y . E $x \bar{\vee} y$	All but x is y . I $x \vee y$
a $x \leq y$ Not all of x is y .	v $x < y$ Some besides x is y .	e $x \vee y$ Some of x is y .	o $x \bar{\vee} y$ Not all but x is y .

And for the four ways of saying the same thing, we shall have :

Universal	$x \leq y$	$\bar{x} \bar{\leq} \bar{y}$	$x \bar{\vee} \bar{y}$	$\bar{x} \vee y$
Particular	$x \leq y$	$\bar{x} < \bar{y}$	$x \vee \bar{y}$	$\bar{x} \bar{\vee} y$

It will be observed that of the universal propositions there are two negative and two affirmative, and that the same is true of particular propositions. The distinction between universal and particular copulas is this: every universal is made up of an odd

number of marks (namely, three), and every particular is made up of an even number of marks (namely, two or four). The rule for the transformation of a statement from one form to another is equally simple: any rotation of a copula necessitates a change of sign in the *subject* (as, $\bar{x} < \bar{y} = x \vee \bar{y}$), and the introduction of a negative sign into the angle of a copula, or the reverse, necessitates a change of sign in both subject and predicate (as, $x \bar{\vee} \bar{y} = \bar{x} \vee y$). It follows that to do both things at once necessitates a change of sign in the *predicate* only (as, $\bar{x} < \bar{y} = \bar{x} \bar{\vee} y$). To illustrate these rules farther, let us start, for instance, with $x \bar{\vee} \bar{y}$, 'No x is non- y '. First put the sign of negation inside the angle, and we get for an equivalent statement (if we change the sign of both subject and predicate) $\bar{x} \bar{\vee} y$, 'All but non- x is y '. Then rotate this last symbol (at the same time changing the sign of the subject) and we have for an equivalent statement $x \leq y$, 'All x is y '. If, however, we start with $x \bar{\vee} \bar{y}$, 'No x is non- y ,' and change the place of the non-oblique line and at the same time rotate the symbol (and change the sign of the predicate, as the rule requires) we get $x \leq y$, 'All x is y ,' the same as before. In applying the rule to particular propositions, it must be observed that a double mark inside an angle, as well as a double negative sign, is the same thing as no mark at all. Start, for instance, with $\bar{x} \bar{\vee} y$, 'Not all but non- x is y ,' and put the sign of negation within the angle, and we get for an equivalent statement (changing the sign of both subject and predicate) $x \vee \bar{y}$, 'Some x is non- y '. Rotate the copula and we get (changing the sign of the subject) $\bar{x} < \bar{y}$, 'Some besides non- x is non- y '. Now take out one of the (double, and therefore invisible) horizontal lines from within this last copula, leaving the other one (and at the same time changing the sign of both subject and predicate) and we get $x \leq y$, 'Not all of x is y '. Otherwise, the change from $\bar{x} \bar{\vee} y$, with which we started, to $x \leq y$, can be effected at once by a simple rotation (and a consequent change of sign of the subject). The rule may also be expressed thus: Given any proposition in any one of these eight copulas, the same proposition expressed with copula of opposite *quality* must have both subject and predicate of opposite sign; expressed with copula of opposite *symmetry*, it must have subject only of opposite sign. With the symmetrical copulas, subject and predicate can be freely interchanged; with the non-symmetrical copulas, subject and predicate change places upon the condition that their quality also is changed. The simple symmetrical copulas, $\vee$ and $\bar{\vee}$, can be inserted anywhere in a logical product, and the worse looking symmetrical copulas can be inserted anywhere in a logical sum. Thus, $a \bar{\vee} bc = ab \bar{\vee} c$, and $a + bc \bar{\vee} d$ ('All but a and bc is d ') = $a \bar{\vee} bc + d$ ('All but a is bc or d ').

IV.

THE LAWS OF THOUGHT.

It is very singular that the form of expression 'Everything is either x or y ' has been treated with such contumely by logicians. It is of common use in daily life. It is necessary to a complete scheme of expression; when it has once been admitted, the problem is readily solved of expressing any proposition whatever, given in terms of S and P , in an exactly equivalent form, with S negative, or with P negative, or with both S and P negative.¹ But, strongest reason of all for not neglecting it, *it is the only form in which one of the Laws of Thought can be expressed.* It is very singular to begin Logic by attributing great importance to the statement that everything is either a or $\bar{a}$, to make frequent use of it in the most fundamental parts of the subject, and then to utterly ignore it as a form of expression the laws for the use of which need to be plainly set forth. The primary Laws of Thought, rightly considered, are four in number, and they are the matrices, so to speak, for the four forms of expression of Table I. The first is the law of the positive term,

$$a \prec a.$$

The second is the law of the negative term,

$$\bar{a} \prec \bar{a}.$$

Either may also be called a law of identity. The other two are the laws which regulate the relations of the negative term and the positive term to each other. They are,

$$a\bar{a} \prec 0 \text{ and } \infty \prec a + \bar{a}.$$

The first expresses the fact that a term and its negative have no part in common, and the second that they together occupy the whole field of discourse. The first is usually called the Law of Contradiction. It is a very unfortunate name for it. The terms a and non- a are called contradictories, and the law of contradiction ought to mean the same thing as the law for contradictories,—that is, the law which furnishes the necessary and sufficient condition that two terms should be mutual contradictories. But it takes *both* the above laws to furnish that

