THE INFLEXION CONIC OF A TRINODAL QUARTIC CURVE

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1. By means of the parameter equations of a trinodal quartic curve Brill* has shown that its six points of inflexion lie on a conic. In the following simple proof the equation of the conic is obtained in one of the systems of coordinates used by Salmon.† Suppose that on the trinodal quartic curve

$$\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + \delta^{\frac{1}{2}} = 0, \quad a\alpha + b\beta + c\gamma + d\delta = 0 \quad (1)$$

the point $(\alpha, \beta, \gamma, \delta)$ is an inflexion, and $(\alpha', \beta', \gamma', \delta')$ any point on the tangent there; the equation

$$(\alpha + \lambda\alpha')^{\frac{1}{2}} + (\beta + \lambda\beta')^{\frac{1}{2}} + (\gamma + \lambda\gamma')^{\frac{1}{2}} + (\delta + \lambda\delta')^{\frac{1}{2}} = 0,$$

which when rationalized is of the fourth degree in λ , will have three zero roots. Therefore, by differentiation,

$$\alpha^{-\frac{1}{2}}\alpha' + \beta^{-\frac{1}{2}}\beta' + \dots = 0, \quad \alpha^{-\frac{3}{2}}\alpha'^2 + \beta^{-\frac{3}{2}}\beta'^2 + \dots = 0,$$

also

$$a\alpha' + b\beta' + c\gamma' + d\delta' = 0.$$

All values of $\alpha', \beta', \gamma', \delta'$ consistent with the first and third of these conditions must satisfy the second condition; hence‡

$$\alpha^{\frac{3}{2}}(\alpha + \mu\alpha^{-\frac{1}{2}})^2 + \beta^{\frac{3}{2}}(b + \mu\beta^{-\frac{1}{2}})^2 + \dots$$

must vanish for all values of μ ; so that at a point of inflexion, in addition to equations (1), we have

$$a^2\alpha^{\frac{3}{2}} + b^2\beta^{\frac{3}{2}} + c^2\gamma^{\frac{3}{2}} + d^2\delta^{\frac{3}{2}} = 0. \quad (2)$$

Now consider the product

$$F(z) \equiv (z - a\alpha^{\frac{1}{2}})(z - b\beta^{\frac{1}{2}})(z - c\gamma^{\frac{1}{2}})(z - d\delta^{\frac{1}{2}}) \equiv z^4 + pz^3 + qz^2 + rz + s.$$

* *Mathematische Annalen*, Bde. XII., XIII.; see also Stahl, *Crelle*, Vol. CI., and Meyer, *Apolarität*, p. 283. Stahl mentions also a dissertation by Grassmann, Berlin. 1875.

† *Higher Plane Curves*, pp. 257, 254.

‡ The conditions are the same as those which in geometry of three dimensions express that a straight line lies wholly upon a quadric, viz., that every plane containing the line touches the quadric.

It is clear that $s = abcd\alpha^3\beta^3\gamma^3\delta^3$, and also that

$$\begin{aligned} & \alpha^3\alpha^2 + b^3\beta^2 + c^3\gamma^2 + d^3\delta^2 + p(\alpha^2\alpha^3 + b^2\beta^3 + \dots) \\ & \quad + q(\alpha\alpha + b\beta + \dots) + r(\alpha^3 + \beta^3 + \dots) + s(\alpha^{-1} + b^{-1} + c^{-1} + d^{-1}) \\ & = \alpha^{-1}F(\alpha\alpha^3) + b^{-1}F(b\beta^3) + c^{-1}F(c\gamma^3) + d^{-1}F(d\delta^3) = 0. \end{aligned} \quad (8)$$

Therefore at the inflexions of the curve (1) we have, from (1), (2), and (8),

$$\alpha^3\alpha^2 + b^3\beta^2 + c^3\gamma^2 + d^3\delta^2 + (bcd + cda + dab + abc)\alpha^3\beta^3\gamma^3\delta^3 = 0.$$

Also, as in Salmon, we deduce from $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 0$ that

$$\alpha^2 + \beta^2 + \dots - 2\alpha\beta - \dots = -8\alpha^3\beta^3\gamma^3\delta^3.$$

Hence the six inflexions of the trinodal quartic (1) lie on

$$8(\alpha^3\alpha^2 + b^3\beta^2 + c^3\gamma^2 + d^3\delta^2) = (bcd + cda + dab + abc)(\alpha^2 + \beta^2 + \dots - 2\alpha\beta - \dots),$$

a conic.

2. We can also prove the theorem and find the equation of the inflexion conic, when the equation of the curve, referred to the nodal triangle, is given.*

If the fixed conic $(a, b, c, f, g, h)(X, Y, Z)^2 = 0$ is osculated by the variable conic $\lambda YZ + \mu ZX + \nu XY = 0$, the six points of osculation lie on the cubic

$$\frac{aX}{aX+hY+gZ} + \frac{bY}{hX+bY+fZ} + \frac{cZ}{gX+fY+cZ} = 0.$$

This cubic does not pass through the vertices of the triangle of reference, but the cubic

$$\begin{aligned} & \alpha X(hX+bY+fZ)(gX+fY+cZ) + \dots \\ & \quad - (LX+MY+NZ)(aX^2+bY^2+\dots+2fYZ+\dots), \end{aligned}$$

which also passes through the six inflexions, will do so, provided $L = gh$, $M = hf$, $N = fg$, and its equation is then

$$AYZ(hY+gZ) + BZX(fZ+hX) + CXY(gX+fY) + DXYZ = 0,$$

where A, B, C denote the corresponding minors of

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad \text{and} \quad D \equiv \frac{1}{2}(af^2 + bg^2 + ch^2 + 3abc - 6fgh).$$

* Salmon, *Higher Plane Curves*, p. 254.

Transforming this equation by the relations

$$X : Y : Z = yz : zx : xy,$$

it follows that the six points of inflexion of the trinodal quartic $(a, b, c, f, g, h)(1/x, 1/y, 1/z)^2 = 0$ lie on the cubic

$$Ax^2(hz + gy) + By^2(fx + hz) + Cz^2(gy + fx) + Dxyz = 0. \quad (4)$$

The conic $1/fx + 1/gy + 1/hz = 0$ has evidently triple contact with (4) at the three nodes, and hence the remaining intersections of cubic and quartic, viz., the six inflexions, must also lie on a conic.

To find its equation, we express that

$$\begin{aligned} (ay^2z^2 + \dots + 2fx^2yz + \dots) + [Ax^2(hz + gy) + \dots + Dxyz][\alpha x + \beta y + \gamma z] \\ \equiv (ghyz + \dots)(ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy). \end{aligned}$$

We easily find that

$$\begin{aligned} u &= \frac{A\alpha}{f}, \quad \dots, \quad \alpha = \frac{f(gG + hH)}{(Ag^2h^2 + \dots) - fghD}, \quad \dots, \\ u' &= -\frac{fF^2 + \frac{1}{2}\Delta(2gh + 3af)}{(Ag^2h^2 + \dots) - fghD}, \quad \dots \end{aligned}$$

Hence the equation of the inflexion conic is

$$A(gG + hH)x^2 + \dots - [2fF^2 + \Delta(gh + \frac{3}{2}af)]yz - \dots = 0,*$$

and the two other points where this conic meets the quartic lie on the line

$$f(gG + hH)x + \dots = 0,$$

and hence they are easily seen to be the points where we have

$$\frac{af}{x} + \frac{bg}{y} + \frac{ch}{z} = 0.$$

* This form is given by Brill, *Math. Ann.*, Vol. XII., p. 106, lines 10-12.