

We have also proved that

$$Sl \frac{1}{(1-a_1x)\left(1-\frac{x}{a_1}\right)(1-a_1a_2x^2)\left(1-\frac{x^2}{a_1a_2}\right) \dots n \text{ product-pairs}}$$

$$= \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4) \dots (1-x^n)(1-x^{n+1})},$$

and we have inferred that the latter expression is also the value of

$$Sl \left\{ \frac{1}{(1-a_{n-1}x)\left(1-\frac{a_{n-2}}{a_{n-1}}x\right) \dots \left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)} \right.$$

$$\times \frac{1}{(1-a_{n-1}b_{n-1}x^2)\left(1-\frac{a_{n-2}b_{n-2}}{a_{n-1}b_{n-1}}x^2\right) \dots \left(1-\frac{a_1b_1}{a_2b_2}x^2\right)\left(1-\frac{x^2}{a_1b_1}\right)} \Bigg\}.$$

Comparing these results with the diagram of the general expression given in § 13, it can be inferred that there is a symmetry of value between two expressions, one having m rows and n factors in each row, the other having n rows and m factors in each row. The establishment of this symmetry can be effected with the aid of the theory of partitions, by which subject, indeed, all these fractions were primarily suggested.

2nd October, 1895.

Note on Matrices. By J. BRILL, M.A. Received October 14th, 1895. Communicated November 14th, 1895.

My object in the following short communication is to obtain the most general form of the differential of a matrix which admits of its being commutative with the matrix itself.

If m be the matrix, and $\lambda_1, \lambda_2, \dots, \lambda_n$ its latent roots, which we shall suppose to be all different, then m satisfies the equation

$$(m-\lambda_1)(m-\lambda_2) \dots (m-\lambda_n) = 0 \dots\dots\dots(1).$$

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Differentiating this, we have

$$\begin{aligned} & (dm - d\lambda_1)(m - \lambda_2) \dots (m - \lambda_n) \\ & + (m - \lambda_1)(dm - d\lambda_2) \dots (m - \lambda_n) \\ & + \&c. = 0. \end{aligned}$$

If we now introduce the condition that dm shall be commutative with m , we may write this equation in the form

$$\begin{aligned} & (dm - d\lambda_1)(m - \lambda_2) \dots (m - \lambda_n) \\ & + (dm - d\lambda_2)(m - \lambda_1)(m - \lambda_3) \dots (m - \lambda_n) \\ & + \&c. \\ & + (dm - d\lambda_n)(m - \lambda_1) \dots (m - \lambda_{n-1}) = 0. \end{aligned}$$

Multiplying this equation by $(m - \lambda_1)$, and taking account of equation (1), we have

$$\begin{aligned} & (dm - d\lambda_2)(m - \lambda_1)^2(m - \lambda_3) \dots (m - \lambda_n) \\ & + \&c. \\ & + (dm - d\lambda_n)(m - \lambda_1)^2(m - \lambda_2) \dots (m - \lambda_{n-1}) = 0 \dots \dots (2). \end{aligned}$$

Now

$$\begin{aligned} m - \lambda_1 &= m - \lambda_2 + \lambda_2 - \lambda_1 \\ &= m - \lambda_2 + \lambda_2 - \lambda_1 \\ &= \&c. \end{aligned}$$

Taking account of these relations, and also of equation (1), we easily reduce equation (2) to the form

$$\begin{aligned} & (dm - d\lambda_2)(m - \lambda_1)(m - \lambda_3) \dots (m - \lambda_n)(\lambda_2 - \lambda_1) \\ & + (dm - d\lambda_3)(m - \lambda_1)(m - \lambda_2)(m - \lambda_4) \dots (m - \lambda_n)(\lambda_3 - \lambda_1) \\ & + \&c. \\ & + (dm - d\lambda_n)(m - \lambda_1)(m - \lambda_2) \dots (m - \lambda_{n-1})(\lambda_n - \lambda_1) = 0. \end{aligned}$$