¹ It is as important as any other form when it comes to the expression of the compound proposition. Either 'No a is b ' (p) or else 'All c is d ' (q) means that $\infty \prec p + q$, that is, that all possible cases in the given universe of thought are summed up in the case of no a being b together with the case of all c being d . If it had not been for the accident of Aristotle's having excluded this form of statement from his simple propositions, it would probably never have occurred to Lotze that the Disjunctive Proposition is the expression for some pure and subtle form of knowledge of a superior kind to that which can be expressed in such words as 'If some a is b , then all c is d '.

condition. Suppose I wish to find out whether the colours red and blue on a certain palette are the exact contradictories of each other: I have to convince myself both that they do not overlap and that they together cover the entire palette. It is these two properties together that constitute their contradictoriness. The first of them, which states that a and $\bar{a}$ have nothing in common, I propose to call the Law of *Mutual Exclusion*; it is true that this is too suggestive of 'Excluded Middle,' which is the old name of the other law, but it is too plainly the right name for the thing to be given up on that account. The other property, which states that a and $\bar{a}$ together fill up the whole universe of discourse, I propose to call the Law of *Collective Exhaustion*, or simply of *Exhaustion*. These two names, besides being exactly descriptive of the properties in question, have the great advantage of lending themselves to the formation of adjectives. If two terms have been discovered to be *mutually exclusive* and *exhaustive*, either can be at once set down as the negative of the other. When it comes to the discussion of logical division, these two descriptive epithets are not only indispensable, but actually in use (Bain, *Logic*, pp. 426-428). Why not use the same names for the same properties when those properties first come into notice? The following are, then, the Laws of Thought in tabular form:—

TABLE V.

Laws of Identity, - - -	{ (o) $a < a$. Positive term. (c) $\bar{a} < \bar{a}$. Negative term.
Laws of Co-relation of Positive and Negative Terms.	{ (0) $a\bar{a} < 0$. Mutual Exclusion. (∞) $\infty < a + \bar{a}$. Exhaustion.

(c) is of somewhat, but not entirely, fictitious interest; in propositions in two different terms, the contraposed form does not differ in itself from the original form. Instead of saying that (0) and (∞) are Laws of Thought, and searching for their origin and sanction in various mysterious regions of the mind, it is better to regard them as furnishing together a definition of the negative term. What do we mean by the negative of anything, if we do not mean *all* of that which is *other than* the thing itself?

V.

ON THE PROVING OF RELATED PROPOSITIONS.

One reason why Logic seems to the student so remote from the reasoning of everyday life is that, besides making much ado over some very simple arguments, it does not attempt to grapple at all with others that are equally simple, and equally regular, and of equally frequent occurrence. 'No ripe grapes are sour,' 'No sour grapes are ripe,' 'Nothing which is sour when ripe is a

grape,' are statements which no child would have any trouble in seeing to be equivalent, as matters of fact, and which it would give Logic very little trouble to take account of. If it were not for some unnatural restrictions, such as that subject and predicate must always be considered single and indivisible, a general *aperçu* of the laws of thinking would be more easily got than now, and would cover a larger field. To use a favourite quotation from Hesiod of Professor Sylvester's, "'Tis strange how much greater the part is than the whole!" For instance, every transposition which can be performed upon the proposition limited to two terms is merely a particular case of the one simple rule for transposition in general. What child would hesitate, if asked whether these two propositions are identical or not—'The undevout astronomer is mad,' and 'Every astronomer is either devout or mad'? The identity between $\bar{a}a \prec m$ and $a \prec m + d$ is established at once by a simple reference to the Laws of Thought (or to the Properties of the Negative); since undevout astronomers are mad, and devout ones are devout, they are all either devout or mad; and, contraversely, if all astronomers are devout or mad, since the undevout ones are not devout, they must be mad. The rule is: *Any term which enters the subject as a factor is-the-same-thing-as its negative in the predicate as an alternative.* (This is the only transposition possible with this copula, without loss; the factor must be a factor of the whole subject and the alternative must be an alternative by itself.) With the simple proviso that, of the two sentences, 'There are no virtuous kings' and 'No kings are virtuous,' the first shall be taken as the normal form instead of the second, every case of identical propositions (except simple commutations), *whether affirmative or negative, universal or particular*, is a case of this one single rule. When the *last* term is brought away, it must be remembered that the unexpressed part of the subject is not the same as the unexpressed part of the predicate. The proposition $a \prec b$ means in full $\infty a \prec b + 0$, that is, 'Everything which is a is either b or else non-existent'. (No consistent Logic of universal propositions is possible except with the convention that they do not imply the existence of their terms.) If, in this proposition, we bring back the b , we have $a\bar{b} \prec 0$, if we bring forward the a , we have $\infty \prec \bar{a} + b$, if we make both changes at once we have $\bar{b} \prec \bar{a}$. And for particular propositions the case is exactly the same.

Is it not better for the student to stretch his reasoning powers to the extent of taking in the logical equivalence of the two sentences, 'All ripe grapes are sweet' and 'All grapes are either sweet or else unripe,' and then to have solved all questions of logical equivalence, rather than to spend the time which he must now devote to the discussion of obversion, contraposition and obverted contraposition, each applied in accordance with different rules to each of the four propositions of Aristotle?