Multiplying this by $(m - \lambda_2)$, it becomes

$$(dm - d\lambda_2)(m - \lambda_1)(m - \lambda_2)^2(m - \lambda_4) \dots (m - \lambda_n)(\lambda_2 - \lambda_1) + \&c. = 0;$$

and, treating this equation in a manner similar to that in which we treated equation (2), we find that it reduces to

$$\begin{aligned} & (dm - d\lambda_2)(m - \lambda_1)(m - \lambda_2)(m - \lambda_4) \dots (m - \lambda_n)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2) \\ & + \&c. \\ & + (dm - d\lambda_n)(m - \lambda_1)(m - \lambda_2) \dots (m - \lambda_{n-1})(\lambda_n - \lambda_1)(\lambda_n - \lambda_2) \\ & = 0. \end{aligned}$$

This process may evidently be continued until we arrive at the result

$$(dm - d\lambda_n)(m - \lambda_1)(m - \lambda_2) \dots (m - \lambda_{n-1})(\lambda_n - \lambda_1) \dots (\lambda_n - \lambda_{n-1}) = 0.$$

Now the product $(\lambda_n - \lambda_1) \dots (\lambda_n - \lambda_{n-1})$

is scalar, and does not vanish. We may therefore divide out by it, and we obtain

$$(dm - d\lambda_n)(m - \lambda_1)(m - \lambda_2) \dots (m - \lambda_{n-1}) = 0.$$

By proceeding in a similar manner, we may also obtain the following:—

$$(dm - d\lambda_1)(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_n) = 0,$$

$$(dm - d\lambda_2)(m - \lambda_1)(m - \lambda_3) \dots (m - \lambda_n) = 0,$$

&c.,

&c.

Thus we readily obtain

$$\begin{aligned} dm \left\{ \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_n)} + \frac{(m - \lambda_1)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) \dots (\lambda_2 - \lambda_n)} + \&c. \right\} \\ = d\lambda_1 \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_n)} + \&c. \end{aligned}$$

Now, by means of Sylvester's Interpolation Theorem, we have that, if $f(m)$ be a function framed on the model of a scalar function, then

$$f(m) = \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_n)} f(\lambda_1) + \&c.;$$

and therefore

$$1 = m^0 = \sum \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_n)}.$$

Thus we obtain, for the most general form of dm that shall be commutative with m , the expression

$$\begin{aligned} d\lambda_1 \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_n)} + d\lambda_2 \frac{(m - \lambda_1)(m - \lambda_3) \dots (m - \lambda_n)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) \dots (\lambda_2 - \lambda_n)} \\ + \&c. + d\lambda_n \frac{(m - \lambda_1)(m - \lambda_2) \dots (m - \lambda_{n-1})}{(\lambda_n - \lambda_1)(\lambda_n - \lambda_2) \dots (\lambda_n - \lambda_{n-1})}. \end{aligned}$$

If there be equalities existing among the latent roots of the matrix, this formula will require modifying in a similar manner to Sylvester's Interpolation Theorem.

[As some points connected with the above paper have been misunderstood, I have thought it advisable to add a few words by way of further explanation. The question arose from an attempt of mine to apply the theory of matrices to obtain solutions of linear differential equations with constant coefficients.* I came across the theorem as applicable to the binary matrix. Struck with the resemblance to Sylvester's Interpolation Theorem, I first of all verified the theorem for the case of the ternary matrix, and then sought a method of establishing it for the general case, with the result given above.

If a matrix be capable of continuous variation, this is only possible through the continuous variation of certain scalar elements involved in its expression. The matrix m satisfies an identical equation, viz. (1). The matrix $m + dm$ will also satisfy an identical equation of similar form, and the main assumption introduced in the above paper is that the latent roots involved in this equation differ infinitesimally from those involved in equation (1). The work will therefore hold only where this is the case. I cannot at present conceive of any case in which the infinitesimal variation of any scalar quantity involved in the expression of a matrix would give rise to a finite change in any of its latent roots. If any such cases should arise, they would need special investigation.

The above work thus furnishes us with a test as to whether, when a matrix is varied by means of the infinitesimal variation of certain scalar elements involved in its expression, the differential of the matrix be commutative with the matrix itself.—*December 14th, 1895.*]

* "On the Application of the Theory of Matrices to the Discussion of Linear Differential Equations with Constant Coefficients," *Proc. Camb. Phil. Soc.*, VIII., 201-210. See also "On Quaternion Functions, with especial reference to the Discussion of Laplace's Equation," *Proc. Camb. Phil. Soc.*, VII., 151-156.